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# THE NUCLEON OF COOPERATIVE GAMES AND AN ALGORITHM FOR MATCHING GAMES

Ulrich Faigle Walter Kern

Department of Applied Mathematics
University of Twente
P.O.Box 217
NL-7500 AE Enschede
The Netherlands

Sándor P. Fekete Winfried Hochstättler

ZPR, Zentrum für paralleles Rechnen
Universität zu Köln
Albertus-Magnus-Platz
D-50923 Köln
Germany

**Abstract**

The *nucleon* is introduced as a new allocation concept for non-negative cooperative n-person transferable utility games. The nucleon may be viewed as the multiplicative analogue of Schmeidler's nucleolus. It is shown that the nucleon of (not necessarily bipartite) matching games can be computed in polynomial time.

**Keywords:** Cooperative game, core, nucleolus, nucleon, matching

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## 1 Introduction

One of the central problems in cooperative game theory is to provide fair allocations to the players in the game. The games that we consider here are cooperative n-person transferable utility games in characteristic function form. Formally, the general setup can be described as follows.

There is a finite set $N = \{1, \ldots, n\}$ of *players*. These players may form *coalitions* $S \subseteq N$ in an arbitrary way. Each coalition $S$ can achieve a *value* $v(S) \in \Re$ (assuming that the players in $S$ "cooperate"). The value $v(N)$ of the *grand coalition* $N$ can thus be understood as the total "profit" arising from the cooperation of all players. The pair $(N, v)$ therefore represents our game in *characteristic form.* An *allocation* is a vector $x \in \Re^N$ with component sum equal to $v(N)$. The allocations we seek should be *fair* in the sense that they assess the strength of individual players relative to $(N, v)$ in an acceptable way.

Many interesting examples of such games have been investigated where the value $v(S)$ of a coalition $S \subseteq N$ is determined as the optimal value of a combinatorial optimization problem the set $S$ of players faces (see, *e.g.*, Curiel [1988]).

In the *matching game*, for instance, we are given the complete graph $K_n$ with $N$ as the set of nodes. A *matching* is a set $M$ of edges such that no two edges in $M$ have a node in common. Each edge $e$ in $K_n$ is assigned a weight $w(e)$ and the value $v(S)$ of a coalition is equal to the weight of a maximal matching in the subgraph induced by $S$. Here each individual player $i \in N$ has value $v(i) = 0$ while value $v(N) > 0$ may well be possible. How should the strength of $i \in N$ be assessed?

There are many notions of "fairness" for allocations (see, *e.g.*, Shubik [1981]). In the following we will only present a few of them.

The idea of the *core* of a game, which essentially goes back to von Neumann and Morgenstern [1944], approaches fairness from the point of view of coalitions. The allocation $x = (x_1, \ldots, x_n)$ is said to be in the *core* of $(N, v)$ if there is no coalition $S \subseteq N$ such that

$$\sum_{i \in S} x_i \; < \; v(S) \, .$$

Note that the vectors $x$ in the core of $(N, v)$ form a polyhedron in $\Re^N$ as they are determined by the linear restrictions

$$\begin{aligned} \sum_{i \in N} x_i \; &= \; v(N) \, . \\ \sum_{i \in S} x_i \; &\geq \; v(S) \;\; \text{for all } S \subseteq N \end{aligned}$$

A game may have an empty core (*e.g.*, the matching game on $K_3$ with unit edge weights). Therefore, relaxations of the concept of a core have received attention. For a given $\epsilon \in \Re$, Shapley and Shubik [1966] consider the modified game $(N, v^\epsilon)$, where

$$v^\epsilon(S) = \begin{cases} v(S) & \text{if } S \in \{\emptyset, N\} \\ v(S) - \epsilon & \text{otherwise.} \end{cases}$$

The *additive $\epsilon$-core* of the game $(N, v)$ is defined to be the core of the game $(N, v^\epsilon)$.

Faigle and Kern [1993] propose an $\epsilon$-correction relative to the value of a proper coalition and arrive at the modified game $(N, v_\epsilon)$, where

$$v_\epsilon(S) = \begin{cases} v(S) & \text{if } S \in \{\emptyset, N\} \\ (1 - \epsilon) v(S) & \text{otherwise.} \end{cases}$$

The *multiplicative $\epsilon$-core* of $(N, v)$ is then the core of the game $(N, v_\epsilon)$. (In the case where each individual player $i \in N$ has value $v(i) = 0$, the multiplicative $\epsilon$-core coincides with the *$\epsilon$-tax core* of Tijs and Driessen [1986]).

There is always some $\epsilon$ yielding a non-empty additive $\epsilon$-core. The same is true for the multipicative $\epsilon$-core whenever $v(N) \geq 0$ (take, *e.g.*, $\epsilon = 1$). For both models, this observation suggests to seek an $\epsilon$ that is as small as possible while still guaranteeing a non-empty $\epsilon$-core (see, *e.g.*, Faigle and Kern [1993] and Faigle *et al.* [1994] for the multiplicative $\epsilon$-core of some combinatorial games).

The concept of the additive $\epsilon$-core is refined by the notion of the *nucleolus* due to Schmeidler [1969]. We want an allocation $x$ that maximizes the *excess*

$$e(x, S) = \sum_{i \in S} x_i \; - v(S)$$

uniformly over all proper coalitions $S$, *i.e.*, we solve the linear program

$$(LP_1) \quad \begin{array}{lcll} \max \epsilon & & & \\ \sum_{i \in N} x_i & = & v(N) & \\ \sum_{i \in S} x_i & \geq & v(S) + \epsilon & \text{for all } S \notin \{\emptyset, N\} \end{array}$$

Denoting by $\epsilon_1$ the optimal objective function value of $(LP_1)$, it follows that $\epsilon = -\epsilon_1$ is the minimal value admitting a non-empty additive $\epsilon$-core.

If $(LP_1)$ has a unique solution $(\epsilon_1, x^*)$, then $x^*$ is the *nucleolus* of the game $(N, v)$. Otherwise, there is a unique collection $\mathcal{S}_1 \subset 2^N$ of coalitions $S$ $(\neq \emptyset, N)$ for which the inequalities in $(LP_1)$ become tight at $\epsilon = \epsilon_1$.

Now, in a second step, we maximize the excess over all remaining coalitions:

$$(LP_2) \quad \begin{array}{lcll} \max \epsilon & & & \\ \sum_{i \in N} x_i & = & v(N) & \\ \sum_{i \in S} x_i & = & v(S) + \epsilon_1 & \text{for all } S \in \mathcal{S}_1 \\ \sum_{i \in S} x_i & \geq & v(S) + \epsilon & \text{otherwise} \end{array}$$

Continuing in this way, we obtain a sequence

$$\epsilon_1 < \epsilon_2 < \ldots < \epsilon_k$$

until, finally, the optimal solution of $(LP_k)$ is unique with an allocation $x^*$, the *nucleolus* of the game.

A more concise (and less algorithmic) description can be given as follows.

With the allocation $x$ we associate the *excess vector* $e(x) \in \Re^{2^n-2}$ as the vector of excesses arranged in non-decreasing order. The nucleolus is then the unique vector $x^*$ that lexicographically maximizes the excess vectors $e(x)$ relative to the game $(N, v)$.

General algorithms for the computation of the nucleolus have been investigated by several researchers (see, *e.g.*, Potters *et al.* [1994]). Relative to special classes of games, these algorithms do generally not guarantee a polynomially bounded running time. On the other hand, Solymosi and Raghavan [1994] could show that the nucleolus of a matching game can be computed in polynomial time in the *bipartite* case, *i.e.*, in the case where the edges of positive weight in the underlying graph do not contain a circuit of odd length. The complexity status of the computational problem for general matching games is open.

We suggest another approach to the allocation problem for general matching games. In Section 2, we introduce the *nucleon* as the multiplicative analogue of the nucleolus for cooperative games in a straightforward way. From a purely mathematical point of view, the nucleon is a meaningful concept for general cooperative n-person games. From a conceptual point of view, however, there might be difficulties in accepting the multiplicative analogue of the excess of a coalition with *negative* value as an appropriate measure of its "satisfaction". Therefore, we will restrict ourselves to games with non-negative characteristic functions.

In the last section, we focus on general matching games and, as an application of our new allocation concept, demonstrate that the nucleon of general matching games can be found in polynomial time.

## 2 The Nucleon of a Game

Let $(N, v)$ be a cooperative n-person game. We will throughout assume that $v(\emptyset) = 0$ holds, *i.e.*, that $(N, v)$ is *normalized.* We will, furthermore, restrict our attention to non-negative games and thus assume that $v(S) \geq 0$ holds for any coalition $S \subseteq N$.

To simplify the presentation, recall the (standard) notation relative to the vector $x \in \Re^N$ and the coalition $S \subseteq N$

$$x(S) := \sum_{i \in S} x_i \; .$$

Let $\mathcal{S}_0 := \{\emptyset, N\}$ and $\alpha \geq 0$. Consider the polyhedron $P_1(\alpha)$ of all vectors $x$ that satisfy the following linear restrictions

$$\begin{array}{llrcll} P_1(\alpha) & :: & x(N) & = & v(N) & \\ & & x(S) & \geq & \alpha v(S) & \quad (S \notin \mathcal{S}_0) \end{array}$$

Letting $\alpha_0 := 0$, we conclude from the non-negativity of $v$

$$P_1(\alpha_0) = P_1(0) \neq \emptyset \; .$$

Moreover, $P_1(1)$ is precisely the (usual) *core* of the game $(N, v)$.

Let

$$\alpha_1 := \max\{\alpha \in \Re \mid P_1(\alpha) \neq \emptyset\}.$$

If $\alpha_1 = \infty$, we have $v(S) = 0$ for all $S \notin \mathcal{S}_0$. The *nucleon* $P^* = P^*(N, v)$ of the game $(N, v)$ is then defined to be the polyhedron

$$P^* := P_1(\alpha_0) \;=\; \{x \in \Re^N \mid x(N) = v(N), x \geq 0\}\ .$$

Otherwise, *i.e.*, if $\alpha_1 < \infty$, let $\mathcal{S}_1$ denote the set of coalitions $S \subset N$ that correspond to "forced equalities" at level $\alpha = \alpha_1$, *i.e.*,

$$\mathcal{S}_1 := \{S \notin \mathcal{S}_0 \mid x(S) = \alpha_1 v(S) \text{ for all } x \in P_1(\alpha_1)\}\ .$$

Assume now that $P_j(\alpha), \alpha_j$, and $\mathcal{S}_j$ have been defined for $j = 1, \ldots, i$. Let then the polyhedron $P_{i+1}(\alpha)$ be defined by the linear constraints

$$\begin{array}{llcll}
P_{i+1}(\alpha) & :: & x(N) & = & v(N) & \\
 & & x(S) & = & \alpha_1 v(S) & (S \in \mathcal{S}_1) \\
 & & & \vdots & & \\
 & & x(S) & = & \alpha_i v(S) & (S \in \mathcal{S}_i) \\
 & & x(S) & \geq & \alpha v(S) & (S \notin \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_i)
\end{array}$$

and set

$$\alpha_{i+1} := \max\{\alpha \in \Re \mid P_{i+1}(\alpha) \neq \emptyset\}\ .$$

If $\alpha_{i+1} = \infty$, then the *nucleon* of $(N, v)$ is defined to be

$$P^* \;:=\; P_i(\alpha_i) \;=\; P_{i+1}(\alpha_i)\ .$$

Otherwise, *i.e.*, if $\alpha_{i+1} < \infty$, set

$$\mathcal{S}_{i+1} := \{S \notin \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_i \mid x(S) = \alpha_{i+1} v(S) \text{ for all } x \in P_{i+1}(\alpha_{i+1})\}$$

and continue.

Apparently, this inductive procedure will stop after a finite number of steps with $\alpha_{k+1} = \infty$ as soon as

$$v(S) = 0 \ \text{ for all } S \notin \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_k.$$

Summarizing, the nucleon is obtained by successively computing

$$
\begin{array}{ccccccccccc}
0=\alpha_0 & < & \alpha_1 & < & \alpha_2 & < & \ldots & < & \alpha_k & < & \alpha_{k+1}=\infty \\
P_0(0) & \supseteq & P_1(\alpha_1) & \supseteq & P_2(\alpha_2) & \supseteq & \ldots & \supseteq & P_k(\alpha_k)=P^* & & .
\end{array}
$$

**Example**. Let $N = \{1,2\}$, $v(N) = 1$, and $v(S) = 0$ otherwise. (This is the simplest case of a matching game on the complete graph $K_2$ with unit edge weight). Then

$$P^* = \{(x_1, x_2) \in \Re^2 \mid x_1 + x_2 = 1,\ x_1, x_2 \geq 0\} .$$

The Example shows that the nucleon does not necessarily consist of a single vector $x^* \in \Re^N_+$. However, if $\{i\} \subseteq \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_k$ holds for all $i \in N$, then the nucleon $P^*$ is a singleton. Indeed, in the latter case, the value $x_i = x(\{i\})$ is then prescribed at a fixed value for every $x \in P^*$. In particular, $P^*$ will have cardinality one if $v(\{i\}) > 0$ for every $i \in N$.

As is the case for the (additive) nucleolus, there is an alternative definition of the nucleon in terms of "multiplicative excess vectors".

Given a game $(N, v)$ and a vector $x \in \Re^N_+$ with $x(N) = v(N)$, define for every coalition $S \notin \mathcal{S}_0$ the *multiplicative excess* via

$$\alpha(x, S) := \begin{cases} x(S)/v(S) & \text{if } v(S) > 0 \\ \infty & \text{if } v(S) = 0. \end{cases}$$

The *excess vector* $\alpha(x)$ is obtained by ordering the $2^n - 2$ excess values $\alpha(x, S)$ in a non-decreasing sequence.

**Proposition 2.1** *The nucleon of the non-negative game $(N, v)$ is the set of all allocation vectors $x \in \Re^N_+$ that lexicographically maximize the excess vector $\alpha(x)$.*

We omit the straightforward proof of the proposition.

Note that our original "algorithmic" definition of the nucleon $P^*$ does not provide an efficient way of computing a vector in $P^*$. Indeed, the sheer computation of $\alpha_1$ in the way suggested by the definition means to solve a linear program with an exponential (in $n$) number of constraints. The question, therefore, arises whether $P^*$ can be efficiently determined at all for interesting classes of games. We give a positive answer to this question for the special class of matching games in Section 4.

## 3 Computational Aspects

Recall that the nucleon $P^* = P_k(\alpha_k)$ consists of all vectors $x$ that satisfy the linear restrictions

$$
\begin{array}{lrcll}
P_k(\alpha_k) & :: \; x(N) & = & v(N) & \\
& x(S) & = & \alpha_1 v(S) & (S \in \mathcal{S}_1) \\
& & \vdots & & \\
& x(S) & = & \alpha_k v(S) & (S \in \mathcal{S}_k) \\
& x & \geq & 0 &
\end{array}
$$

The number $k$ in the preceding definition of the nucleon $P^*$ may, in general, be exponential in $n$. Intuitively, this can happen when all "new" equations

$$x(S) \;=\; \alpha_i v(S) \quad (S \in \mathcal{S}_i)$$

are already implied by the previous equations for $S \in \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_{i-1}$. Then $\alpha_i > \alpha_{i-1}$ while $\dim P_i(\alpha_i) = \dim P_{i-1}(\alpha_{i-1})$. We want to derive an iterative computational procedure for $P^*$ that avoids steps that are redundant in that sense. We will show that $P^*$ can be found in at most $n$ iterations.

For any $\mathcal{S} \subseteq 2^N$, denote by $< \mathcal{S} >$ the *span* of $\mathcal{S}$, *i.e.*,

$$< \mathcal{S} > := \; \{T \subseteq N \mid I|_T \in lin(I|_S \mid S \in \mathcal{S})\},$$

where $I|_S$ denotes the incidence vector of $S \subseteq N$ and $lin(.)$ denotes the linear hull operator.

With this terminology, we may describe $P^*$ equivalently via

$$
\begin{array}{lrcll}
P^* & :: \; x(N) & = & v(N) & \\
& x(S) & = & \alpha_1 v(S) & (S \in \mathcal{S}_1 \setminus < \mathcal{S}_0 >) \\
& & \vdots & & \\
& x(S) & = & \alpha_k v(S) & (S \in \mathcal{S}_k \setminus < \mathcal{S}_0 \cup \ldots \cup \mathcal{S}_{k-1} >) \\
& x & \geq & 0 &
\end{array}
$$

This representation of $P^*$ suggests the following iterative computational procedure:

Let $\mathcal{T}_0 := \{\emptyset, N\}$ and define for $\beta \geq 0$ the polyhedron $Q_1(\beta)$ via

$$
\begin{array}{lrcll}
Q_1(\beta) & :: \; x(N) & = & v(N) & \\
& x(T) & \geq & \beta v(T) & (T \notin \mathcal{T}_0)
\end{array}
$$

and set

$$\beta_1 := \max\{\beta \in \Re \mid Q_1(\beta) \neq \emptyset\} .$$

If $\beta_1 = \infty$, then

$$P^* = \{x \in \Re^N_+ \mid x(N) = v(N)\} .$$

If $\beta_1 < \infty$, let $\mathcal{T}_1$ denote the set of coalitions that correspond to forced equalities at level $\beta = \beta_1$ (thus $\mathcal{T}_1 = \mathcal{S}_1$).

Assume now, inductively, that $Q_j(\beta), \beta_j$, and $\mathcal{T}_j$ have been defined for $j = 1, \ldots, i$. Let then the polyhedron $Q_{i+1}(\beta)$ be presented by the constraints

$$\begin{array}{lcrcll} Q_{i+1}(\beta) & :: & x(N) & = & v(N) & \\ & & x(T) & = & \beta_1 v(T) & (T \in \mathcal{T}_1) \\ & & & \vdots & & \\ & & x(T) & = & \beta_i v(T) & (T \in \mathcal{T}_i) \\ & & x(T) & \geq & \beta v(T) & (T \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_i >) \end{array}$$

and set

$$\beta_{i+1} := \max\{\beta \in \Re \mid Q_{i+1}(\beta) \neq \emptyset\} .$$

If $\beta_{i+1} = \infty$, then $P^* = Q_i(\beta_i)$ and we stop. Otherwise, define $\mathcal{T}_{i+1}$ to be the set of coalitions $T$ that become tight at level $\beta = \beta_{i+1}$:

$$\mathcal{T}_{i+1} := \{T \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_i > \mid x(T) = \beta_{i+1} v(T) \text{ for all } x \in Q_{i+1}(\beta_{i+1})\} .$$

From the alternative description of $P^*$ above, it is apparent that the sequence $(Q_i(\beta_i))$ is a subsequence of $(P_i(\alpha_i))$ and that $(\beta_i)$ is a subsequence of $(\alpha_i)$.

Note, moreover, that in each iterative step in the computation of the sequence $(Q_i(\beta_i))$ equality constraints are added that are independent from the previous equality constraints. Hence we conclude for the dimension

$$\dim Q_{i+1}(\beta_{i+1}) < \dim Q_i(\beta_i) ,$$

which implies that $P^*$ is determined after at most $n$ iterations.

Because we are interested in efficient algorithms for the compution of the nucleon $P^*$, we also want to demonstrate that the parameters $\beta_i$ do not grow “too big” in the course of the iterative procedure.

Recall that the *size* $\ll r \gg$ of a rational number $r$ is defined to be the number of bits in a binary representation of $r$. Then we observe

**Proposition 3.1** *Let $\beta_1 < \ldots < \beta_l$ and $\mathcal{T}_1, \ldots, \mathcal{T}_l$ be given as above ($l \leq n$), and let $\mathcal{T} := \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_l$. Then the size $\ll \beta_i \gg$ of each $\beta_i$ is bounded by a polynomial in $n, |\mathcal{T}|$, and $\max_{T \in \mathcal{T}} \ll v(T) \gg$.*

**Proof:** It follows directly from the definition that $(\beta_1, \ldots, \beta_l)$ is the unique lexicographically maximal vector $(b_1, \ldots, b_l)$ such that the linear system

$$\begin{array}{rcll} x(N) & = & v(N) & \\ x(T) & = & b_1 v(T) & (T \in \mathcal{T}_1) \\ & \vdots & & \\ x(T) & = & b_l v(T) & (T \in \mathcal{T}_l) \\ x & \geq & 0 & \end{array}$$

has a solution $x \in \Re^N$. Hence we can obtain $(\beta_1, \ldots, \beta_l)$ from the unique lexicographically maximal solution $(b_1^*, \ldots, b_l^*, x_1^*, \ldots, x_n^*)$ of the above system.

The latter, however, represents a vertex of the feasibility region. Standard results from linear programming, therefore, imply that each component is polynomially bounded in the size of the system (see, *e.g.*, Grötschel, Lovász and Schrijver [1988]).

The size of the linear system is bounded by

$$\mathcal{O}((n + \max_{T \in \mathcal{T}} \ll v(T) \gg) \cdot |\, \mathcal{T} \,|) \; ,$$

which proves the Proposition.

□

# 4 The Nucleon of a Matching Game

A *matching game* is defined on the graph $\mathcal{G} = (N, E)$ with an edge weighting $w : E \to \Re$. The characteristic function $v$ is given for each coalition $S \subseteq N$ via

$$v(S) \; = \; \text{value of a maximal weighted matching in } \mathcal{G}|_S \; ,$$

where $\mathcal{G}|_S$ is the subgraph of $\mathcal{G}$ induced by $S$.

Since a matching of maximal weight will never contain a negative edge we may assume w.l.o.g. that the weighting $w$ is non-negative. Adding edges with weight zero, if necessary, we can similarly assume that $\mathcal{G}$ is the complete graph $K_n$.

Recall from Section 3 the inductively defined polyhedra $Q_i(\beta)$:

$$\begin{array}{llcll} Q_i(\beta) & :: \; x(N) & = & v(N) & \\ & \quad x(T) & = & \beta_1 v(T) & (T \in \mathcal{T}_1) \\ & & \vdots & & \\ & \quad x(T) & = & \beta_{i-1} v(T) & (T \in \mathcal{T}_{i-1}) \\ & \quad x(T) & \geq & \beta v(T) & (T \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1} >) \end{array}$$

Our aim is to show that the defining equations and inequalities for $Q_i(\beta)$ can be replaced by a polynomial number of equations and inequalities if we want to compute the nucleon of a matching game. Essentially, it will turn out that we may restrict our attention to the value an allocation $x$ takes on one- and two-element coalitions.

Let $Q \subseteq \Re^N$ be a set of vectors. We say that $Q$ *fixes* the set $S \subseteq N$ if $x(S) = y(S)$ holds for all $x, y \in Q$.

With the terminology of Section 3, we define for $i = 1, \ldots, l$,

$$\mathcal{F}_i \; := \; \{S \subseteq N \mid S \text{ is fixed by } Q_i(\beta_i)\} \; .$$

**Lemma 4.1** *For $i = 1, \ldots, l$,* $\mathcal{F}_i \; = < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_i >$ .

**Proof:** By definition, we have

$$\begin{array}{llcll} Q_i(\beta_i) & :: \; x(N) & = & v(N) & \\ & \quad x(T) & = & \beta_1 v(T) & (T \in \mathcal{T}_1) \\ & & \vdots & & \\ & \quad x(T) & = & \beta_i v(T) & (T \in \mathcal{T}_i) \\ & \quad x(T) & \geq & \beta_i v(T) & (T \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1} >) \end{array}$$

Moreover, each of the inequalities for $x(T)$, $T \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1} >$, can be made strict. Thus the relative interior $Q_i^o(\beta_i)$ is described by the constraints

$$\begin{array}{llcll} Q_i^o(\beta_i) & :: \; x(N) & = & v(N) & \\ & \quad x(T) & = & \beta_1 v(T) & (T \in \mathcal{T}_1) \\ & & \vdots & & \\ & \quad x(T) & = & \beta_i v(T) & (T \in \mathcal{T}_i) \\ & \quad x(T) & > & \beta_i v(T) & (T \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1} >) \end{array}$$

It is now clear that the set of coalitions fixed by the relative interior $Q_i^o(\beta_i)$ is precisely $< \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_i >$ . Hence $\mathcal{F}_i \subseteq < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_i >$. The converse containment is straightforward.

□

While Lemma 4.1 is valid for arbitrary games, we will from now on assume that $v$ arises from the matching game on $\mathcal{G}$ relative to the edge weighting $w$. We denote by $\mathcal{N}_i$ resp. $\mathcal{E}_i$ the one- resp. two-element coalitions in $\mathcal{F}_i$. We will usually think of $\mathcal{N}_i$ as a subset of $N$ and of $\mathcal{E}_i$ as a subset of $E$.

**Proposition 4.1** *For $i = 1, \ldots, l$, $\mathcal{T}_i \subseteq < \mathcal{T}_0 \cup \mathcal{E}_i \cup \mathcal{N}_i >$.*

**Proof:** Suppose that there exists some $S \in < \mathcal{T}_0 \cup \mathcal{E}_i \cup \mathcal{N}_i >$ so that $S \notin \mathcal{T}_i$. Choose such an $S$ with $|S|$ minimal. Let $M$ be a matching of maximal weight in $\mathcal{G}|_S$, *i.e.*, $v(S) = w(M)$. Because $\mathcal{G}$ is a complete graph and $w$ is non-negative, we can also assume that $M$ is a maximum cardinality matching, *i.e.*, $S = N(M)$ if $|S|$ is even and $S = \{t\} \cup N(M)$ for some $t \in N$ if $|S|$ is odd. (For any set $A$ of edges, we denote by $N(A)$ the nodes of $\mathcal{G}$ covered by $A$). We will derive a contradiction to the existence of such a coalition $S$.

*Case 1:* $M \subseteq \mathcal{E}_i$.

If $|S|$ is even, then $S \in < \mathcal{E}_i > \subseteq < \mathcal{T}_0 \cup \mathcal{E}_i \cup \mathcal{N}_i >$, which contradicts the choice of $S$.

If $|S|$ is odd, then $S \in \mathcal{F}_i$ and $M \subseteq \mathcal{F}_i$ imply $\{t\} = S \setminus N(M) \in \mathcal{F}_i$. So $t \in \mathcal{N}_i$ and, therefore, $S = t \cup N(M) \in < \mathcal{T}_0 \cup \mathcal{E}_i \cup \mathcal{N}_i >$, again contradicting the choice of $S$.

*Case 2:* There exists some $e \in M \setminus \mathcal{E}_i$.

Consider $S' := S \setminus e$. If $S' \in \mathcal{F}_i$, then $S \in \mathcal{T}_i \subseteq \mathcal{F}_i$ implies that also $e \in \mathcal{F}_i$ must hold, contrary to our assumption on $e$. So $S' \notin \mathcal{F}_i$ and, in particular, $S' \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1} >$.

By the definition of $Q_i(\beta_i)$, we know for all $x \in Q_i(\beta_i)$,

$$x(S') \geq \beta_i v(S') \ .$$

On the other hand, we have $e \notin < \mathcal{T}_0 \cup \ldots \cup \mathcal{T}_{i-1} >$ and, therefore, for all $x \in Q_i(\beta_i)$,

$$x(e) \geq \beta_i w(e) \ .$$

Since $S \in \mathcal{T}_i$, we furthermore know for all $x \in Q_i(\beta_i)$, $x(S) = \beta_i v(S)$ .

Summarizing, we conclude for all $x \in Q_i(\beta_i)$, $x(e) = \beta_i w(e)$ ,*i.e.*, $e \in \mathcal{F}_i$, contrary to the choice of $e$.

□

Proposition 4.1 calls our attention to the sequences

$$\begin{array}{ccccccc} \emptyset = \mathcal{E}_0 & \subseteq & \mathcal{E}_1 & \subseteq & \dots & \subseteq & \mathcal{E}_l \\ \emptyset = \mathcal{N}_0 & \subseteq & \mathcal{N}_1 & \subseteq & \dots & \subseteq & \mathcal{N}_l \end{array}$$

In each iterative step ($i \to i+1$) some edges $e \in \mathcal{E}_{i+1} \setminus \mathcal{E}_i$ become fixed by $Q_{i+1}(\beta_{i+1})$ to some non-negative value $c(e)$, say, until eventually all edges with non-zero weight are fixed.

Similarly, some nodes $t \in \mathcal{N}_{i+1} \setminus \mathcal{N}_i$ are fixed at some value $c(t) \geq 0$. The nucleon $P^*$ is determined by

$$\begin{array}{lrcll} P^* & :: \; x(N) & = & v(N) & \\ & x(e) & = & c(e) & (e \in \mathcal{E}_l) \\ & x(t) & = & c(t) & (t \in \mathcal{N}_l) \\ & x & \geq & 0 & \end{array}$$

Furthermore, as a consequence of Proposition 4.1, we can describe $Q_i(\beta)$ via

$$\begin{array}{lrcll} Q_i(\beta) & :: \; x(N) & = & v(N) & \\ & x(e) & = & c(e) & (e \in \mathcal{E}_{i-1}) \\ & x(t) & = & c(t) & (t \in \mathcal{N}_{i-1}) \\ & x(T) & \geq & \beta v(T) & (T \notin < \mathcal{T}_0 \cup \mathcal{E}_{i-1} \cup \mathcal{N}_{i-1} >) \end{array}$$

Our next goal is to replace the exponentially many inequalities in the preceding description of $Q_i(\beta)$ by polynomially many inequalities.

For $i \geq 1$ and $\beta \geq 0$, let $\mathcal{G}_{i-1} = (N, \mathcal{E}_{i-1})$ be the subgraph containing only those edges that are fixed after the iterative step $i-1$. For $e \in E$, let $\mathcal{G}_{i-1} \setminus e$ denote the graph obtained from $\mathcal{G}_{i-1}$ by removing the two endpoints of $e$ and all incident edges. Similarly, for $t \in N$, let $\mathcal{G}_{i-1} \setminus t$ be the subgraph obtained from $\mathcal{G}_{i-1}$ by removing $t$ and all incident edges.

Relative to the original weighting $w : E \to \Re$ and the weighting $c : \mathcal{E}_l \to \Re_+$, we define a new weighting $w_\beta : \mathcal{E}_{i-1} \to \Re$ on $\mathcal{G}_i$ by

$$w_\beta(f) \;:=\; \beta w(f) - c(f) \; .$$

For $e \in E$, let $M_\beta^e$ denote some fixed (possibly empty) matching in $\mathcal{G}_{i-1} \setminus e$ of maximal weight with respect to the weighting $w_\beta$. Let $S_\beta^e := \{e\} \cup N(M_\beta^e)$ be the associated coalition.

Similarly for $t \in N$, denote by $M_\beta^t$ some fixed matching of maximal weight with respect to $w_\beta$ in the graph $\mathcal{G}_{i-1} \setminus t$ and let $S_\beta^t := \{t\} \cup N(M_\beta^t)$ be the associated coalition.

Define the polyhedron $Q_i^*(\beta)$ by

$$
\begin{array}{lllll}
Q^*(\beta) & :: & x(N) & = & v(N) & \\
& & x(e) & = & c(e) & (e \in \mathcal{E}_{i-1}) \\
& & x(t) & = & c(t) & (t \in \mathcal{N}_{i-1}) \\
& & x(S^e_\beta)) & \geq & \beta v(S^e_\beta) & (e \notin \mathcal{E}_{i-1}) \\
& & x(S^t_\beta)) & \geq & \beta v(S^t_\beta) & (t \notin \mathcal{N}_{i-1})
\end{array}
$$

**Proposition 4.2** $Q_i(\beta) \; = \; Q^*_i(\beta)$ *for all* $\beta \geq 0$.

**Proof:** By the choice of $M^e_\beta$, we have $N(M^e_\beta) \in < \mathcal{E}_{i-1} >$. So $e \notin \mathcal{E}_{i-1}$, *i.e*, $e \notin \mathcal{F}_{i-1}$, implies $S^e_\beta \notin \mathcal{F}_{i-1}$. In particular, $e \notin \mathcal{E}_{i-1}$ yields $S^e_\beta \notin < \mathcal{T}_0 \cup \mathcal{E}_{i-1} \cup \mathcal{N}_{i-1} >$. Therefore, all the inequalities occurring in the definition of $Q^*_i(\beta)$ also occur in the description of $Q_i(\beta)$. A similar argument holds for $t \notin \mathcal{N}_{i-1}$. Thus

$$Q_i(\beta) \; \subseteq \; Q^*_i(\beta) \quad \text{for all} \quad \beta \geq 0.$$

Conversely, let $x \in Q^*_i(\beta)$ be arbitrary and let $S \notin < \mathcal{T}_0 \cup \mathcal{E}_{i-1} \cup \mathcal{N}_{i-1} >$ . We show that $x(S) \geq \beta v(S)$ holds, which implies $x \in Q_i(\beta)$.

Let $M$ be a matching of maximum weight relative to $w$ in $\mathcal{G}|_S$. So $v(S) = w(M)$. Assume again that $M$ is of maximal cardinality, *i.e.*, $S = N(M)$ or $S = t \cup N(M)$, depending on whether $|S|$ is even or odd.

*Case 1:* $M \; \subseteq \; \mathcal{E}_{i-1}$.

Then $S = t \cup N(M)$ for some $t \notin \mathcal{N}_{i-1}$ (otherwise, we would have $S \in < \mathcal{E}_{i-1} \cup \mathcal{N}_{i-1} >$, a contradiciton to our assumption on $S$). Since $x \in Q^*_i(\beta)$, we know that

$$x(S^t_\beta) \; \geq \; \beta v(S^t_\beta) \; .$$

Hence

$$
\begin{array}{lllll}
x(t) & \geq & \beta v(S^t_\beta) & - & x(M^t_\beta) \\
& \geq & \beta w(M^t_\beta) & - & x(M^t_\beta) \\
& = & w_\beta(M^t_\beta) & & \\
& \geq & w_\beta(M) & & \\
& = & \beta w(M) & - & x(M) \\
& = & \beta v(S) & - & x(M)
\end{array}
$$

Thus

$$x(S) = x(t) + x(M) \geq \beta v(S) \; .$$

*Case 2:* There exists some $e \in M \setminus \mathcal{E}_{i-1}$.

Observe first that $x \geq 0$ holds for all $x \in Q^*_i(\beta)$. Thus it suffices to show that $x(M) \geq \beta w(M)$ holds.

Let $U := M \cap \mathcal{E}_{i-1}$ and $V := M \setminus U$. For each $e \in V$, we have $x(S^e_\beta) \geq \beta v(S^e_\beta)$. So

$$\begin{array}{rcllll} x(e) & \geq & \beta v(S^e_\beta) & - & x(M^e_\beta) & \\ & \geq & \beta w(e) & + & \beta w(M^e_\beta) & - \ x(M^e_\beta) \\ & = & \beta w(e) & + & w_\beta(M^e_\beta) & \end{array}$$

Summing up the above inequalities for all $e \in V$ and using $w_\beta(M^e_\beta) \geq 0$ and $w_\beta(M^e_\beta) \geq w_\beta(U)$, we see

$$x(V) \geq \beta w(V) + w_\beta(U) = \beta w(V) + \beta w(U) - x(U)\,.$$

Thus

$$x(M) \geq \beta(w(U) + w(V)) = \beta v(M)\ .$$

□

We have achieved our goal. Given $\beta \geq 0$, we are able to represent $Q_i(\beta) = Q^*_i(\beta)$ with polynomially (in $n$) many equations and inequalities. Note that computing the coalitions $S^t_\beta$ and $S^e_\beta$ amounts to solving maximum weight matching problems with respect to the weighting $w_\beta$, which can be done in polynomial time (see, *e.g.*, Lovász and Plummer [1986]), provided the weights $w_\beta$ have polynomial size.

One difficulty, however, remains. The coalitions of type $S^e_\beta$ and $S^t_\beta$ in the description of $Q^*_i(\beta)$ very much depend on the value of $\beta$. The idea, therefore, is to compute

$$\beta_i \;=\; \max\{\beta \mid Q^*_i(\beta) \neq \emptyset\}$$

by binary search.

**Lemma 4.2** *For $i = 1, \ldots, l$,*

$$1/n \leq \beta_i \leq v(N)/w_{min} =: M\ ,$$

*where $w_{min}$ is the smallest non-zero weight $w(e)$, $e \in E$.*

**Proof:** By definition,

$$\begin{array}{lclcll} Q^*_1(\beta) & :: & x(N) & = & v(N) & \\ & & x(e) & \geq & \beta w(e) & \quad (e \in E) \end{array}$$

The vector $x = (v(N)/n, \ldots, v(N)/n)$ shows that $Q^*_1(1/n)$ is non-empty.

Hence $1/n \leq \beta_1 < \ldots < \beta_l$. (In fact, one can show that $2/3 \leq \beta_1$ holds (see Faigle and Kern [1993])).

On the other hand, each $\mathcal{T}_i$ contains at least some coalition $T_i$, say, with $v(T_i) > 0$ (otherwise $\beta_i = \infty$).

But then $x(T_i) = \beta_i v(T_i) \geq \beta_i w_{min}$ implies

$$\beta_i \leq x(T_i)/w_{min} \leq x(N)/w_{min} = M \ .$$

$\square$

We are now in the position to state our main result.

**Theorem 4.1** *The nucleon $P^*$ of a matching game on the graph $\mathcal{G} = (N, E)$ with edge weighting $w$ can be computed in time polynomial in $n = |N|$ and the size $\ll w \gg$ of $w$.*

**Proof:** Let $s := \max\{\ll w(e) \gg \ | \ e \in E\}$. We know from Proposition 3.1 that each $\ll \beta_i \gg$ is polynomially bounded, say $\ll \beta_i \gg \leq \ p(n, s)$, for some suitable polynomial $p$.

It remains to deal with the size of $c : (\mathcal{E}_l \cup \mathcal{N}_l) \to \Re$.

If $e \in \mathcal{E}_i \setminus \mathcal{E}_{i-1}$, then $x(e) = c(e)$ is determined by the equation

$$x(S^e_{\beta_i}) = \beta_i v(S^e_{\beta_i}) \ .$$

Therefore, the nucleon $P^*$ may alternatively be described via

$$\begin{array}{llcll} P^* & :: \quad x(N) & = & v(N) & \\ & \quad x(S^e_{\beta_1}) & = & \beta_1 v(S^e_{\beta_1}) & (e \in \mathcal{E}_1) \\ & \quad x(S^t_{\beta_1}) & = & \beta_1 v(S^t_{\beta_1}) & (e \in \mathcal{N}_1) \\ & & \vdots & & \\ & \quad x(S^e_{\beta_l}) & = & \beta_l v(S^e_{\beta_l}) & (e \in \mathcal{E}_l) \\ & \quad x(S^t_{\beta_l}) & = & \beta_l v(S^t_{\beta_l}) & (e \in \mathcal{N}_l) \\ & \quad x & \geq & 0 & \end{array}$$

This system has size polynomial in $n$ and $s$. Consequently, any basic solution $x$ of that system has size polynomial in $n$ and $s$. But each basic solution $x$ fixes $c(e) = x(e)$ and $c(t) = x(t)$ for every $e \in \mathcal{E}_l$ and $t \in \mathcal{N}_l$. Therefore, the size $\ll c \gg$ of $c$ is polynomial.

The latter fact ensures that we can successively compute $\beta_1, \ldots, \beta_l$ in time polynomial in $n$ and $s$ by applying binary search to determine $\beta_i$ in each iterative step.

$\square$

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