Single-letter languages accepted by alternating and probabilistic pushdown automata

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Abstract

We consider 1-way k-tape alternating pushdown automata (k-apa) and 1-way k-tape alternating finite automata (k-afa). We say that an alternating automaton accepts a language L with $f(n)$-bounded maximal (respectively, minimal) leaf-size if arbitrary (respectively, at least one) accepting tree for any $w \in L$ has no more than $f(|w|)$ leaves. The main results of the article are the following. For 1-way k-tape alternating counter automata, 1-tape alternating pushdown automata and 1-way synchronized alternating finite automata, if theses automata accepts language L over one-letter alphabet with $o(\log n)$-bounded maximal leaf-size or $o(\log\log n)$-bounded minimal leaf-size then the language L is semilinear. Moreover, if a language L is accepted with $o(\log\log(n))$-bounded minimal (respectively, $o(\log(n))$-bounded maximal) leafsize then it is accepted by constant-bounded minimal (respectively, maximal) leafsize by the same automaton.

To show that this bound is optimal we prove that 1-apa and 4-tape afa can accept a non-semilinear languages over one-letter alphabet with $O(\log\log n)$-bounded minimal leaf-size. For maximal leaf-size our bound is optimal due to King's results.

Languages over one-letter alphabet accepted by probabilistic one-way 1-tape pushdown automata with isolated cutpoint are proved to be regular.

1 Introduction

In this article we regard languages over one-letter input alphabet accepted by 1-way alternating automata of various types and 1-way probabilistic automata.

There are many reasons why to investigate languages over one-letter alphabet separately, specially accepted by k-tape and k-head alternating automata. The alternation adds very much computational power in contrast to powerful nondeterministic automata investigated by Ibarra [Ib78] has shown that k-tape languages over one letter alphabet accepted by 2-way nondeterministic automata with restricted number of reversals of input heads and with finite number reversal restricted counters and unrestricted pushdown are semilinear languages. The languages accepted by such automata are very close to problems regarding vector addition systems with states [HRHY86].

King [Ki81] has shown that non-regular languages can be accepted already by 2-head 1-way alternating finite automata (2-head 1afa).

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In [Ge88] was shown that the class of languages accepted by k-head 1afa is very complicated and for $k > 2$ emptiness problem is undecidable.
There are many examples showing that capabilities of automata in one letter alphabet differ from one in multi-letter alphabet. Therefore also is of interest to investigate k-tape one-letter languages.
One of the most interesting characteristics of computation by alternating automata is leafsize [Ki81]. Leafsize, in a sense, reflects the number of processors which run in parallel in reading a given input. A sequence of results regarding hierarchies of languages over multi-letter alphabet accepted by different leafsize bound was got. There are two definitions of leafsize bound — minimal and maximal we shall further define more precisely.
In [Hr85] is shown that there is a language that can be accepted by 3-head 1afa with $n^{1/2}$-bounded leafsize and can not be accepted with any k-head 1afa with $o(n/\log(n))$ bounded leafsize ($k \geq 1$). Minimal leafsize bound is used.
In [Ge88] is shown hierarchy for k-tape languages accepted by constant-bounded leafsize ($k \geq 2$). Minimal leafsize bound is used.
In [Na89] is shown hierarchy for 1-tape 1-way alternating pushdown automata by constant-bounded leafsize and hierarchy for $f(n)$-bounded leafsize for some functions $f(n)$ in the gap between $\log(n)$ and $n/\log(n)$. Minimal leafsize bound is used.
In [ITH89] is shown hierarchy by leafsize for languages accepted by 2-dimensional automata with small space and leafsize smaller than $\log(n)$. Maximal leafsize bound is used.
In this article we show that for languages over one-letter alphabet maximal and minimal leafsize bounds for the same language can differ exponentially.
Some further results demonstrates that these bounds are optimal.
In the automata theory there is an important problem to compare different generalizations of the deterministic automata. In context of previous results arises related question on capabilities of 1-way probabilistic pushdown automata in one-letter alphabet.
In the last section of our paper we present one more non-trivial result that languages over one letter alphabet accepted by 1-way probabilistic pushdown automata (ppa) with isolated cutpoint are regular. So we have got an interesting contrast to result by Freivalds [Fr81a] about possibility to recognize some non-context-free language over multi-letter alphabet by ppa with arbitrarily small error probability (hence this language can not be accepted by nondeterministic pushdown automata).

2 Definitions

One-way finite k-tape automaton has a finite set of states and k read-only tapes and k input heads which can never shift left (one head on each of tapes). On each tape is written a word in input alphabet Σ and on the right of each word is written the end marker $\# \notin \Sigma$. Languages accepted by these automata are subsets $L \subseteq (\Sigma^*)^k$.
We consider one-letter case $\Sigma = \{0\}$ only. A language $L \subseteq (\{0\}^k)^*$ is called semilinear if the set $\{(n_1, \ldots, n_k) | (0^{n_1}, \ldots, 0^{n_k}) \in L\}$ is semilinear subset of N^k. It is known that semilinear one-letter languages can be accepted by nondeterministic one-way finite multitape automata. Alternating one-way finite multitape automata can accept a non-semilinear language $\{(0^{2^n}, 0^{2^n})\}$ (it follows from [Ki81a]).
Let $N = \{0, 1, 2, \ldots\}$ the set of natural numbers, and $[0, \infty)$ be the set of all nonnegative real numbers.

Definition 2.1 *Let $\mathcal{U}$ be an alternating or synchronized alternating automaton and $f : N \to [0, \infty)$ be a function. We say that $\mathcal{U}$ accepts a language L with $f(n)$-bounded minimal leaf-size if*

1. *there is no accepting tree for any input word $w \notin L$,*
2. *there is an accepting tree for each $w \in L$, and*
3. *at least one accepting tree for each $w \in L$ has no more than $f(|w|)$ leaves.*

Let $[0, \infty] = [0, \infty) \cup \{\infty\}$.

Definition 2.2 *Let $\mathcal{U}$ be an alternating or synchronized alternating automaton and $f : N \to [0, \infty]$ be a function. We say that $\mathcal{U}$ accepts a language L with $f(n)$-bounded maximal leaf-size if*

1. *there is no accepting tree for any input word $w \notin L$,*
2. *there is an accepting tree for each $w \in L$, and*
3. *arbitrary accepting tree for each $w \in L$ has no more than $f(|w|)$ leaves.*

(In the last definition we allow the values of $f(n)$ being ∞ since for some input words there can be infinitely many accepting trees without maximum of the number of leaves.)
The minimal leaf-size [Ki81, ITT83, MITT85, Hr85, Ge88, Na89] is used more often rather than the maximal one [ITH89].

3 Alternating multitape pushdown automata

In this section we show semilinearity of languages over one-letter alphabet accepted with small amount of leaves.
The main results of the paper are two following theorems.
Let $A = \{k$-tape alternating counter automata, 1-tape alternating pushdown automata$\}$.

Theorem 3.1 *If a language $L \subseteq (\{0\}^*)^k$ is accepted by an automaton of type A with $f(n) = o(\log n)$-bounded maximal leaf-size then L is semilinear.*

Theorem 3.2 *If a language $L \subseteq (\{0\}^*)^k$ is accepted by an automaton of type A with $f(n) = o(\log \log n)$-bounded minimal leaf-size then L is semilinear.*

These theorems will follow from Lemma 3.3 and Theorems 3.4, 3.5.

Lemma 3.3 *If a language $L \subseteq (\{0\}^*)^k$ is accepted by an automaton of type A with const-bounded minimal leaf-size (const $< \infty$) then L is semilinear.*

Theorem 3.4 *If a language $L \subseteq (\{0\}^*)^k$ is accepted by an automaton $\mathcal{U}$ of type A with $f(n) = o(\log n)$-bounded maximal leaf-size then L is accepted by $\mathcal{U}$ also with $g(n)$-bounded maximal leaf-size where $g(n) \le const < \infty$ for all sufficiently large n.*

Theorem 3.5 *If a language $L \subseteq (\{0\}^*)^k$ is accepted by an automaton $\mathcal{U}$ of type A with $f(n) = o(\log \log n)$-bounded minimal leaf-size then L is accepted by $\mathcal{U}$ with const-bounded minimal leaf-size (const $< \infty$).*

To prove Lemma 3.3 and Theorems 3.4, 3.2, we need a very formal definition of k-tape alternating pushdown automata (k-apa).
By technical reasons we define k-apa in a non-traditional way. We assume that the input tape is infinite to the right. The head positions we identify with vector $v \in N^k$. Initial position is zero-vector. A vector of words $(0^{n_1}, 0^{n_2}, \ldots, 0^{n_k})$ is accepted by a computation path if the automaton enters an accepting configuration at heads position $(n_1, \ldots, n_k) \in N^k$. (It may be presumed that our definition restricts the capabilities of k-apa since in contrast to usual definition our automaton has not the endmarker. In fact, it is not so since reading of the endmarker can be simulated by nondeterministic "guessing".)
k-apa with input alphabet $\{0\}$ is a system

$$A = \langle Q, q_0, q_{accept}, U, \Gamma, \gamma_0, \varphi_E, \varphi_U, \psi \rangle \text{ where}$$

- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $q_{accept} \in Q$ is the accepting state,
- $U \subseteq Q$ is the set of universal states,
- Γ is a finite pushdown alphabet,
- $\gamma_0 \in \Gamma$ is the bottom symbol,
- φ_E, φ_U and ψ are the components of the transition function:

$$\begin{aligned}
&\varphi_e : (Q \setminus U) \times \Gamma \to 2^Q \\
&\varphi_U : U \times \Gamma \to Q \times Q, \\
&\psi : (Q \setminus U) \times \Gamma \to \Gamma \cup \{POP\} \cup \{0,1\}^k.
\end{aligned}$$

(To simplify our proofs we assume that transition function can be split in functions φ_E, φ_U and ψ where φ_E describes existential state transitions, φ_U describes universal state transitions, and ψ deterministically defines change of pushdown and moves of the input heads. Values of φ_U are pairs (q', q''). Thus $\mathcal{U}$ never has more than two universal choices. Change of pushdown and moving the input heads are not allowed simultaneously. ψ is not defined on $U \times Q$. Thus we assume that neither moving the input heads nor change of pushdown is allowed simultaneously with universal branching.)
Configuration of k-apa we define as triple (q, v, z) where $q \in Q$ is a state, $v \in N^k$ is a vector (head positions), and $z \in \gamma_0\Gamma^*$ is the content of pushdown. Initial configuration is (q_0, o, γ_0) where o is zero-vector.
Configuration (q, v, z) is called universal (existential) if $q \in U$ $(q \in Q \setminus U)$.
In the set of configurations we define the following relations:

1. $(q, v, z\gamma) \overset{\forall 1}{\leftarrow} (q', v', z')$ where $\gamma \in \Gamma$, iff $q \in U$, $v' = v$, $z' = z\gamma$, and $\varphi_U(q, \gamma) = (q', q'')$ for some q'';

2. $(q, v, z\gamma) \overset{\forall 2}{\leftarrow} (q', v', z')$ where $\gamma \in \Gamma$, iff $q \in U$, $v' = v$, $z' = z\gamma$, and $\varphi_U(q, \gamma) = (q'', q')$ for some q'';

3. $(q, v, z\gamma) \overset{\exists}{\leftarrow} (q', v', z')$ where $\gamma \in \Gamma$, iff $q \in Q \setminus U$, $q' \in \varphi_E(q, \gamma)$, and one of three following conditions holds:

(a) $\psi(q,\gamma) = \gamma' \in \Gamma$, $v' = v$, and $z' = z\gamma\gamma'$;

(b) $\psi(q,\gamma) = POP$, $v' = v$, and $z' = z$;

(c) $\psi(q,\gamma) = \alpha \in \{0,1\}^k$, $v' = v + \alpha$, and $z' = z\gamma$.

Function ψ satisfies requirements

$$\psi(q,\gamma_0) \neq POP \quad \text{and} \quad \psi(q,\gamma) \neq \gamma_0$$

for arbitrary $q \in Q$ and $\gamma \in \Gamma$ (γ can never be popped or pushed).
A computation (computation tree) of $\mathcal{U}$ is a finite, nonempty labeled tree with the properties

(a) each node π is labeled with a configuration $c(\pi)$ and a path word $p(\pi) \in \{1,2\}^*$;

(b) the root π_0 of the tree is labeled with empty word $w(\pi_0) = e$;

(c) if π is an internal (a non-leaf) node of the tree, and $p(\pi)$ is universal, then π has exactly two children π_1, π_2 such that $c(\pi) \overset{\forall 1}{\leftharpoondown} c(\pi_1)$, $c(\pi) \overset{\forall 2}{\leftharpoondown} c(\pi_2)$, $p(\pi_1) = p(\pi)1$, and $p(\pi_2) = p(\pi)2$.

(d) if π is an internal node of the tree, and $c(\pi)$ is existential then π has exactly one child π_1 such that $c(\pi) \overset{\exists}{\leftharpoondown} c(\pi_1)$ and $p(\pi_1) = p(\pi)$.

In the set of leaves of the computation tree we define an order $(\pi_1 < \pi_2) \Leftrightarrow (p(\pi_1) < p(\pi_2))$ where for $w_1, w_2 \in \{1,2\}^*$

$$(w_1 < w_2) \Leftrightarrow (\exists w, w_1', w_2' \in \{1,2\}^*)\ (w_1 = w1w_1' \ \&\ w_2 = w2w_2').$$

We say that $\mathcal{U}$ accepts a vector of words $(0^{n_1}, 0^{n_2}, \ldots, 0^{n_k})$ iff there is a computation whose root is labeled with configuration (q_0, o, γ_0) and whose leaves each are labeled with $(q_{accept}, v, \gamma_0)$ where $v = (n_1, \ldots, n_k) \in N^k$ (clearly, the nontraditional requirement of the pushdown being empty in the moment of accepting is unessential).
To prove our results we are forced to generalize the concept of accepting by allowing configurations being distinct at different leaves of the accepting tree.
We say that a computation of k-apa $\mathcal{U}$ accepts a vector $(v_1, \ldots, v_m) \in N^{mk}$ where $m \geq 1$ and $v_i \in N^k$ iff the root of this computation is labeled with (q_0, o, γ_0), this computation has exactly m leaves $\pi_1, \ldots, \pi_m$ where $\pi_1 < \ldots < \pi_m$, and π_i $(i = 1, \ldots, m)$ is labeled with (q_0, v_i, γ_0).
Clearly, $\mathcal{U}$ accepts $(0^{n_1}, 0^{n_2}, \ldots, 0^{n_k})$ iff it has a computation accepting a vector $mult_m(v) \in N^{mk}$ where $(n_1, n_2, \ldots, n_k) \in N^k$.
Next we use the following notion:

- $accept(\mathcal{U}) \subseteq N^k$ — the set of vectors $(n_1, n_2, \ldots, n_k)$ such that $(0^{n_1}, 0^{n_2}, \ldots, 0^{n_k})$ is accepted by the k-apa $\mathcal{U}$;

- $accept(\mathcal{U}, m) \subseteq N^k$ — the set of vectors $(n_1, n_2, \ldots, n_k)$ such that $(0^{n_1}, 0^{n_2}, \ldots, 0^{n_k})$ is accepted by a computation of $\mathcal{U}$ with exactly m leaves;

- $S_{mk}(\mathcal{U}) \subseteq N^{mk}$ — the set of (mk)-dimensional vectors accepted by $\mathcal{U}$ $(m \geq 1)$.

Note that

(a) k-apa $\mathcal{U}$ accepts a semilinear language iff $accept(\mathcal{U})$ is a semilinear subset of N ;

(b) $accept(\mathcal{U}) = \bigcup_{m \geq 1} accept(\mathcal{U}, m)$;

(c) $accept(\mathcal{U}, m) = diag_k(S_{mk}(\mathcal{U}))$;

(d) $\mathcal{U}$ accepts a language with $f(n)$-bounded minimal leaf-size iff $(\forall n \in N)$ $accept(\mathcal{U}) \cap \{0, \ldots, n\}^k \subseteq \bigcup_{m \leq f(n)} accept(\mathcal{U}, m)$;

(e) $\mathcal{U}$ accepts a language with $f(n)$-bounded maximal leaf-size iff $(\forall n \in N)(\forall m > f(n))$ $accept(\mathcal{U}, m) \cap \{0, \ldots, n\}^k = \emptyset$.

The proofs of Lemma 3.3 and Theorems 3.4 and 3.5 are based on the following Lemmas 3.6.

Lemma 3.6 *For any k-tape apa $\mathcal{U}$ there is a constant $c \geq 1$ such that for arbitrary $m \geq 1$ $S_{mk}(\mathcal{U}) \in \mathcal{SL}_{mk}(c^m, c)$.*

Lemma 3.7 *There is a constant $c \geq 1$ such that for arbitrary $m \geq 1$, $q \in Q$, $\gamma \in \Gamma$ and $\bar{q} \in Q^m$*

$$shift(\mathcal{R}_{idle}(q, \gamma, \bar{q})) \in \mathcal{SL}_{mk}(c^m, c).$$

Lemma 3.8 *There is a constant $d \geq 1$ such that for any $m \geq 1$, $q, q' \in Q$, $\bar{q}, \bar{q}' \in Q^m$, and $\gamma, \gamma' \in \Gamma$*

$$\begin{aligned} &\{mult_m(shift(R_1)) + shift(R_2) | R_1 \in \mathcal{R}_{push}(q, \gamma, q', \gamma') \ \& \\ &\quad R_2 \in \mathcal{MR}_{pop}(\bar{q}', \gamma', \bar{q}, \gamma) \ \& \ push(R_1) = pop(R_2)\} \in \mathcal{SL}(d^m, d). \end{aligned} \quad (1)$$

Lemma 3.9 *For any automaton $\mathcal{U}$ of type A there is a constant $c \geq 1$ such that for arbitrary $m \geq 1$*

$$accept(\mathcal{U}, m) \in \mathcal{SL}_k(c^m, c^m).$$

4 Alternating pushdown automata

In this section we show that there is a non-regular language over one-letter alphabet accepted by 1-way alternating pushdown automaton (apa) with $f(n) = O(\log\log(n))$-bounded minimal leaf-size.
We use $(b|a)$ to denote $a \equiv 0 \pmod{b}$ and $(b \not| a)$ to denote $a \not\equiv 0 \pmod{b}$. For arbitrary $d \geq 1$ we define $b(d) = 2^{\lceil \log_2(d) \rceil}$. We define a language

$$L = \{0^n | (\exists d > 0)((d \not| n) \& (b(d)|n))\}.$$

We say that apa $\mathcal{M}$ is $\exists\forall$-apa automaton if each path in accepting tree from existential initial state to leaf contains at most one alternation.

Theorem 4.1 *There is a $\exists\forall$-apa $\mathcal{M}$ accepting L with $f(n) = O(\log\log(n))$-bounded minimal leaf-size.*

5 Probabilistic pushdown automata

Another interesting generalization of deterministic automata are probabilistic automata.
R. Freivalds has shown that one-way probabilistic pushdown automata are more powerful than deterministic one. For example, it is proved in [Fr81] that the following language L_k can be recognized by one-way probabilistic one-counter automaton with arbitrarily small error probability ε: $L \subseteq \{a_1, b_1, \ldots, a_k, b_k\}^*$ consists of all words w such that for all $i \in \{1, \ldots, k\}$

the number of entries a_i in the word w equals the number of entries b_i in w. For $k \geq 3$ the language L_k can not be accepted by any one-way nondeterministic pushdown automaton.
However, probabilistic pushdown automata are less investigated than nondeterministic and alternating one. We are not able to characterize the class of one-letter languages accepted with isolated cutpoint by probabilistic multitape pushdown or finite automata. We have only one result in this direction, which is stated in the following theorem:

Theorem 5.1 *If a language $L \subseteq \{0\}^*$ is accepted by a ppa with an isolated cutpoint then L is regular.*

The probability of the word w being accepted by a ppa $\mathcal{U}$ we denote by $\chi_{\mathcal{U}}(w)$. We say that $\mathcal{U}$ accepts a language $L \subseteq \Sigma^*$ with δ-isolated cutpoint λ ($0 \leq \lambda \leq 1$, $d > 0$) iff $(\forall w \in L)$ $\chi_{\mathcal{U}}(w) \geq \lambda + \delta$ and $(\forall w \in \Sigma^* \setminus L)$ $\chi_{\mathcal{U}}(w) \leq \lambda - \delta$. Since we consider one-letter alphabet in this section, we can identify the input word with the length of this word. Following this approach, we introduce the notion of probabilistic pushdown automaton without input. The difference between the usual ppa and ppa without input is that ppa without input has the counter instead of the input tape. The counter never can be decreased.
ppa without input is a system $\mathcal{U} = \langle Q, q_0, F, \Gamma, \gamma_0, \varphi \rangle$ where

- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $F \subseteq Q \times \Gamma$ is the set of accepting pairs,
- Γ is a finite pushdown alphabet,
- $\gamma_0 \in \Gamma$ is the bottom symbol,
- $\varphi : Q \times \Gamma \times Q \times \{0,1\} \times (\Gamma \cup \{POP, IDLE\}) \to [0,1]$ is a transition function such that for arbitrary $q \in Q$ and $\gamma \in \Gamma$

  $$\sum_{q' \in Q} \sum_{x \in \{0,1\}} \sum_{\gamma' \in \Gamma \cup \{POP, IDLE\}} \delta(q, \gamma, q', x, \gamma') = 1,$$

  and for all $q, q' \in Q$, $x \in \{0,1\}$ and $\gamma \in \Gamma$

  $$\varphi(q, \gamma_0, q', x, POP) = 0,$$
  $$\varphi(q, \gamma, q', x, \gamma_0) = 0.$$

Configuration of $\mathcal{U}$ at any moment is a triple (q, n, w) where $q \in Q$ is the current state, $n \in N$ is the value of the counter, and $w \in \gamma_0 \Gamma^*$ is the content of pushdown. Initial configuration of $\mathcal{U}$ is $(q_0, 0, \gamma_0)$.
If the current configuration of $\mathcal{U}$ is $(n, q, w\gamma)$ where $\gamma \in \Gamma$ then it enters a new configuration $(n + x, q', w')$ with probability

$$\begin{cases} \varphi(q, \gamma, q', x, \gamma') & \text{if } w' = w\gamma\gamma' \text{ where } \gamma' \in \Gamma; \\ \varphi(q, \gamma, q', x, IDLE) & \text{if } w' = w\gamma; \\ \varphi(q, \gamma, q', x, POP) & \text{if } w' = w; \\ 0 & \text{otherwise.} \end{cases}$$

We say that $\mathcal{U}$ has accepted the number $n \in N$ if it enters a configuration $(q, n, w\gamma)$ where $\gamma \in \Gamma$ and $(q, \gamma) \in F$.

The probability of the number n being accepted by $\mathcal{U}$ we denote by $\chi_{\mathcal{U}}(n)$. We say that $\mathcal{U}$ accepts a set $L \subseteq N$ with δ-isolated cutpoint λ $(0 \leq \lambda \leq 1, \delta > 0)$ if $(\forall n \in L)$ $\chi_{\mathcal{U}}(n) \geq \lambda + \delta$ and $(\forall n \in N \setminus L)$ $\chi_{\mathcal{U}}(n) \leq \lambda - \delta$.
Note that there is some disparity in the definitions of traditional ppa and ppa without input. The usual ppa has the end marker # on the input tape which allows to process the content of pushdown after reading the input word. ppa without input has not such possibility: the number n being accepted is decided immediately in the moment of reaching a configuration (n, q, w) where $q \in Q$ and $w \in \gamma_0\Gamma^*$. However, the next lemma shows that ppa are not more powerful than ppa without input. This allows us to consider ppa without input instead of usual ppa in the proof of the main result.

Lemma 5.2 *For arbitrary ppa $\mathcal{M}$ with input alphabet $\{0\}$ there is a ppa without input $\mathcal{U}$ such that*

$$(\forall n \in N)\ \chi_{\mathcal{M}}(0^n) = \chi_{\mathcal{U}}(n).$$

Outline of proof. If $\mathcal{D}$ is a probabilistic finite automaton (pfa) and q, q' are states and w is an input word, then we denote by $p_{\mathcal{D}}(q, w, q')$ the probability the automaton $\mathcal{D}$ entering the state q' after reading w, provided it's initial state is q.
The performance of ppa $\mathcal{M}$ after reaching the end marker # can be simulated by a probabilistic finite automaton (pfa) $\mathcal{A}$ with the input word $mi(w)$ where w is the content of pushdown and $mi(w)$ is the mirror image of w. Let Q be the set of states of $\mathcal{A}$. We take a pfa $\mathcal{B}$ with a set of states Q' and fixed initial state q_0 having the following property: for arbitrary $q', q'' \in Q$ there is a subset $S(q, q') \subseteq Q'$ such that for arbitrary input word w

$$p_{\mathcal{A}}(q', mi(w), q'') = \sum_{q \in S(q', q'')} p_{\mathcal{B}}(q_0, w, q).$$

The existence of such pfa $\mathcal{B}$ follows from M. Nasu and N. Honda's result [NH68] asserting that the class of word functions $f : \Gamma^* \to [0, 1]$ realizable by pfa is closed under the transposition.
The main idea in the construction of the ppa without input $\mathcal{U}$ is following: $\mathcal{U}$ simultaneously simulates ppa M on the input word and pfa $\mathcal{B}$ on the current content of the pushdown. The pushdown of $\mathcal{U}$ consist of two tapes. The first tape is used for simulation of $\mathcal{M}$. The second tape is used to write the current state of $\mathcal{B}$. This construction allows $\mathcal{U}$ to "remember" the previous state of $\mathcal{B}$ after popping the pushdown top symbol.
The proof of the main result needs one more lemma.

Lemma 5.3 *Let*

- *$\eta_i, i \in N, \ldots$ be a Markov chain with a finite set of states H;*
- *(η_i, ν_i), $i \in N$, be a Markov chain with denumerable set of states $H \times N$ having the chain η_i, $i \in N$, as a component;*
- *$Z_1, Z_2, Z_3, \ldots$ be a sequence of events such that*
  - *(a) $P\{(\eta_0, \nu_0) = (h_0, 0)\} = 1$ where $h_0 \in H$ is the initial state of the chain $\eta_i, i \in N$;*
  - *(b) $(\forall i \in N)$ $P\{\nu_{i+1} \geq \nu_i\} = 1$;*
  - *(c) there is a function $f : H \times H \times N \to [0, 1]$ such that transition probability from $(h, n) \in H \times N$ to $(h', n') \in H \times N$ for $n' \geq n$ equals $f(h, h', n' - n)$;*

(d) *there is a function* $g : H \times (N \setminus \{0\}) \to [0,1]$ *such that for any integers* $t \geq 1$, $n > n' \geq 0$ *and any event* A *depending only on* η_i *and* ν_i $(i = 0, \ldots, t-1)$ *it follows from* $P(A \cap \{(\eta_t, \nu_t) = (h, n')\}) > 0$ *that* $P\{Z_n | A \cap \{(\eta_t, \nu_t) = (h, n')\}\} = g(h, n - n')$.

Then there is an integer $d \geq 1$ *such that* $\lim_{n \to \infty}(P(Z_{n+d}) - P(Z_n)) = 0$.

The main idea of proof is to reduce the general case to special one where all the states in $H \setminus \{h_0\}$ are absorbing.
Proof of Theorem 5.1. Let $\mathcal{M}$ be the given ppa and let $\mathcal{M}$ accept a language $L \subseteq \{0\}^*$ with a δ-isolated cutpoint λ $(0 \leq \lambda \leq 1, \delta > 0)$. By lemma 5.2, there is a ppa without input $\mathcal{U} = (Q, q_0, F, \Gamma, \gamma_0, \varphi)$ such that $(\forall n \in N)\ \chi_{\mathcal{M}}(0^n) = \chi_{\mathcal{U}}(n)$. Hence, for any $n \in N$

$$(0^n \in L) \Rightarrow (\chi_{\mathcal{U}}(n) \geq \lambda + \delta) \text{ and } (0^n \notin L) \Rightarrow (\chi_{\mathcal{U}}(n) \leq \lambda - \delta).$$

Let $C = Q \times N \times (\gamma_0 \Gamma^*)$ be the set of configurations of $\mathcal{U}$.
For two configurations $c', c'' \in C$ we denote by $\mu(c', c'')$ the probability of $\mathcal{U}$ next configuration being c'', provided it's current configuration is c'. Probabilities $\mu(c', c'')$ are completely determined by transition function φ. One step transition from c' to c'' is possible iff $\mu(c', c'') > 0$.
We denote by Ω the set of all sequences $(c_0, c_1, c_2, \ldots)$ where $(\forall i \in N)\ c_i \in C$, $c_0 = (q_0, 0, \gamma_0)$ and $(\forall i \in N)\ \mu(c_i, c_{i+1}) > 0$. We call Ω the possibility space of $\mathcal{U}$.
The behavior of $\mathcal{U}$ from the initial configuration can be viewed as a stochastic process $\xi_o, \xi_1, \xi_0, \ldots$, where ξ_t $(t = 0, 1, 2, \ldots)$ is the configuration of $\mathcal{B}$ in the moment t and $\xi_0 = (q_0, 0, \gamma_0)$. This is a Markov chain with denumerable set of states C and fixed initial state $(q_0, 0, \gamma_0)$. Any probabilistic event concerning the work of $\mathcal{U}$ can be expressed in form $(\xi_0, \xi_1, \xi_2, \ldots) \in S$ where $S \subseteq \Omega$. We identify this event with the set S.
For any $\omega = (c_0, c_1, c_2, \ldots) \in \Omega$ and any $k \in N$ we define a finite sequence $pref_k(\omega) = (c_0, c_1, \ldots, c_k)$.
Using conventional methodology [KSK76] we define probabilistic space $\langle \Omega, \mathcal{B}, P \rangle$ where

- $\mathcal{B} \subseteq 2^\Omega$ (2^Ω is the set of all subsets of Ω) is minimal σ-algebra containing all sets in form

  $$S(c_0, c_1, \ldots, c_k) = \{w \in \Omega | pref_k(\omega) = (c_0, c_1, \ldots, c_k)\}$$

  where $k \in N$, $c_0 = (q_0, 0, \gamma_0)$ and for $1 \leq i \leq k$ it holds $\mu(c_{i-1}, c_i) > 0$, and

- $P : \mathcal{B} \to [0,1]$ is a probability on $\langle \Omega, B \rangle$ such that

  $$P(S(c_0, c_1, \ldots, c_k)) = \mu(c_0, c_1)\mu(c_1, c_2) \ldots \mu(c_{k-1}, c_k)$$

  (this requirement determines $P(A)$ for all $A \in \mathcal{B}$).

Let for $n \in N$

$$Z_n = \{(c_1, c_2, c_3, \ldots) \in \Omega | (\exists i \in N)(\exists (q, \gamma) \in F)(\exists w \in \Gamma^*)\ c_i = (q, n, w\gamma)\}$$

be the event "the number n is accepted by $\mathcal{U}$". Clearly, $Z_n \in \mathcal{B}$ and it's probability is $P(Z_n) = \chi_{\mathcal{U}}(n)$.
For any configuration $c \in C$, where $c = (q, n, w\gamma)$ and $\gamma \in \Gamma$, we define $word(c) = w\gamma$ and $pair(c) = (q, \gamma)$.
We say that $t \in N$ is a bottom-moment of a sequence $(c_0, c_1, c_2, \ldots) \in \Omega$ if $(\forall i \geq t)\ |word(c_i)| \geq |word(c_t)|$. $t \in N$ being the bottom-moment means that after the moment t neither of $word(c_t)$ symbols will be popped out of the pushdown. Clearly $t = 0$ is a bottom-moment.

Obviously, an arbitrary sequence $\omega \in \Omega$ has infinitely many bottom-moments. We define increasing sequence of bottom-moments

$$\tau_0(\omega) < \tau_1(\omega) < \tau_2(\omega) < \ldots$$

where $\tau_0(\omega) = 0$ and for any $i \in N$ there is no bottom-moment t such that $\tau_i(\omega) < t < \tau_{i+1}(\omega)$. For any $\omega = (c_0, c_1, c_2, \ldots) \in \Omega$ and $i \in N$ we define the i-th bottom-pair

$$\eta_i(\omega) = pair(c_{\tau_i}).$$

It is easy to see that τ_i, $i = 1, 2, 3, \ldots$, and η_i, $i = 1, 2, 3, \ldots$, are sequences of random variables on $\langle \Omega, \mathcal{B}, P \rangle$. Note that $pref_k(\omega') = pref_k(\omega'')$ and $\tau_i(\omega') \leq k$ does not imply $\tau_i(\omega') = \tau_i(w'')$, i. e. τ_i and η_i depend also on "future" (not only on "past"). In spite of that, it can be proved that $\eta_0, \eta_1, \eta_2, \ldots$ is a Markov chain with a finite set of states $H = Q \times \Gamma$ and fixed initial state $h_0 = (q_0, \gamma_0)$.
Let ν_i be the value of the counter in the i-th bottom-moment. It is obvious that $(\eta_0, \nu_0), (\eta_1, \nu_1), (\eta_2, \nu_2), \ldots$ is a Markov chain with the set of states $H \times N$ and the initial state $(h_0, 0)$.
Clearly, $(\forall i \in N)$ $P\{\nu_{i+1} \geq \nu_i\} = 1$. It is easy to see that transition probability from $(h, n) \in h \times N$ to $(h', n') \in h \times N$ where $n' \geq n$ depends only on h, h' and $n' - n$, i. e. it equals $f(h, h', n' - n)$ where $f : H \times H \times N \to [0, 1]$ is a function.
Obviously, for any integers $t \geq 1$, $n > n' \geq 0$ and any event A depending only on η_i and ν_i $(i = 0, \ldots, t-1)$ the conditional probability $P\{Z_n | A \cap \{(\eta_t, \nu_t) = (h, n')\}\}$ depends only on h and $n' - n$, i. e.

$$P(A \cap \{(\eta_t, \nu_t) = (h, n')\}) > 0 \Rightarrow P\{Z_n | A \cap \{(\eta_t, \nu_t) = (h, n')\}\} = g(h, n - n')$$

where $g : H \times (N \setminus \{0\}) \to [0, 1]$ is a function. Hence, by Lemma 5.3, there is an integer $d \geq 1$ such that $\lim_{n \to \infty}(P(Z_{n+d}) - P(Z_n)) = 0$. Since the cutpoint λ is isolated, this implies $(P(Z_{n+d}) > \lambda) \Leftrightarrow (P(Z_n) > \lambda)$ for all sufficiently large n. It follows from this that L is regular.

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