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Choice numbers of graphs

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Contents

1 Introduction

A graph $G = (V, E)$ is $(a : b)$-*choosable* if for every family of sets $\{S(v) : v \in V\}$, where $|S(v)| = a$ for all $v \in V$, there are subsets $C(v) \subseteq S(v)$, where $|C(v)| = b$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$. The *kth choice number* of G, denoted by $ch_k(G)$, is the minimum integer n so that G is $(n : k)$-choosable. A graph $G = (V, E)$ is k-*choosable* if it is $(k : 1)$-choosable. The *choice number* of G, denoted by $ch(G)$, is equal to $ch_1(G)$.

The concept of $(a : b)$-choosability was defined and studied by Erdős, Rubin and Taylor in [8]. In the present paper we prove several results concerning $(a : b)$-choosability, a number of which generalize known results regarding choice numbers of graphs that appear in [4] and [2]. The following theorem examines the behavior of $ch_k(G)$ when k is large.

Theorem 1.1 *Let G be a graph. For every $\epsilon > 0$ there exists an integer k_0 such that $ch_k(G) \leq k(\chi(G) + \epsilon)$ for every $k \geq k_0$.*

In [8] the authors ask the following question:
If G is $(a : b)$-choosable, and $\frac{c}{d} > \frac{a}{b}$, does it follow that G is $(c : d)$-choosable?
The following corollary gives a negative answer to this question.

Corollary 1.2 *If $l > m \geq 3$, then there is a graph G which is $(a : b)$-choosable but not $(c : d)$-choosable where $\frac{c}{d} = l$ and $\frac{a}{b} = m$.*

Let K_{m*r} denote the complete r-partite graph with m vertices in each vertex class, and let $K_{m_1,\ldots,m_r}$ denote the complete r-partite graph with m_i vertices in the ith vertex class. It is shown in [2] that there exist two positive constants c_1 and c_2 such that for every $m \geq 2$ and for every $r \geq 2$, $c_1 r \log m \leq ch(K_{m*r}) \leq c_2 r \log m$. The following theorem generalizes the upper bound.

Theorem 1.3 *If $r \geq 1$ and $m_i \geq 2$ for every i, $1 \leq i \leq r$, then*

$$ch_k(K_{m_1,\ldots,m_r}) \leq 948r(k + \log \frac{m_1 + \cdots + m_r}{r}).$$

The following are two applications of this theorem.

Corollary 1.4 *For every graph G and $k \geq 1$*

$$ch_k(G) \leq 948\chi(G)(k + \log (\frac{|V|}{\chi(G)} + 1)).$$

The second corollary generalizes a result from [2] concerning the choice numbers of random graphs for the common model $G_{n,p}$ (see, e.g., [7]), in which the graph is obtained by taking each pair of the n labeled vertices $1, 2, \ldots, n$ to be an edge, randomly and independently, with probability p.

Corollary 1.5 *For every two constants $k \geq 1$ and $0 < p < 1$, the probability that $ch_k(G_{n,p}) \leq 475 \log(1/(1-p)) n \frac{\log \log n}{\log n}$ tends to 1 as n tends to infinity.*

A theorem which appears in [4] reveals the connection between the choice number of a graph G and its orientations. We present here a generalization of this theorem for a special case.

Theorem 1.6 *Let $D = (V, E)$ be a digraph and $k \geq 1$. For each $v \in V$, let $S(v)$ be a set of size $k(d_D^+(v) + 1)$, where $d_D^+(v)$ is the outdegree of v. If D contains no odd directed (simple) cycle, then there are subsets $C(v) \subseteq S(v)$, where $|C(v)| = k$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$. There is a polynomial time algorithm in $|V|$ and k which finds the subsets $C(v)$.*

Corollary 1.7 *Let G be an undirected graph. If G has an orientation D which contains no odd directed (simple) cycle in which the maximum outdegree is d, then G is $(k(d+1) : k)$-choosable for every $k \geq 1$.*

Corollary 1.8 *An even cycle is $(2k : k)$-choosable for every $k \geq 1$.*

The last corollary enables us to prove a generalization of a variant of Brooks Theorem which appears in [8].

Corollary 1.9 *If a connected graph G is not K_n, and not an odd cycle, then $ch_k(G) \leq k\Delta(G)$ for every $k \geq 1$, where $\Delta(G)$ is the maximum degree of G.*

For a graph $G = (V, E)$, define $M(G) = \max(|E(H)|/|V(H)|)$, where $H = (V(H), E(H))$ ranges over all subgraphs of G. The following two corollaries are generalizations of results which appear in [4].

Corollary 1.10 *Every bipartite graph G is $(k(\lceil M(G) \rceil + 1) : k)$-choosable for all $k \geq 1$.*

Corollary 1.11 *Every bipartite planar graph G is $(3k : k)$-choosable for all $k \geq 1$.*

The following are additional applications.

Corollary 1.12 *If every induced subgraph of a graph G has a vertex of degree at most d, then G is $(k(d+1):k)$-choosable for all $k \geq 1$.*

Corollary 1.13 *If G is a triangulated graph, then $ch_k(G) = k\chi(G) = k\omega(G)$ for every $k \geq 1$, where $\omega(G)$ is the clique number of G.*

The list-chromatic conjecture asserts that for every graph G, $ch(L(G)) = \chi(L(G))$, where $L(G)$ is the line graph of G. The list-chromatic conjecture is easy to establish for trees, graphs of degree at most 2, and $K_{2,m}$. It has also been verified for snarks [11], $K_{3,3}$, $K_{4,4}$, $K_{6,6}$ [4], and 2-connected cubic planar graphs. The following corollary shows that the list-chromatic conjecture is true for graphs which contain no C_n for every $n \geq 4$.

Corollary 1.14 *If a graph G contains no C_n for every $n \geq 4$, then $ch(L(G)) = \chi(L(G))$.*

The *core* of a graph G is the graph obtained from G by deleting nodes of degree 1 successively until there are no nodes of degree 1. The graph $\Theta_{a,b,c}$ consists of two distinguished nodes u and v together with three paths of lengths a,b, and c, which are node disjoint except that each path has u at one end, and v at the other end. The following theorem from [8] gives a characterization of the 2-choosable graph:

Theorem 1.15 *A connected graph G is 2-choosable if, and only if, the core of G belongs to $\{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\}$.*

In [8] the authors ask the following question:
If G is $(a:b)$-choosable, does it follow that G is $(am:bm)$-choosable?
The following theorem gives a partial solution to this question by using theorem 1.15.

Theorem 1.16 *If a graph G is 2-choosable, then G is also $(4:2)$-choosable.*

Theorem 1.17 *Suppose that k and m are positive integers and that k is odd. If a graph G is $(2mk:mk)$-choosable, then G is also $2m$-choosable.*

A graph $G = (V, E)$ is f-*choosable* for a function $f : V \mapsto N$ if for every family of sets $\{S(v) : v \in V\}$, where $|S(v)| = f(v)$ for all $v \in V$, there is a proper vertex-coloring of G assigning to each vertex $v \in V$ a color from $S(v)$. It is shown in [8] that the following problem is Π_2^p-complete: (for terminology see [10])

BIPARTITE GRAPH $(2,3)$-CHOOSABILITY (BG $(2,3)$-CH)
INSTANCE: A bipartite graph $G = (V, E)$ and a function $f : V \mapsto \{2, 3\}$.
QUESTION: Is G f-choosable?

We consider the following decision problem:

BIPARTITE GRAPH k-CHOOSABILITY (BG k-CH)
INSTANCE: A bipartite graph $G = (V, E)$.
QUESTION: Is G k-choosable?

If follows from theorem 1.15 that this problem is solvable in polynomial time for $k = 2$.

Theorem 1.18 **BIPARTITE GRAPH k-CHOOSABILITY** *is Π_2^p-complete for every constant $k \geq 3$.*

A graph $G = (V, E)$ is *strongly k-colorable* if every graph obtained from G by adding to it a union of vertex disjoint cliques of size at most k (on the set V) is k-colorable. An analogous definition of *strongly k-choosable* is made by replacing colorability with choosability. The *strong chromatic number* of a graph G, denoted by $s\chi(G)$, is the minimum k such that G is strongly k-colorable. Define $s\chi(d) = \max(s\chi(G))$, where G ranges over all graphs with maximum degree at most d. The definition of strongly k-colorable given in [1] is slightly different. It is claimed there that if G is strongly k-colorable, then it is strongly $(k+1)$-colorable as well. However, it is not known how to prove this if we use the definition from [1].

Theorem 1.19 *If G is strongly k-colorable, then it is strongly $(k+1)$-colorable as well.*

We give a weaker version of this theorem for choosability.

Theorem 1.20 *If G is strongly k-choosable, then it is strongly km-choosable as well.*

Theorem 1.21 *Let $G = (V, E)$ be a graph, and suppose that km divides $|V|$. If the choice number of any graph obtained from G by adding to it a union of vertex disjoint k-cliques (on the set V) is k, then the choice number of any graph obtained from G by adding to it a union of vertex disjoint km-cliques is km.*

Corollary 1.22 *Let n and k be positive integers, and let G be a $(3k+1)$-regular graph on $3kn$ vertices. Assume that G has a decomposition into a Hamiltonian circuit and n pairwise vertex disjoint $3k$-cliques. Then $ch(G) = 3k$.*

It is proved in [1] that there is a constant c such that for every d, $3\lfloor d/2 \rfloor < s\chi(d) \leq cd$. The following theorem improves the lower bound.

Theorem 1.23 *For every $d \geq 1$, $s\chi(d) \geq 2d$.*

2 A solution to a problem of Erdős, Rubin and Taylor

In this section we prove an upper bound for the kth choice number of a graph when k is large and apply this bound to settle a problem raised in [8].

Proof of Theorem 1.1 Let $G = (V, E)$ be a graph and $\epsilon > 0$. Denote $r = \chi(G)$, and let $V = V_1 \cup \ldots \cup V_r$ be a partition of the vertices, such that each V_i is a stable set. For each $v \in V$, let $S(v)$ be a set of $\lfloor k(\chi(G) + \epsilon) \rfloor$ distinct colors. Let $S = \cup_{v \in V} S(v)$ be the set of all colors. Put $R = \{1, 2, \ldots, r\}$ and let $f : S \mapsto R$ be a random function, obtained by choosing, for each color $c \in S$, randomly and independently, the value of $f(c)$ according to a uniform distribution on R. The colors c for which $f(c) = i$ will be the ones to be used for coloring the vertices in V_i. To complete the proof, it thus suffices to show that with positive probability for every i, $1 \leq i \leq r$, and for every vertex $v \in V_i$ there are at least k colors $c \in S(v)$ so that $f(c) = i$.

Fix an i and a vertex $v \in V_i$, and define $X = |S(v) \cap f^{-1}(i)|$. The probability that there are less than k colors $c \in S(v)$ so that $f(c) = i$ is equal to $Pr(X < k)$. Since X is a random variable with distribution $B(\lfloor k(r+\epsilon) \rfloor, 1/r)$, by Chebyshev's inequality (see, e.g., [3])

$$Pr(X < k) \leq Pr(|X - \frac{\lfloor k(r+\epsilon) \rfloor}{r}| \geq \frac{\lfloor k\epsilon \rfloor}{r}) \leq \frac{\lfloor k(r+\epsilon) \rfloor \frac{1}{r}(1 - \frac{1}{r})}{(\frac{\lfloor k\epsilon \rfloor}{r})^2} = O(\frac{1}{k}).$$

It follows that there is an integer k_0 such that $P(X < k) < 1/|V|$ for every $k \geq k_0$. There are $|V|$ possible choices of i, $1 \leq i \leq r$ and $v \in V_i$, and hence, the probability that for some i and some $v \in V_i$ there are less than k colors $c \in S(v)$ so that $f(c) = i$ is smaller than 1, completing the proof. $\square$

Note that it is not true that for every graph G there exists an integer k_0 such that $ch_k(G) \leq k\chi(G)$ for every $k \geq k_0$. For example, take the graph $G = K_{3,3}$ which has chromatic number 2.

The graph G is not 2-choosable and therefore by theorem 1.17 it is not $(2k : k)$-choosable for every k odd. This means that $ch_k(G) > k\chi(G)$ for every k odd.

Proof of Corollary 1.2 Suppose that $l > m \geq 3$, and let G be a graph such that $ch(G) = l + 1$ and $\chi(G) = m - 1$ (it is proved in [13] that for every $l \geq m \geq 2$ there is a graph G, where $ch(G) = l$ and $\chi(G) = m$). By theorem 1.1, for $\epsilon = 1$ there exist an integer k such that G is $(k(\chi(G) + 1) : k)$-choosable. We have that G is $(km : k)$-choosable but not $(l : 1)$-choosable, as needed. □

3 An upper bound for the kth choice number

In this section we establish an upper bound for $ch_k(K_{m_1,\ldots,m_r})$, and use it to prove two consequences. The following lemma appears in [3].

Lemma 3.1 *If X is a random variable with distribution $B(n, p)$, $0 < p \leq 1$, and $k < pn$ then*

$$Pr(X < k) < e^{-\frac{(np-k)^2}{2pn}}.$$

In the rest of this section we denote $t = \frac{m_1+\cdots+m_r}{r}$, $t_1 = \frac{m_1+\cdots+m_{r/2}}{r/2}$, and $t_2 = \frac{m_{r/2+1}+\cdots+m_r}{r/2}$. Notice that $t = (t_1 + t_2)/2$, and therefore $\log t_1 t_2 \leq 2 \log t$.

Lemma 3.2 *If $1 \leq r \leq t$, $k \geq 1$, and $m_i \geq 2$ for every i, $1 \leq i \leq r$, then $ch_k(K_{m_1,\ldots,m_r}) \leq 4r(k + \log t)$.*

Proof Let $V_1, V_2, \ldots, V_r$ be the vertex classes of $K = K_{m_1,\ldots,m_r}$, where $|V_i| = m_i$ for all i, and let $V = V_1 \cup \ldots \cup V_r$ be the set of all vertices of K. For each $v \in V$, let $S(v)$ be a set of $\lfloor 4r(k + \log t)\rfloor$ distinct colors. Put $R = \{1, 2, \ldots, r\}$ and let $f : S \mapsto R$ be a random function, obtained by choosing, for each color $c \in S$, randomly and independently, the value of $f(c)$ according to a uniform distribution on R. The colors c for which $f(c) = i$ will be the ones to be used for coloring the vertices in V_i. To complete the proof it thus suffices to show that with positive probability for every i, $1 \leq i \leq r$, and every vertex $v \in V_i$ there are at least k colors $c \in S(v)$ so that $f(c) = i$.

Fix an i and a vertex $v \in V_i$, and define $X = |S(v) \cap f^{-1}(i)|$. The probability that there are less than k colors $c \in S(v)$ so that $f(c) = i$ is equal to $Pr(X < k)$. Since X is a random variable with distribution $B(\lfloor 4r(k + \log t)\rfloor, 1/r)$, by lemma 3.1

$$Pr(X < k) < e^{-\frac{(E(X)-k)^2}{2E(X)}} \leq e^{-\frac{(4(k+\log t)-1-k)^2}{8(k+\log t)}} < e^{-\frac{16(k+\log t)^2-8(k+1)(k+\log t)}{8(k+\log t)}} \leq e^{-2\log t} = \frac{1}{t^2} \leq \frac{1}{rt},$$

where the last inequality follows from the fact that $r \le t$. There are rt possible choices of i, $1 \le i \le r$ and $v \in V_i$, and hence, the probability that for some i and some $v \in V_i$ there are less than k colors $c \in S(v)$ so that $f(c) = i$ is smaller than 1, completing the proof. $\square$

Lemma 3.3 *Suppose that r is even, $r > t$, $k \ge 1$, $d \ge 244$, and $m_i \ge 2$ for every i, $1 \le i \le r$. If $ch_k(K_{m_1,\ldots,m_{r/2}}) \le d(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_1)$ and $ch_k(K_{m_{r/2+1},\ldots,m_r}) \le d(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_2)$, then $ch_k(K_{m_1,\ldots,m_r}) \le dr(k + \log t)$.*

Proof Let $V_1, V_2, \ldots, V_r$ be the vertex classes of $K = K_{m_1,\ldots,m_r}$, where $|V_i| = m_i$ for all i, and let $V = V_1 \cup \ldots \cup V_r$ be the set of all vertices of K. For each $v \in V$, let $S(v)$ be a set of $\lfloor dr(k + \log t) \rfloor$ distinct colors. Define $R = \{1, 2, \ldots, r\}$, and let $S = \cup_{v \in V} S(v)$ be the set of all colors. Put $R_1 = \{1, 2, \ldots, r/2\}$ and $R_2 = \{r/2 + 1, \ldots, r\}$. Let $f : S \mapsto \{1, 2\}$ be a random function obtained by choosing, for each $c \in S$ randomly and independently, $f(c) \in \{1, 2\}$ where for all $j \in \{1, 2\}$

$$Pr(f(c) = j) = \frac{k + \log t_j}{2k + \log t_1 t_2}.$$

The colors c for which $f(c) = 1$ will be used for coloring the vertices in $\cup_{i \in R_1} V_i$, whereas the colors c for which $f(c) = 2$ will be used for coloring the vertices in $\cup_{i \in R_2} V_i$.

For every vertex $v \in V$, define $C(v) = S(v) \cap f^{-1}(1)$ if v belongs to $\cup_{i \in R_1} V_i$, and $C(v) = S(v) \cap f^{-1}(2)$ if v belongs to $\cup_{i \in R_2} V_i$. Because of the assumptions of the lemma, it remains to show that with positive probability,

$$|C(v)| \ge d(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_j) \tag{1}$$

for all $j \in \{1, 2\}$ and $v \in \cup_{i \in R_j} V_i$.

Fix a $j \in \{1, 2\}$ and a vertex $v \in \cup_{i \in R_j} V_i$, and define $X = |C(v)|$. The expectation of X is

$$\lfloor dr(k + \log t) \rfloor \frac{k + \log t_j}{2k + \log t_1 t_2} \ge (dr(k + \log t) - 1)\frac{k + \log t_j}{2k + 2\log t} \ge d\frac{r}{2}(k + \log t_j) - 1 = T.$$

If follows from lemma 3.1 and the inequality $E(X) \ge T$ that

$$Pr(X < T - T^{2/3}) < e^{-\frac{(E(X) - T + T^{2/3})^2}{2E(X)}} \le e^{-\frac{1}{2}T^{1/3}} \le e^{-\frac{1}{2}(d\frac{r}{2})^{1/3}}.$$

Since $|\cup_{i \in R_j} V_i| \le rt < r^2$, the probability that $|C(v)| < T - T^{2/3}$ holds for some $v \in \cup_{i \in R_j} V_i$ is at most

$$r^2 \cdot e^{-\frac{1}{2}(d\frac{r}{2})^{1/3}} < 1/2,$$

where the last inequality follows from the fact that $d \geq 244$. One can easily check that

$$T - T^{2/3} = T(1 - \frac{1}{T^{1/3}}) \geq d\frac{r}{2}(k + \log t_j)(1 - \frac{1}{5r^{1/3}}),$$

and therefore, with positive probability (1) holds for all $j \in \{1, 2\}$ and $v \in \cup_{i \in R_j} V_i$. $\Box$

Proof of Theorem 1.3 Define for every r which is a power of 2

$$f(r) = \prod_{j=0}^{\log_2 r} (1 - \frac{1}{5 \cdot 2^{j/3}}) / \prod_{j=0}^{2} (1 - \frac{1}{5 \cdot 2^{j/3}}).$$

We claim that for every r which is a power of 2

$$ch_k(K_{m_1,\ldots,m_r}) \leq \frac{244r(k + \log t)}{f(r)}. \tag{2}$$

The proof is by induction on r.

Case 1: $r \leq t$.

The result follows from lemma 3.2 since

$$\frac{244}{f(r)} \geq 244 \prod_{j=1}^{2} (1 - \frac{1}{5 \cdot 2^{j/3}}) > 4.$$

Case 2: $r > t$.

Notice that $t \geq 2$, and therefore $r \geq 4$. By the induction hypothesis

$$ch_k(K_{m_1,\ldots,m_{r/2}}) \leq \frac{244(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_1)}{f(r)}$$

and

$$ch_k(K_{m_{r/2+1},\ldots,m_r}) \leq \frac{244(1 - \frac{1}{5r^{1/3}})\frac{r}{2}(k + \log t_2)}{f(r)}.$$

Since $r \geq 4$, we have $244/f(r) \geq 244$ and it follows from lemma 3.3 that (2) holds, as claimed.

It is easy to check that

$$\prod_{j=3}^{\log_2 r} (1 - \frac{1}{5 \cdot 2^{j/3}}) \geq 1 - \sum_{j=3}^{\log_2 r} \frac{1}{5 \cdot 2^{j/3}} \geq 1 - \frac{1}{10(1 - 2^{-1/3})},$$

and therefore $244/f(r) \leq 474$. If follows from (2) that for every r which is a power of 2

$$ch_k(K_{m_1,\ldots,m_r}) \leq 474r(k + \log t). \tag{3}$$

Returning to the general case, assume that $r \geq 1$. Choose an integer r' which is a power of 2 and $r < r' \leq 2r$. By applying (3), we get

$$ch_k(K_{m_1,\ldots,m_r}) \leq ch_k(K_{m_1,\ldots,m_r,\underbrace{2,\ldots,2}_{r'-r}})$$

$$\leq 474r'(k + \log \frac{m_1 + \cdots + m_r + 2(r'-r)}{r'}) \leq 948r(k + \log \frac{m_1 + \cdots + m_r}{r}),$$

completing the proof. $\square$

Denote $K = K_{m,\underbrace{s,\ldots,s}_{r}}$, where $m \geq 2$ and $s \geq 2$. Every induced subgraph of K has a vertex of degree at most rs, and therefore by corollary 1.10 $ch_k(K) \leq k(rs+1)$ for all $k \geq 1$. Note that this upper bound for $ch_k(K)$ does not depend of m, which means that a good lower bound for $ch_k(K_{m_1,\ldots,m_r})$ has a more complicated form than the upper bound given in theorem 1.3.

Proof of Corollary 1.4 Let $G = (V, E)$ be a graph and $k \geq 1$. Denote $r = \chi(G)$, and let $V = V_1 \cup \ldots \cup V_r$ be a partition of the vertices, such that each V_i is a stable set. Denote $m_i = |V_i|$ for all i, $1 \leq i \leq r$. By theorem 1.1

$$ch_k(G) \leq ch_k(K_{m_1+1,\ldots,m_r+1}) \leq 948r(k + \log \frac{m_1 + \cdots + m_r + r}{r}) = 948\chi(G)(k + \log (\frac{|V|}{\chi(G)} + 1)),$$

as needed. $\square$

Proof of Corollary 1.5 As proved by Bollobás in [6], for a fixed probability p, $0 < p < 1$, almost surely (i.e., with probability that tends to 1 as n tends to infinity), the random graph $G_{n,p}$ has chromatic number

$$(\frac{1}{2} + o(1)) \log (1/(1-p)) \frac{n}{\log n}.$$

By corollary 1.4, for every $\epsilon > 0$ almost surely

$$ch_k(G_{n,p}) \leq 948(\frac{1}{2} + \epsilon) \log (1/(1-p)) \frac{n}{\log n}(k + \log (\frac{3 \log n}{\log (1/(1-p))} + 1)).$$

The result follows since k and p are constants. $\square$

Note that in the proof of the last corollary we have not used any knowledge concerning independent sets of $G_{n,p}$, as was done in [2] for the proof of the special case.

4 Choice numbers and orientations

Let $D = (V, E)$ be a digraph. We denote the set of out-neighbors of v in D by $N_D^+(v)$. A set of vertices $K \subseteq V$ is called a *kernel* of D if K is an independent set and $N_D^+(v) \cap K \neq \emptyset$ for every vertex $v \notin K$. Richardson's theorem (see, e.g., [5]) states that any digraph with no odd directed cycle has a kernel.

Proof of Theorem 1.6 Let $D = (V, E)$ be a digraph which contains no odd directed (simple) cycle and $k \geq 1$. For each $v \in V$, let $S(v)$ be a set of size $k(d_D^+(v) + 1)$. We claim that the following algorithm finds subsets $C(v) \subseteq S(v)$, where $|C(v)| = k$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$.

1. $S \leftarrow \cup_{v \in V} S(v)$, $W \leftarrow V$ and for every $v \in V$, $C(v) \leftarrow \emptyset$.

2. Choose a color $c \in S \cap \cup_{v \in W} S(v)$ and put $S \leftarrow S - \{c\}$.

3. Let K be a kernel of the induced subgraph of D on the vertex set $\{v \in W : c \in S(v)\}$.

4. $C(v) \leftarrow C(v) \cup \{c\}$ for all $v \in K$.

5. $W \leftarrow W - \{v \in K : |C(v)| = k\}$.

6. If $W = \emptyset$, stop. If not, go to step 2.

During the algorithm, W is equal to $\{v \in V : |C(v)| < k\}$, and S is the set of remaining colors. We first prove that in step 2, $S \cap \cup_{v \in W} S(v) \neq \emptyset$. When the algorithm reaches step 2, it is obvious that $W \neq \emptyset$. Suppose that $w \in W$ in this step, and therefore $|C(w)| < k$. It follows easily from the definition of a kernel that every color from $S(w)$, which has been previously chosen in step 2, belongs either to $C(w)$ or to $\cup_{v \in N_D^+(w)} C(v)$. Since

$$|C(w)| + |\bigcup_{v \in N_D^+(w)} C(v)| < k + k \cdot d_D^+(v) = |S(w)|,$$

not all the colors of $S(w)$ have been used. This means that $S \cap S(w) \neq \emptyset$, as needed. It follows easily that the algorithm always terminates.

Upon termination of the algorithm, $|C(v)| = k$ for all $v \in V$. In step 4 the same color is assigned to the vertices of a kernel which is an independent set, and therefore $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$. This proves the correctness of the algorithm.

In step 4, the operation $C(v) \leftarrow C(v) \cup \{c\}$ is performed for at least one vertex. Upon termination $| \cup_{v \in V} C(v)| \leq k|V|$, which means that the algorithm performs at most $k|V|$ iterations. There is a polynomial time algorithm for finding a kernel in a digraph with no odd directed cycle. Thus, the algorithm is of polynomial time complexity in $|V|$ and k, completing the proof. $\Box$

Proof of Corollary 1.7 This is an immediate consequence of theorem 1.6, since $k(d_D^+(v)+1) \leq k(d+1)$ for every $v \in V$. $\Box$

Proof of Corollary 1.8 The result follows from 1.7 by taking the cyclic orientation of the even cycle. $\Box$

The proof of corollary 1.9 is similar to the proof of the special case which appears in [8]. A graph $G = (V, E)$ is *k-degree-choosable* if for every family of sets $\{S(v) : v \in V\}$, where $|S(v)| = kd(v)$ for all $v \in V$, there are subsets $C(v) \subseteq S(v)$, where $|C(v)| = k$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$.

Lemma 4.1 *If a graph $G = (V, E)$ is connected, and G has a connected induced subgraph $H = (V', E')$ which is k-degree-choosable, then G is k-degree-choosable.*

Proof For each $v \in V$, let $S(v)$ be a set of size $kd(v)$. The proof is by induction on $|V|$. In case $|V| = |V'|$ there is nothing to prove. Assuming that $|V| > |V'|$, let v be a vertex of G which is at maximal distance from H. This guarantees that $G-v$ is connected. Choose any subset $C(v) \subseteq S(v)$ such that $|C(v)| = k$, and remove the colors of $C(v)$ from all the vertices adjacent to v. The choice can be completed by applying the induction hypothesis on $G - v$. $\Box$

Lemma 4.2 *If $c \geq 2$, then $\Theta_{a,b,c}$ is k-degree-choosable for every $k \geq 1$.*

Proof Suppose that $\Theta_{a,b,c}$ has vertex set $V = \{u, v, x_1, \ldots, x_{a-1}, y_1, \ldots, y_{b-1}, z_1, \ldots, z_{c-1}\}$ and contains the three paths $u - x_1 - \cdots - x_{a-1} - v$, $u - y_1 - \cdots - y_{b-1} - v$, and $u - z_1 - \cdots - z_{c-1} - v$. For each $w \in V$, let $S(w)$ be a set of size $kd(w)$. For the vertex u we choose a subset $C(u) \subseteq S(u) - S(z_1)$ of size k. For each node according to the sequence $x_1, \ldots, x_{a-1}, y_1, \ldots, y_{b-1}, v, z_{c-1}, \ldots, z_1$ we choose a subset of k colors that were not chosen in adjacent earlier nodes. $\Box$

For the proof of corollary 1.9, we shall need the following lemma which appears in [8].

Lemma 4.3 *If there is no node which disconnects G, then G is an odd cycle, or $G = K_n$, or G contains, as a node induced subgraph, an even cycle without chord or with only one chord.*

Proof of Corollary 1.9 Suppose that a connected graph G is not K_n, and not an odd cycle. If G is not a regular graph, then every induced subgraph of G has a vertex of degree at most $\Delta(G)-1$, and by corollary 1.12 $ch_k(G) \le k\Delta(G)$ for all $k \ge 1$. If G is a regular graph, then there is a part of G not disconnected by a node, which is neither an odd cycle nor a complete graph. It follows from lemma 4.3 that G contains, as a node induced subgraph, an even cycle or a particular kind of $\Theta_{a,b,c}$ graph. We know from corollary 1.8 and lemma 4.2 that both an even cycle and $\Theta_{a,b,c}$ are k-degree-choosable for every $k \ge 1$. The result follows from lemma 4.1. $\Box$

Proof of Corollary 1.10 It is proved in [4] that a graph $G = (V, E)$ has an orientation D in which every outdegree is at most d if and only if $M(G) \le d$. Therefore, there is an orientation D of G in which the maximum outdegree is at most $\lceil M(G) \rceil$. Since D contains no odd directed cycles, the result follows from corollary 1.7. $\Box$

Proof of Corollary 1.11 $M(G) \le 2$, since any bipartite (simple) graph on r vertices contains at most $2r-2$ edges. The result follows from corollary 1.10. $\Box$

Proof of Corollary 1.12 We claim that if every induced subgraph of a graph $G = (V, E)$ has a vertex of degree at most d, then G has an acyclic orientation in which the maximum outdegree is d. The proof is by induction on $|V|$. If $|V| = 1$, the result is trivial. If $|V| > 1$, let v be a vertex of G with degree at most d. By the induction hypothesis, $G - v$ has an acyclic orientation in which the maximum outdegree is d. We complete this orientation of $G - v$ by orienting every edge incident to v from v to its appropriate neighbor and obtain the desired orientation of G, as claimed. The result follows from corollary 1.7. $\Box$

An undirected graph G is called *triangulated* if G does not contain an induced subgraph isomorphic to C_n for $n \ge 4$. Being triangulated is a hereditary property inherited by all the induced subgraphs of G. A vertex v of G is called *simplicial* if its adjacency set $Adj(v)$ induces a complete subgraph of G. It is proved in [12] that every triangulated graph has a simplicial vertex.

Proof of Corollary 1.13 Suppose that G is a triangulated graph, and let H be an induced subgraph of G. Since H is triangulated, it has a simplicial vertex v. The set of vertices $\{v\} \cup Adj_H(v)$ induces a complete subgraph of H, and therefore v has degree at most $\omega(G)-1$ in H. It follows from corollary 1.10 that $ch_k(G) \le k\omega(G)$ for every $k \ge 1$. For every graph G and $k \ge 1$, $ch_k(G) \ge k\omega(G)$ and hence $ch_k(G) = k\omega(G)$ for every $k \ge 1$. Since G is triangulated, it is also perfect, which means that $\chi(G) = \omega(G)$, as needed. $\Box$

Proof of Corollary 1.14 It is easy to see that $L(G)$ is triangulated if and only if G contains no C_n for every $n \geq 4$. The result follows from corollary 1.13. $\square$

The validity of the list-chromatic conjecture for graphs of class 2 with maximum degree 3 (and in particular for snarks) follows easily from corollary 1.9. Suppose that G is a graph of class 2 with $\Delta(G) = 3$. Let C be a connected component of $L(G)$. If C is not a complete graph, and not an odd cycle, then $ch(C) \leq \Delta(C) \leq \Delta(L(G)) \leq 4$. If C is a complete graph or an odd cycle, then it is easy to see that $\Delta(C) \leq 2$, and therefore by corollary 1.10 $ch(C) \leq \Delta(C) + 1 \leq 3$. It follows that $ch(L(G)) \leq 4$. Since G is a graph of class 2, $ch(L(G)) \geq \chi(L(G)) = \Delta(G) + 1 = 4$, and hence, $ch(L(G)) = \chi(L(G)) = 4$.

5 Properties of $(2k : k)$-choosable graphs

Let A and B be sets of size 4. We denote $p(A, B) = \{(C, D) : C \subseteq A, D \subseteq B, |C| = |D| = 2\}$. Suppose that $S \subseteq p(A_1, B_1)$ and that $T \subseteq p(A_2, B_2)$. We say that S and T are isomorphic if there exist two bijections $f : A_1 \mapsto A_2$ and $g : B_1 \mapsto B_2$ so that $(C, D) \in S$ iff $(f(C), g(D)) \in T$ for every $C \subseteq A$ and $D \subseteq B$, where $|C| = |D| = 2$.

Let A and B be sets of size 4, and suppose that $S \subseteq p(A, B)$. Suppose that $H_1, \ldots, H_6$ are all the subsets of A of size 2. For each i, $1 \leq i \leq 6$, we denote $c(H_i) = \{G : (H_i, G) \in S\}$ and $d_i = |c(H_i)|$. The sequence $(d_1, \ldots, d_6)$ is called the degree sequence of S. We say that S is special if it has the following properties:

1. Its degree sequence is $(6, 5, 5, 3, 3, 1)$.
2. If H and G are the two subsets of A for which $|c(H)| = |c(G)| = 3$, then $|H \cap G| = 1$. Denote $H = \{1, 2\}$, $G = \{1, 3\}$, and $A = \{1, 2, 3, 4\}$.
3. $c(H) = c(G)$.
4. $c(H)$ has either the form $\{\{5, 6\}, \{5, 7\}, \{5, 8\}\}$ or the form $\{\{5, 6\}, \{5, 7\}, \{6, 7\}\}$.
5. Either $|c(\{2, 3\})| = 1$ and $|c(\{1, 4\})| = 6$, or $|c(\{2, 3\})| = 6$ and $|c(\{1, 4\})| = 1$.

We say that S has property P_1 iff $comp(H)$ has the form $\{\{5, 6\}, \{5, 7\}, \{5, 8\}\}$ and that it has property P_2 iff $|comp(\{2, 3\})| = 1$.

Suppose that $K_{2,2}$ has vertex set $V = X \cup Y$, where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and it has exactly the edges $\{x_i, y_j\}$. For each $v \in V$, let $S(v)$ be a set of size 4. By $C(v)$ we denote a subset of $S(v)$ of size 2. We say that $C(x_1)$ and $C(x_2)$ are compatible if there exist two subsets $C(y_1)$ and $C(y_2)$, so that $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$. A subset $C(x_1) \subseteq S(x_1)$ is called bad if $C(x_1)$ is not compatible with any $C(x_2)$. An analogous definition is made for $C(x_2)$. We say that a family of sets $\{S(v) : v \in V\}$ is defected if there exist two bad subsets $C(x_1)$ and $C(x_2)$. We denote by *incomp*(x_1, x_2) the set of incompatible pairs $(C(x_1), C(x_2))$.

Lemma 5.1 *If the family of sets $\{S(v) : v \in V\}$ is defected and $C(x_1)$ is bad, then both $S(y_1)$ and $S(y_2)$ intersect $C(x_1)$ and at least one them contains $C(x_1)$.*

Proof Suppose that neither $S(y_1)$ nor $S(y_2)$ contain $C(x_1)$. Remove the colors of $C(x_1)$ from $S(y_1)$ and $S(y_2)$. Now both $S(y_1)$ and $S(y_2)$ have size at least 3. We can assume the worst case, in which both $S(y_1)$ and $S(y_2)$ are subsets of $S(x_2)$, and therefore $|S(y_1) \cap S(y_2)| \geq 2$. Let C be a subset of $S(y_1) \cap S(y_2)$ of size 2. Choose a subset $C(x_2) \subseteq S(x_2) - C$. We have that $C(x_1)$ and $C(x_2)$ are compatible in contrast to the fact that $C(x_1)$ is bad. This proves that at least one of $S(y_1)$ and $S(y_2)$ contains $C(x_1)$.

Suppose that $S(y_1) \cap C(x_1) = \emptyset$. Choose a subset $C(y_2) \subseteq S(y_2) - C(x_1)$ and a subset $C(x_2) \subseteq S(x_2) - C(y_2)$. We have that $C(x_1)$ and $C(x_2)$ are compatible in contrast to the fact that $C(x_1)$ is bad. This proves that both $S(y_1)$ and $S(y_2)$ intersect $C(x_1)$. $\square$

Lemma 5.2 *If the family of sets $\{S(v) : v \in V\}$ is defected, then both $S(x_1)$ and $S(x_2)$ contain exactly one bad subset. Furthermore, at least one of the following is valid:*

1. *The set incomp(x_1, x_2) is special and has properties P_1 and P_2.*
2. *incomp(x_1, x_2) has degree sequence $(6, 5, 5, 3, 2, 2)$.*
3. *$|$incomp$(x_1, x_2)| = 21$.*

Proof The set $S(x_1)$ contains a bad subset, which we denote by $C(x_1) = \{1, 2\}$. Without loss of generality, we can assume by lemma 5.1 that $C(x_1) \subseteq S(y_1)$ and that $S(y_2)$ intersects $C(x_1)$. Denote $S(y_1) = \{1, 2, 3, 4\}$. Since $C(x_1)$ is bad, we must have that $|(S(y_1) \cap S(y_2)) - C(x_1)| < 2$.
Case 1: $C(x_1) \subseteq S(y_2)$ and $|S(y_1) \cap S(y_2)| = 3$.

Denote $S(y_2)=\{1,2,3,5\}$. Since $C(x_1)$ is bad, surely $\{3,4,5\}\subseteq S(x_2)$. The set $S(x_2)$ contains a bad subset, which we denote by $C(x_2)$, and therefore $\{1,2\}\cap S(x_2)\neq\emptyset$. We can assume, without loss of generality, that $S(x_2)=\{1,3,4,5\}$. Since $C(x_2)$ is bad and $|S(y_1)\cap S(y_2)|=3$, we must have that $C(x_2)\subseteq S(y_1)\cap S(y_2)$. Hence, $C(x_2)=\{1,3\}$ and $S(x_1)=\{1,2,4,5\}$. We have that

$$S(x_1)=\{1,2,4,5\}, S(x_2)=\{1,3,4,5\}, S(y_1)=\{1,2,3,4\}, S(y_2)=\{1,2,3,5\}.$$

The set $incomp(x_1,x_2)$ is special and has properties P_1 and P_2.

Case 2: $C(x_1)\subseteq S(y_2)$ and $|S(y_1)\cap S(y_2)|=2$.

Denote $S(y_2)=\{1,2,5,6\}$. Since $C(x_1)$ is bad, surely $|S(x_2)\cap\{3,4,5,6\}|\geq 3$. Suppose without loss of generality that $\{3,4,5\}\subseteq S(x_2)$. The set $S(x_2)$ contains a bad subset, which we denote by $C(x_2)$, and therefore $\{1,2\}\cap S(x_2)\neq\emptyset$. We can assume, without loss of generality, that $S(x_2)=\{1,3,4,5\}$. Since $C(x_2)$ is bad and $|S(y_1)\cap S(y_2)|=2$, we must have that $C(x_2)\cap\{1,2\}\neq\emptyset$, and therefore $1\in C(x_2)$. We can assume, without loss of generality, that $C(x_2)=\{1,3\}$. Since $C(x_2)$ is bad, we must have that $4\in S(x_1)$ and $S(x_1)\cap\{5,6\}\neq\emptyset$. Suppose without loss of generality that $S(x_1)=\{1,2,4,5\}$. This is a contradiction to the fact that $C(x_2)$ is bad.

Case 3: $|C(x_1)\cap S(y_2)|=1$ and $|S(y_1)\cap S(y_2)|=2$.

We can assume, without loss of generality, that $1\in S(y_2)$. Denote $S(y_2)=\{1,3,5,6\}$. Since $C(x_1)$ is bad, surely $S(x_2)=\{3,4,5,6\}$. The set $S(x_2)$ contains a bad subset, which we denote by $C(x_2)$. Since $C(x_2)$ is bad and $|S(y_1)\cap S(y_2)|=2$, we must have that $C(x_2)\cap\{1,3\}\neq\emptyset$, and therefore $3\in C(x_2)$. If $C(x_2)=\{3,4\}$, then we must have that $S(x_1)=\{1,2,5,6\}$, so

$$S(x_1)=\{1,2,5,6\}, S(x_2)=\{3,4,5,6\}, S(y_1)=\{1,2,3,4\}, S(y_2)=\{1,3,5,6\}.$$

The set $incomp(x_1,x_2)$ has degree sequence $(6,5,5,3,2,2)$. Otherwise, suppose without loss of generality that $C(x_2)=\{3,5\}$. We must have that $S(x_1)=\{1,2,4,6\}$, so

$$S(x_1)=\{1,2,4,6\}, S(x_2)=\{3,4,5,6\}, S(y_1)=\{1,2,3,4\}, S(y_2)=\{1,3,5,6\}.$$

In this case $|incomp(x_1,x_2)|=21$.

Case 4: $|C(x_1)\cap S(y_2)|=1$ and $|S(y_1)\cap S(y_2)|=1$.

We can assume, without loss of generality, that $1\in S(y_2)$. Denote $S(y_2)=\{1,5,6,7\}$. Since $C(x_1)$ is bad, we must have that $S(x_2)\cap\{3,4\}\neq\emptyset$ and $|S(x_2)\cap\{5,6,7\}|\geq 2$. Suppose without loss of generality that $\{3,5,6\} \subseteq S(x_2)$. Since $C(x_1)$ is bad, we must have that either $S(x_2) = \{4,3,5,6\}$ or $S(x_2) = \{7,3,5,6\}$. It is easy to see that in both cases we have a contradiction to the fact that $S(x_2)$ contains a bad subset. $\square$

For every i, $1 \leq i \leq m$, let A_i be a sequence of 4 distinct elements. The sequence $A_1, \ldots, A_m$ is called valid if whenever $c \in A_i \cap A_{i+1}$, then c appears in the same position in both A_i and A_{i+1}. A valid sequence $A_1, \ldots, A_m$ is called legal if whenever $c \in A_{i+1} - A_i$, then $c \notin A_j$ for every j, $1 \leq j \leq i$. By a subsequence of $A_1, \ldots, A_m$ we mean a sequence of the form $A_i, A_{i+1}, \ldots, A_j$, where $1 \leq i \leq j \leq m$.

Let $A_1, \ldots, A_m$ be a valid sequence. The pair (A_i, A_{i+1}) contains a change in the kth position if the elements which appear in the kth position of A_i and A_{i+1} are different. The sequence $A_1, \ldots, A_m$ contains a change in the kth position if there exists a pair (A_i, A_{i+1}) which contains a change in the kth position.

Let $A_1, \ldots, A_m$ be a sequence. By C_i we denote a subset of A_i of size 2. We say that C_1 and C_m are compatible if there exist subsets $\{C_k : 1 < k < m\}$ so that $C_p \cap C_{p+1} = \emptyset$ for every p, $1 \leq p < m$. A subset C_1 is called bad if C_1 is not compatible with any C_m. A subset C_1 is called good if C_1 is compatible with every C_m. We denote by $comp(C_1; A_1, \ldots, A_m)$ the set which consists of all the subsets C_m which are compatible with C_1, and by $comp(A_1, \ldots, A_m)$ the set of all the compatible pairs (C_1, C_m). By $good(A_1, \ldots, A_m)$ we denote the set which consists of all the good subsets that A_1 contains.

Lemma 5.3 *If the valid sequence $D_1, \ldots, D_r$ contains a change in at least 3 positions and there is no i, $1 < i < r-1$, for which $D_i = D_{i+1}$, then it contains a subsequence $A_1, \ldots, A_m$, so that the sequence A_1 contains at least one good subset. Furthermore, the sequence $A_1, \ldots, A_m$ has at least one of the following properties:*

1. $|good(A_1, \ldots, A_m)| \geq 3$.
2. $|comp(A_1, \ldots, A_m)| > 23$.
3. *The set $comp(A_1, \ldots, A_m)$ is special. If m is odd, then $comp(A_1, \ldots, A_m)$ has exactly one of the properties P_1 and P_2. If m is even then $comp(A_1, \ldots, A_m)$ has either both or none of the properties P_1 and P_2.*

Proof We consider the following cases.

Case 1: For some i, $|D_i \cap D_{i+1}| \leq 1$.

In this case $|good(D_i, D_{i+1})| \geq 3$.

Case 2: For every j, $|D_j \cap D_{j+1}| \leq 2$, and for some i, $|D_i \cap D_{i+1}| = 2$.

Assume without loss of generality that the pair (D_i, D_{i+1}) contains a change in the first and second positions. At least one of the pairs (D_{i-1}, D_i) and (D_i, D_{i+1}) contains a change in some position. Suppose that the pair (D_{i-1}, D_i) contains a change in some position. The proof in case the pair (D_i, D_{i+1}) contains a change in some position is similar. If the pair (D_{i-1}, D_i) contains a change in at least one of the first and second positions, then surely $|good(D_{i-1}, D_i, D_{i+1})| \geq 3$. If the only position in which the pair (D_{i-1}, D_i) contains a change is either the third or the fourth position, then $comp(D_{i-1}, D_i, D_{i+1})$ is special, has property P_2, and does not have property P_1. If the pair (D_{i-1}, D_i) contains a change in the third and fourth positions, then $|comp(D_{i-1}, D_i, D_{i+1})| = 27$.

Case 3: For every j, $|D_j \cap D_{j+1}| \leq 1$.

Let $B_1, \ldots B_k$ be a subsequence of $D_1, \ldots, D_r$ which contains a change in at least 3 positions, but no proper subsequence of $B_1, \ldots, B_k$ has this property. This implies that the three pairs (B_1, B_2), (B_2, B_3) and (B_{k-1}, B_k) contain a change in three different positions. We can assume, without loss of generality, that the three pairs contain a change in the first, second and third positions respectively. Suppose that $2 \leq i \leq k-2$, and consider the pair (B_i, B_{i+1}). If this pair contains a change in the first position, then the sequence $B_2, \ldots, B_m$ contains a change in at least 3 positions. If this pair contains a change in the third or fourth position, then the sequence $B_1, \ldots, B_{i+1}$ contains a change in at least 3 positions. Hence, the pair (B_i, B_{i+1}) contains a change in the second position. If $k = 4$ then the set $comp(B_1, \ldots, B_4)$ is special and does not have neither property P_1 nor property P_2. If $k > 4$ then $|good(B_1, \ldots, B_4)| = 3$. $\square$

Lemma 5.4 *If the set $comp(A_1, \ldots, A_m)$ is special, then both the set $comp(A_1, A_1, \ldots, A_m)$ and the set $comp(A_1, \ldots, A_m, A_m)$ are special. The set $comp(A_1, A_1, \ldots, A_m)$ has property P_1 iff the set $comp(A_1, \ldots, A_m)$ has property P_1. The set $comp(A_1, A_1, \ldots, A_m)$ has property P_2 iff the set $comp(A_1, \ldots, A_m)$ does not have property P_2. The set $comp(A_1, \ldots, A_m, A_m)$ has property P_1 iff the set $comp(A_1, \ldots, A_m)$ does not have property P_1. The set $comp(A_1, \ldots, A_m, A_m)$ has property P_2 iff the set $comp(A_1, \ldots, A_m)$ has property P_2.*

Lemma 5.5 *If* A_1, A_2, A_2, A_3 *is a legal sequence, then*

$$comp(A_1, A_2, A_2, A_3) = comp(A_1, A_3).$$

Proof Let $k_1, \ldots k_n$ be all the positions in which A_1, A_2, A_2, A_3 does not contain a change. It is easy to verify that $(C, D) \in comp(A_1, A_3)$ iff there is no i for which C contains the k_ith element of A_1 and D contains the k_i element of A_3. The same property holds also for $comp(A_1, A_2, A_2, A_3)$. $\square$

Lemma 5.6 *If* $A_i, \ldots, A_j$ *is a subsequence of* $A_1, \ldots, A_m$*, then*

$$|comp(A_1, \ldots, A_m)| \geq |comp(A_i, \ldots, A_j)|.$$

Proof By induction on m. If $m = j - i + 1$, there is nothing to prove. Suppose that $m > j - i + 1$. Assume that $i > 1$. The proof in case $j < m$ is similar. Hence,

$$|comp(A_1, \ldots, A_m)| \geq |comp(A_2, A_2, \ldots, A_m)| = |comp(A_2, \ldots, A_m)| \geq |comp(A_i, \ldots, A_j)|,$$

where the last inequality follows from the induction hypothesis. $\square$

Lemma 5.7 *If* $A_i, \ldots, A_j$ *is a subsequence of* $A_1, \ldots, A_m$*, then*

$$|good(A_1, \ldots, A_m)| \geq |good(A_i, \ldots, A_j)|.$$

Proof Similar to the proof of lemma 5.6. $\square$

Lemma 5.8 *Suppose that* $i \geq 0$*, and denote by* F *the sequence* $A_{i+1}, \ldots, A_m$ *together with an additional* A_{i+1} *as the first element of the sequence in case* $i \equiv 1 \pmod 2$*. If* $A_{i+1}, \ldots, A_m$ *is a subsequence of* $A_1, \ldots, A_m$ *and* $|comp(A_{i+1}, \ldots, A_m)| = |comp(A_1, \ldots, A_m)|$*, then* $comp(A_1 \ldots, A_m)$ *is isomorphic to* $comp(F)$*.*

Proof We can assume that $A_1, \ldots, A_m$ is a valid sequence. Suppose that $i \equiv 1 \pmod 2$. The proof in case $i \equiv 0 \pmod 2$ is similar. Suppose that $C_1 \subseteq A_1$. Denote by T the subset of A_{i+1} that appears in the two positions in which C_1 does not appear in A_1. Since $A_1, \ldots, A_{i+1}$ is a valid sequence, we have that C_1 is compatible with T. Hence,

$$V = comp(C_1; A_1, \ldots, A_m) \supseteq comp(T; A_{i+1}, \ldots, A_m) = W.$$

Since $|comp(A_{i+1}, \ldots, A_m)| = |comp(A_1, \ldots, A_m)|$, we must have that $V = W$. It is easy to see now that $comp(A_1, \ldots, A_m)$ is isomorphic to $comp(A_{i+1}, A_{i+1}, \ldots, A_m)$. $\square$

Lemma 5.9 *Suppose that $i, j \geq 0$. Denote by F the sequence $A_{i+1}, \ldots, A_{m-j}$ together with an additional A_{i+1} as the first element of the sequence in case $i \equiv 1 \pmod 2$ and an additional A_{m-j} as the last element of the sequence in case $j \equiv 1 \pmod 2$. If $A_{i+1}, \ldots, A_{m-j}$ is a subsequence of $A_1, \ldots, A_m$ and $|comp(A_{i+1}, \ldots, A_{m-j})| = |comp(A_1, \ldots, A_m)|$, then $comp(A_{i+1} \ldots, A_{m-j})$ is isomorphic to $comp(F)$.*

Proof Apply lemma 5.8 twice. □

Lemma 5.10 *Suppose that r is odd and that $r \geq 3$. If the valid sequence $D_1, \ldots, D_r$ contains a change in at least 3 positions, then the sequence D_1 contains at least one good subset. Furthermore, at least one of the following is valid:*

1. $|good(D_1, \ldots, D_r)| \geq 3$.
2. $|comp(D_1, \ldots, D_r)| > 23$.
3. *The set $comp(D_1, \ldots, D_m)$ is special and has exactly one of the properties P_1 and P_2.*

Proof We can assume, without loss of generality, that $D_1, \ldots, D_r$ is legal. Due to lemma 5.5, we can assume that there is no i, $1 < i < r-1$, for which $D_i = D_{i+1}$. It follows from lemma 5.3 that the sequence $D_1, \ldots, D_r$ contains a subsequence $A_1, \ldots, A_m$, so that the sequence A_1 contains at least one good subset. It follows from lemma 5.7 that $|good(D_1, \ldots, D_r)| \geq 1$. According to lemma 5.3, we consider the following cases:

Case 1: $|good(A_1, \ldots, A_m)| \geq 3$.

It follows from lemma 5.7 that $|good(D_1, \ldots, D_r)| \geq 3$.

Case 2: $|comp(A_1, \ldots, A_m)| \geq 27$.

It follows from lemma 5.6 that $|comp(D_1, \ldots, D_r)| > 23$.

Case 3: The set $comp(A_1, \ldots, A_m)$ is special.

We know that if m is odd, then $comp(A_1, \ldots, A_m)$ has exactly one of the properties P_1 and P_2. Furthermore, if m is even then $comp(A_1, \ldots, A_m)$ has either both or none of the properties P_1 and P_2. If $|comp(D_1, \ldots, D_r)| > |comp(A_1, \ldots, A_m)|$, then $|comp(D_1, \ldots, D_m)| > 23$. Suppose that $|comp(D_1, \ldots, D_r)| = |comp(A_1, \ldots, A_m)|$. It follows from lemma 5.9 that $comp(D_1, \ldots, D_r)$ is isomorphic to $comp(F)$ for some sequence F. Since r is odd and using lemma 5.4, it is easy to see that $comp(D_1, \ldots, D_r)$ is special and has exactly one of the properties P_1 and P_2. □

Proof of Theorem 1.16 It is easy to see that a graph G is $(4:2)$-choosable iff its core is $(4:2)$-choosable. Due to theorem 1.15, we need to prove that for every $m \geq 1$, $\Theta_{2,2,2m}$ is $(4:2)$-choosable. Suppose that m is odd and that $m \geq 3$. Assume that $\Theta_{2,2,m-1}$ has vertex set $V = \{u, v, z_1, \ldots, z_m\}$ and contains the three paths $z_1 - z_2 - \cdots - z_m$, $z_1 - u - z_m$, and $z_1 - v - z_m$. For each $w \in V$, let $S(w)$ be a set of size 4. We denote $A_i = S(z_i)$ for every i, $1 \leq i \leq m$. We can assume that $A_1, \ldots, A_m$ is a valid sequence.

Suppose first that the sequence $A_1, \ldots, A_m$ contains a change in at most 2 positions. This means that there is a set C of size 2 so that $C \subseteq A_i$ for every i, $1 \leq i \leq m$. From A_i when i is odd, choose the subset C. Complete the choice by choosing a subset of $S(w) - C$ for every other vertex w.

Suppose next that the sequence $A_1, \ldots, A_m$ contains a change in at least 3 positions. The graph induced by the set of vertices $\{z_1, z_m, u, v\} = W$ is isomorphic to $K_{2,2}$. Denote $x_1 = z_1$, $x_2 = z_m$, $y_1 = u$, and $y_2 = v$. We use the same terminology as before.

Case 1: $\{S(w) : w \in W\}$ is not defected.

Suppose without loss of generality that $S(z_1)$ contains no bad subset. If follows from lemma 5.10 that $|good(A_1, \ldots, A_m)| \geq 1$, and therefore a choice is possible.

Case 2: $\{S(w) : w \in W\}$ is defected.

According to lemma 5.10, we consider the following cases:

Case 2a: $|good(A_1, \ldots, A_m)| \geq 3$.

It follows from lemma 5.2 that $S(z_1)$ contains exactly one bad subset, and therefore a choice is possible.

Case 2b: $|comp(D_1, \ldots, D_r)| > 23$.

It follows from lemma 5.2 that $|incomp(z_1, z_m)| \leq 23$, and therefore a choice is possible.

Case 2c: The set $comp(D_1, \ldots, D_m)$ is special.

We know that $comp(D_1, \ldots, D_m)$ has exactly one of the properties P_1 and P_2. It is easy to see from lemma 5.2 that the set $incomp(z_1, z_m)$ does not contain the set $comp(D_1, \ldots, D_m)$, and therefore a choice is possible. $\square$

Proof of Theorem 1.17 Suppose that $G = (V, E)$ is $(2mk : mk)$-choosable for k odd. We prove that G is $2m$-choosable as well. For each $v \in V$, let $S(v)$ be a set of size $2m$. With every color c we associate a set $F(c)$ of size k, such that $F(c) \cap F(d) = \emptyset$ if $c \neq d$. For every $v \in V$, we define $T(v) = \cup_{c \in S(v)} F(c)$. Since G is $(2mk : mk)$-choosable, there are subsets $C(v) \subseteq T(v)$, where $|C(v)| = mk$ for all $v \in V$, and $C(u) \cap C(v) = \emptyset$ for every two adjacent vertices $u, v \in V$.

Fix a vertex $v \in V$. Since k is odd, there is a color $c \in S(v)$ for which $|C(v) \cap F(c)| > k/2$, so we define $f(v) = c$. In case u and v are adjacent vertices for which $c \in S(u) \cap S(v)$, it is not possible that both $|C(u) \cap F(c)|$ and $|C(v) \cap F(c)|$ are greater than $k/2$. This proves that f is a proper vertex-coloring of G assigning to each vertex $v \in V$ a color in $S(v)$. □

6 The complexity of graph choosability

Let $G = (V, E)$ be a graph. We denote by G' the graph obtained from G by adding a new vertex to G, and joining it to every vertex in V. Consider the following decision problem:

GRAPH k-COLORABILITY

INSTANCE: A graph $G = (V, E)$.

QUESTION: Is G k-colorable?

The standard technique to show a polynomial transformation from GRAPH k-COLORABILITY to GRAPH $(k+1)$-COLORABILITY is to use the fact that $\chi(G') = \chi(G) + 1$ for every graph G. However, it is not true that $ch(G') = ch(G) + 1$ for every graph G. To see that, we first prove that $K'_{2,4}$ is 3-choosable.

Suppose that $K'_{2,4}$ has vertex set $V = \{v, x_1, x_2, y_1, y_2, y_3, y_4\}$, and contains exactly the edges $\{x_i, y_j\}$, $\{v, x_i\}$, and $\{v, y_j\}$. For each $w \in V$, let $S(w)$ be a set of size 3.

Case 1: All the sets are the same.

A choice can be made since $K'_{2,4}$ is 3-colorable.

Case 2: There is a set $S(x_i)$ which is not equal to $S(v)$.

Without loss of generality, suppose that $S(v) \neq S(x_1)$. For the node v, choose a color $c \in S(v) - S(x_1)$, and remove c from the sets of the other vertices. We can assume that every set $S(y_j)$ is of size 2 now.

Suppose first that $S(x_1)$ and $S(x_2)$ are disjoint. The number of different sets consisting of one color from each of the $S(x_i)$ is at least 6, and therefore we can choose colors $c_i \in S(x_i)$, such that $\{c_1, c_2\}$ does not appear as a set of $S(y_j)$. We complete the choice by choosing for every vertex y_j

a color from $S(y_j) - \{c_1, c_2\}$. Suppose next that $c \in S(x_1) \cap S(x_2)$. For every vertex x_i we choose c, and for every vertex y_j we choose a color from $S(y_j) - \{c\}$.

Case 3: There is a set $S(y_j)$ which is not equal to $S(v)$.

Without loss of generality, suppose that $S(v) \neq S(y_1)$. For the node v, choose a color $c \in S(v) - S(y_1)$, and remove c from the sets of the other vertices. Suppose first that $S(x_1)$ and $S(x_2)$ are disjoint. The number of different sets consisting of one color from each of the $S(x_i)$ is at least 4, and since $|S(y_1)| = 3$ we can choose colors $c_i \in S(x_i)$, such that $S(y_j) - \{c_1, c_2\} \neq \emptyset$ for every vertex y_j. We can complete the choice as in case 2. In case $S(x_1)$ and $S(x_2)$ are not disjoint, we proceed as in case 2.

This completes the proof that $K'_{2,4}$ is 3-choosable. It follows from theorem 1.15 and corollary 1.12 that $ch(K_{2,4}) = 3$, and therefore $ch(K'_{2,4}) = ch(K_{2,4}) = 3$. The following lemma exhibits a construction which increases the choice number of a graph in exactly 1.

Lemma 6.1 *Let $G = (V, E)$ be a graph. If H is the disjoint union of $|V|$ copies of G, then $ch(H') = ch(G) + 1$.*

Proof Let H be the disjoint union of the graphs $\{G_i : 1 \leq i \leq |V|\}$, where each G_i is a copy of G. Suppose that H' is obtained from H by joining the new vertex v to all the vertices of H.

We claim that if G is k-choosable, then H' is $(k+1)$-choosable. For each $w \in V(H')$, let $S(w)$ be a set of size $k+1$. Choose a color $c \in S(v)$, and remove c from the sets of the other vertices. We can complete the choice since G is k-choosable.

We now prove that if H' is k-choosable, then G is $(k-1)$-choosable. It is easy to see that this is true when G is a complete graph. If G is not a complete graph, then by corollary 1.9 $ch(G) < |V|$, and therefore $ch(H') \leq |V|$. Hence, we can assume that $k \leq |V|$. For each $w \in V$, let $S(w)$ be a set of size $k-1$, such that $S(w) \cap \{1, 2, \ldots, |V|\} = \emptyset$. For every i, $1 \leq i \leq |V|$, on the vertices of the graph G_i we put the sets $S(w)$ together with the additional color i. The vertex v is given the set $\{1, 2, \ldots, k\}$. Let f be a proper vertex-coloring of H' assigning to each vertex a color from its set. Denote $f(v) = i$, then f restricted to G_i is a proper vertex-coloring of G assigning to each vertex $w \in V$ a color in S(w). $\square$

Lemma 6.2 BIPARTITE GRAPH 3-CHOOSABILITY *is Π_2^p-complete.*

Proof It is easy to see that **BG 3-CH** $\in \Pi_2^p$. We transform **BG** $(2,3)$**-CH** to **BG 3-CH**. Let $G = (V, E)$ and $f : V \mapsto \{2, 3\}$ be an instance of **BG** $(2,3)$**-CH**. We shall construct a bipartite graph W such that W is 3-choosable if and only if G is f-choosable.

Let H be the disjoint union of the graphs $\{G_{i,j} : 1 \leq i, j \leq 3\}$, where each $G_{i,j}$ is a copy of G. Let (X, Y) be a bipartition of the bipartite graph H. The graph W is obtained from H by adding two new vertices u and v, joining u to every vertex $w \in X$ for which $f(w) = 2$, and joining v to every vertex $w \in Y$ for which $f(w) = 2$.

Since H is bipartite, W is also a bipartite graph. It is easy to see that if G is f-choosable, then W is 3-choosable. We now prove that if W is 3-choosable, then G is f-choosable. For every $w \in V$, let $S(w)$ be a set of size $f(w)$, such that $S(w) \cap \{1, 2, 3\} = \emptyset$. For every i and j, $1 \leq i, j \leq 3$, on the vertices of the graph $G_{i,j}$ we put the sets $S(w)$ with the vertices for which f is equal to 2 receiving another color as follows: to the vertices which belong to X we add the color i, whereas to the vertices which belong to Y we add the color j. The vertices u and v are both given the set $\{1, 2, 3\}$. Let f be a proper vertex-coloring of H' assigning to each vertex a color from its set. Denote $f(u) = i$ and $f(v) = j$, then f restricted to $G_{i,j}$ is a proper vertex-coloring of G assigning to each vertex $w \in V$ a color in $S(w)$. □

Proof of Theorem 1.18 The proof is by induction on k. For $k = 3$, the result follows from lemma 6.2. Assuming that the result is true for k, $k \geq 3$, we prove it is true for $k + 1$. It is easy to see that **BG** $(k+1)$**-CH** $\in \Pi_2^p$. We transform **BG** k**-CH** to **BG** $(k+1)$**-CH**. Let $G = (V, E)$ be an instance of **BG** k**-CH**. We shall construct a bipartite graph W such that W is $(k+1)$-choosable if and only if G is k-choosable.

Let H be the disjoint union of the graphs $\{G_{i,j} : 1 \leq i, j \leq (k+1)^2\}$, where each $G_{i,j}$ is a copy of G. Let (X, Y) be a bipartition of the bipartite graph H. The graph W is obtained from H by adding two new vertices u and v, joining u to every vertex of X, and joining v to every vertex of Y.

It is easy to see that if G is k-choosable, then W is $(k+1)$-choosable. In a similar way to the proof of lemma 6.2, we can prove that if W is $(k+1)$-choosable, then G is k-choosable. □

7 The strong choice number

Let $G = (V, E)$ be a graph, and let $V_1, \dots, V_r$ be pairwise disjoint subsets of V. We denote by $[G, V_1, \dots, V_r]$ the graph obtained from G by adding to it the union of cliques induces by each V_i, $1 \le i \le r$.

Suppose that $G = (V, E)$ is a graph with maximum degree at most 1. We claim that G is strongly k-choosable for every $k \ge 2$. To see that, let $V_1, \dots, V_r$ be pairwise disjoint subsets of V, each of size at most k. The graph $[G, V_1, \dots, V_r]$ has maximum degree at most k, and therefore by corollary 1.9 it is k-choosable.

Proof of Theorem 1.19 Let $G = (V, E)$ be a strongly k-colorable graph. Let $V_1, \dots, V_r$ be pairwise disjoint subsets of V, each of size at most $k+1$. Without loss of generality, we can assume that $V_1, \dots, V_m$ are subsets of size exactly $k+1$, and $V_{m+1}, \dots, V_r$ are subsets of size less than $k+1$. Let H be the graph $[G, V_1, \dots, V_r]$. To complete the proof, it suffices to show that H is $(k+1)$-colorable. For every i, $1 \le i \le m$, we define $W_i = V_i - \{c\}$ for an arbitrary element $c \in V_i$, whereas for every j, $m+1 \le j \le r$, we define $W_i = V_i$. Since $[G, W_1, \dots, W_r]$ is k-colorable, there exists an independent set S of H which is composed of exactly one vertex from each V_i, $1 \le i \le m$. For every i, $1 \le i \le m$, we define $W_i = V_i - S$, whereas for every j, $m+1 \le j \le r$, we define $W_i = V_i$. Since $[G, W_1, \dots, W_r]$ is k-colorable, we can obtain a proper $(k+1)$-vertex coloring of H by using k colors for $V - S$ and another color for S. $\square$

Lemma 7.1 *Suppose that $k, l \ge 1$. If $\mathcal{F}$ is a family of $k+l$ sets of size $k+l$, then it is possible to partition $\mathcal{F}$ into a family $\mathcal{F}_1$ of k sets and a family $\mathcal{F}_2$ of l sets, to choose for each set $S \in \mathcal{F}_1$ a subset $S' \subseteq S$ of size k, and to choose for each set $T \in \mathcal{F}_2$ a subset $T' \subseteq T$ of size l, so that $S' \cap T' = \emptyset$ for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$.*

Proof Suppose that $\mathcal{F} = \{C_1, \dots, C_{k+l}\}$, and define $C = \cup_{i=1}^{k+l} C_i$. For every partition π of C into the two subsets A and B, we denote $\mathcal{R}(\pi) = \{V \in \mathcal{F} : |V \cap A| > k\}$, $\mathcal{L}(\pi) = \{V \in \mathcal{F} : |V \cap B| > l\}$, and $\mathcal{M}(\pi) = \{V \in \mathcal{F} : |V \cap A| = k \ \ and \ \ |V \cap B| = l\}$. We now start with the partition of C into the two subsets $A = C$ and $B = \emptyset$, and start moving one element at a time from A to B until we obtain a partition π_1 of C into the two subsets A and B and a partition π_2 into the two subsets $A' = A - \{c\}$ and $B' = B \cup \{c\}$, such that $|\mathcal{R}(\pi_1)| > k$ and $|\mathcal{R}(\pi_2)| \le k$. It is easy to that

$\mathcal{L}(\pi_2) \subseteq \mathcal{L}(\pi_1) \cup \mathcal{M}(\pi_1)$, and therefore $|\mathcal{L}(\pi_2)| < l$. We now partition $\mathcal{M}(\pi_2)$ into two subsets $\mathcal{M}_1$ and $\mathcal{M}_2$, such that $\mathcal{F}_1 = \mathcal{R}(\pi_2) \cup \mathcal{M}_1$ has size k and $\mathcal{F}_2 = \mathcal{L}(\pi_2) \cup \mathcal{M}_2$ has size l. For every set $S \in \mathcal{F}_1$ we choose a subset $S' \subseteq S \cap A'$ of size k, whereas for every $T \in \mathcal{F}_2$ we choose a subset $T' \subseteq T \cap B'$ of size l. Since A' and B' are disjoint, we have that $S' \cap T' = \emptyset$ for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$. $\Box$

Lemma 7.2 *Suppose that $k, m \geq 1$. If $\mathcal{F}$ is a family of km sets of size km, then it is possible to partition $\mathcal{F}$ into the m subsets $\mathcal{F}_1, \ldots, \mathcal{F}_m$, each of size k, and to choose for each set $S \in \mathcal{F}$ a subset $S' \subseteq S$ of size k, so that $S' \cap T' = \emptyset$ for every $i \neq j$, $S \in \mathcal{F}_i$ and $T \in \mathcal{F}_j$.*

Proof By induction on m. For $m = 1$ the result is trivial. Assuming that the result is true for m, $m \geq 1$, we prove it is true for $m + 1$. Let $\mathcal{F}$ be a family of $k(m+1)$ sets of size $k(m+1)$. By lemma 7.1, it is possible to partition $\mathcal{F}$ into a family $\mathcal{F}_1$ of k sets and a family $\mathcal{F}_2$ of km sets, to choose for each $S \in \mathcal{F}_1$ a subset $S' \subseteq S$ of size k, and to choose for each set $T \in \mathcal{F}_2$ a subset $T' \subseteq T$ of size km, so that $S' \cap T' = \emptyset$ for every $S \in \mathcal{F}_1$ and $T \in \mathcal{F}_2$. The proof is completed by applying the induction hypothesis on $\mathcal{F}_2$. $\Box$

Proof of Theorem 1.20 Let $G = (V, E)$ be a strongly k-choosable graph. Let $V_1, \ldots, V_r$ be pairwise disjoint subsets of V, each of size at most km. Let H be the graph $[G, V_1, \ldots, V_r]$. To complete the proof, it suffices to show that H is km-choosable. For each $v \in V$, let $S(v)$ be a set of size km. By lemma 7.2, for every i, $1 \leq i \leq r$, is it possible to partition V_i into the m subsets $V_{i,1}, \ldots, V_{i,m}$, each of size at most k, and to choose for each vertex $v \in V_i$ a subset $C(v) \subseteq S(v)$ of size k, so that $C(u) \cap C(v) = \emptyset$ for every $p \neq q$, $u \in V_{i,p}$ and $v \in V_{i,q}$. Since the graph $[G, V_{1,1}, \ldots, V_{r,m}]$ is k-choosable, we can obtain a proper vertex-coloring of H assigning to each vertex a color from its set. $\Box$

Proof of Theorem 1.21 Apply lemma 7.2 as in proof of theorem 1.20. $\Box$

Proof of Corollary 1.22 If is proved in [9] that if G is a 4-regular graph on $3n$ vertices and G has a decomposition into a Hamiltonian circuit and n pairwise vertex disjoint triangles, then $ch(G) = 3$. The result follows from theorem 1.21. $\Box$

Proof of Theorem 1.23 Since $s\chi(1) = 2$, we can assume that $d > 1$. Suppose first that d is even, and denote $d = 2r$. Construct a graph G with $12r - 3$ vertices, partitioned into 8 classes, as follows. Let these classes be $A, B_1, B_2, C_1, C_2, D_1, D_2, E$, where $|A| = |D_1| = |D_2| = 2r$, $|B_1| = |B_2| = r$,

$|C_1| = |C_2| = r - 1$, and $|E| = 2r - 1$. Each vertex in A is joined by edges to each member of B_1 and each member of B_2. Each member of D_1 is adjacent to each member of D_2. Consider the following partition of the set of vertices of G into three classes of cardinality $4r - 1$ each:

$$V_1 = B_1 \cup C_1 \cup D_1, V_2 = B_2 \cup C_2 \cup D_2, V_3 = A \cup E.$$

We claim that $H = [G, V_1, V_2, V_3]$ is not $(4r-1)$-colorable. In a proper $(4r-1)$-vertex coloring of H, every color used for coloring the vertices of A must appear on a vertex of $C_1 \cup D_1$ and on a vertex of $C_2 \cup D_2$. Since $|C_1 \cup C_2| < |A|$, there is a color used for coloring the vertices of A which appears on both D_1 and D_2. But this is impossible as each vertex in D_1 is adjacent to each member of D_2. Thus $s\chi(G) > 4r - 1$ and as the maximum degree in G is $2r$, this shows that $s\chi(2r) \geq 4r$.

Suppose next that d is odd, and denote $d = 2r + 1$. Construct a graph G with $12r + 3$ vertices, partitioned into 8 classes, as follows. Let these classes be named as before, where $|A| = |D_1| = |D_2| = 2r + 1$, $|B_1| = r + 1$, $|C_1| = r - 1$, $|B_2| = |C_2| = r$, and $|E| = 2r$. In the same manner we can prove that $[G, V_1, V_2, V_3]$ is not $(4r+1)$-colorable. Thus $s\chi(G) > 4r + 1$ and as the maximum degree in G is $2r + 1$, this shows that $s\chi(2r + 1) \geq 4r + 2$, completing the proof. □

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