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Semidefinite Programming and its Applications to NP Problems

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1 Abstract

The graph homomorphism problem is a canonical NP-complete problem. It generalizes various other well-studied problems such as graph coloring and finding cliques. To get a better insight into a combinatorial problem, one often studies relaxations of the problem. We define *fractional homomorphisms* and *pseudo-homomorphisms* as natural relaxations of graph homomorphisms. In their paper [3], Feige and Lovász defined a semidefinite relaxation of the homomorphism problem, which allowed them to obtain polynomial time algorithms for certain special cases of the problem. Their relaxation is defined in terms of the solution to a semidefinite program. Hence a characterization of their relaxation in terms of known combinatorial notions is desirable. We show that our pseudo-homomorphism is equivalent to the relaxation defined by Feige and Lovász [3]. Our definition of pseudo-homomorphism involves the classical ϑ number first defined by Lovász [13]. Although general graph homomorphism does not admit a simple forbidden subgraph characterization, surprisingly we can show that there is a simple forbidden subgraph characterization (the forbidden subgraph is a clique in this case) of the fractional homomorphism. As a byproduct, we obtain a simpler proof of the NP hardness of the fractional chromatic number, first proved by Grötschel, Lovász and Schrijver using the ellipsoid method [5] Finally, we briefly discuss how to apply these techniques to general NP problems and describe a unified setting in which a wide variety of seemingly disparate polynomial time problems can be decided.

2 Introduction

A *homomorphism* between two graphs G and H is a function from the vertex set of G to the vertex set of H, which maps adjacent vertices to adjacent vertices. We say that G is *homomorphic* to H if such a homomorphism exists, and in this case we will write $G \to H$. Determining whether $G \to H$ for input graphs G and H is known to be NP-complete (cf. [9]). This problem is a generalization of a large number of well-studied problems. For example, determining if a graph G is k-colorable is equivalent to setting H to be a complete graph on k vertices and determining if $G \to H$. Determining if a graph H has a k-clique is equivalent to setting G to a complete graph on k vertices and determining if $G \to H$. Therefore it is desirable to identify families of instances for which a polynomial algorithm exists. Recent research in semidefinite optimization shows that it is a powerful tool towards obtaining approximation algorithms for NP hard problems (see [4, 11]). We use semidefinite programming to study problems in NP such as the graph homomorphism problem. We define the notions *pseudo-homomorphism* and *fractional homomorphism* which are, respectively, semidefinite and linear relaxations of a certain integer program corresponding to the graph homomorphism problem. In fact, the pseudo-homomorphism is defined in terms of the classical ϑ number of [13]. As semidefinite programs can be solved in polynomial time, pseudo-homomorphism is a polynomial time notion. It turns out, as we will show in a later section, that our notion of pseudo-homomorphism coincides with another semidefinite relaxation defined by Feige and Lovász in [3]. Feige and Lovász used this relaxation to show polynomial time solvability of various special cases of graph homomorphism.

Although the general graph homomorphism does not admit of a simple forbidden subgraph characterization, we show a surprisingly simple such characterization for the fractional homomorphism which also yields several other interesting consequences. First, we obtain a much simpler proof of the NP-hardness of fractional clique number, a result which was first proven by [5] using the ellipsoid algorithm. A longstanding conjecture in graph theory is the graph product conjecture [6]. Using our characterization of the fractional homomorphism, we show that the above conjecture is true if we replace homomorphism by fractional homomorphism.

The notions of pseudo-homomorphism and fractional homomorphism can be generalized to all sets in NP. Observing that the natural generalization of pseudo-homomorphism to other NP sets can be computed in polynomial time, we obtain polynomial time algorithms for certain NP problems when the instances are drawn from certain families.

2.1 Definitions

Let G be a finite undirected graph without loops. The set of the vertices, and edges of the graph G is denoted by $V(G)$, and $E(G)$, respectively. For $(u,v) \in E(G)$, we also use $u \sim v$ and say that they are *adjacent*. We say that two vertices of G are *incident* if they are either adjacent or are equal. The *complementary* graph $\overline{G}$ of the graph G be the graph with $V(\overline{G}) = V(G)$ and $(u,v) \in E(\overline{G})$ iff $(u,v) \notin E(G)$. All vectors will be row vectors.

Definition 1 *Let G and H be two graphs. The* hom-product $G \circ H$ *is defined as the graph with* $V(G \circ H) = V(G) \times V(H)$, *in which* $((s,u),(t,w)) \in E(G \circ H)$ *iff* $(s \neq t)$ *and* $((s,t) \in E(G)$ *implies* $(u,w) \in E(H))$.

We first formulate the clique number, $\omega(G)$, and the chromatic number, $\chi(G)$, of a graph G, as integer linear programs. In these formulations we take v to range over the vertex set of G and I to range over all maximal independent sets in G.

$$\begin{array}{ll} \omega(G) = \max \sum_v x_v & \chi(G) = \min \sum_I y_I \\ \sum_{v \in I} x_v \leq 1 \text{ for each } I & \sum_{\mathcal{I} \ni v} y_{\mathcal{I}} \geq 1 \text{ for each } v \\ x_v \in \{0,1\} & y_I \in \{0,1\} \end{array}$$

If we replace the integrality conditions $x_v \in \{0,1\}$, $y_{\mathcal{I}} \in \{0,1\}$ by $x_v \geq 0$, $y_{\mathcal{I}} \geq 0$ respectively, we obtain linear programs with optima $\omega_f(G)$, the fractional clique number of G, and $\chi_f(G)$, the fractional chromatic number of G. Note that these linear programs are dual of each other and hence $\omega_f(G) = \chi_f(G)$ by the duality theorem of linear programming (see [1]). Therefore we have the inequalities:

$$\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G).$$

To compute any of these numbers is NP hard. The NP hardness of the fractional chromatic number was proved by Grötschel, Lovász and Schrijver in [5]. They also proved that one can compute in polynomial time a real number ϑ (see also [12]) which satisfies

$$\omega(G) \leq \vartheta \leq \omega_f(G).$$

One possible definition for ϑ is as follows [15]:

Let $\mathcal{S}$ be the set of all $|V(G) \times V(G)|$ positive semidefinite matrices, let I denote the unit matrix and J denote the matrix of all 1's. The $*$ product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the number $A * B = \sum_{i,j} a_{ij} b_{ij}$. Then

$$\vartheta = \max\{B * J | B \in \mathcal{S}; B * I = 1; \forall (s,t) \notin E(G) : b_{st} = 0\}.$$

Another semidefinite programming upper bound for ω is the real number $\vartheta_{1/2}$ of Schrijver [14], which can be expressed as:

$$\vartheta_{1/2} = \max\{B * J | B \in \mathcal{S}; B * I = 1; \forall (s,t) \notin E(G) : b_{st} = 0; \forall s,t \in V(G) : b_{st} \geq 0\}.$$

We need a notation that explicitely expresses the dependence of $\vartheta, \vartheta_{1/2}$ on the graph G. For historical reasons [15, 13, 12]we write $\vartheta = \vartheta(\overline{G})$ and $\vartheta_{1/2} = \vartheta_{1/2}(\overline{G})$.

3 Overview

In section 4, we define the notions of fractional homomorphism and pseudo-homomorphism, and establish certain relationships between them. In section 5 we give a forbidden subgraph characterization of the fractional homomorphism. This characterization yields a much simpler proof of the NP-hardness of fractional clique number, which was first proved using the ellipsoid method [5]. Using this characterization, we also show a fractional version of the well known graph product conjecture [6]. In section 6 we show that the notion of pseudo-homomorphism is equivalent to the notion of hoax as defined in [3] and give a necessary condition for existence of a pseudo-homomorphism. In section 7, we discuss the applications of this theory to other problems in NP.

4 Relaxations of Homomorphisms

Theorem 2 *For any graphs G, H, the following inequalities hold:*

$$\omega(G \circ H) \leq \vartheta_{1/2}(\overline{G \circ H}) \leq \vartheta(\overline{G \circ H}) \leq \omega_f(G \circ H) = \chi_f(G \circ H) \leq \chi(G \circ H) \leq |V(G)|,$$

and $\omega(G \circ H) = |V(G)|$ iff there is a homomorphism from G to H.

Proof: The first and the second inequality follows from [15]. The third inequality follows from Lovász's Theorem 10 in [13]. From definition of a fractional clique and from the duality of linear programming, it follows that $\omega_f = \chi_f \leq \chi$. The last inequality follows from the fact that we can color the vertices of $G \circ H$ by the vertices of the graph G, by assigning to a vertex $(s,u) \in V(G \circ H)$ the color $s \in V(G)$. From the definition of the hom-product it follows that no two adjacent vertices obtain the same color. It remains to prove that $G \to H$ iff $\omega(G \circ H) = |V(G)|$. Let us assume that f is a homomorphism from G to H. One can easily observe that vertices $(s, f(s)), s \in V(G)$ form a clique in $G \circ H$ of size $|V(G)|$. On the other hand if we have a clique C in $G \circ H$ of size $|V(G)|$, then the first coordinates have to span all the vertices of G because $(s,u) \not\sim (s,w)$, and hence C can be regarded as a function $V(G) \to V(H)$. We define a homomorphism f from G to H as follows: $f(s) = u$ iff $(s,u) \in C$. To prove that f is a homomorphism, let s, t be vertices in G. Then $(s, f(s))$ and $(t, f(t))$ are from the clique C and hence they are adjacent. From the definition of the hom-product, it follows that $s \sim t$ implies $f(s) \sim f(t)$. We conclude that f is a homomorphism.

□

The previous theorem suggests the following relaxations of homomorphisms:
Let G, H be two graphs.

Definition 3 *We say that G is* fractionally homomorphic *to H (denote by $G \to_f H$) if $\omega_f(G \circ H) = |V(G)|$.*

Definition 4 *We say that G is* pseudo-homomorphic *to H (denote by $G \to_p H$) if* $\vartheta(\overline{G \circ H}) = |V(G)|$.

Definition 5 *We say that G is* pseudo$_{1/2}$-homomorphic *to H (denote by $G \to_{p/2} H$) if* $\vartheta_{1/2}(\overline{G \circ H}) = |V(G)|$.

Corollary 6 $(G \to H) \Rightarrow (G \to_{p/2} H) \Rightarrow (G \to_p H) \Rightarrow (G \to_f H)$.

Later we will show that we cannot reverse the first and the last implications in general. This is not surprising because $G \to_p H$, and $G \to_{p/2} H$ are polynomial while both $G \to H$ and $G \to_f H$ are NP hard. However for certain families of graphs we can prove the reverse implications, and so obtain a polynomial algorithm for these families.

5 Fractional Homomorphisms

In this section we will give a forbidden subgraph characterization of fractional homomorphism, which will enable us to prove a fractional version of the graph product conjecture [6]. This also gives an alternative proof of the NP hardness of computing the fractional clique number of a graph.

Theorem 7 *$G \to_f H$ iff $\omega(G) \le \omega_f(H)$.*

Proof: Assume first that there is a clique C with $\lfloor \omega_f(H)+1 \rfloor$ vertices in G. We will show that $\omega_f(G \circ H) < |V(G)|$ by constructing a dual fractional coloring smaller than $|V(G)|$. Let w be a minimum fractional coloring of H. Its value is $\chi_f(H) = \omega_f(H)$. All independent sets in $G \circ H$ are of the form $I = \bigcup_{i=1}^k s_i \times A_i$ where $s_1, \ldots, s_k$ form a clique in G and for $i \ne j, \forall a_i \in A_i, a_j \in A_j : a_i \not\sim a_j$. This is so, because two vertices $(s,u),(t,w)$ from $G \circ H$ are not connected by an edge whenever $(s = t) \wedge (u \ne w)$, or $(s \sim t) \wedge (u \not\sim w)$.
Now we construct a fractional coloring w^* of $G \circ H$ as follows: $w^*_{C \times T} = w_T$ for any maximal independent set T from H, $w^*_{u \times V(H)} = 1$ for $u \notin C$, and $w^*_I = 0$ otherwise. One can observe that this is a feasible fractional coloring and its objective value is at most $\omega_f(H) + (|V(G)| - \lfloor \omega_f(H)+1 \rfloor) < |V(G)|$.
On the other hand if we assume that $\omega(G) \le \omega_f(H)$, we will show that $\omega_f(G \circ H) = |V(G)|$. Let x be a maximum fractional clique in H, and w a maximum fractional coloring of H dual to x. From complementary slackness [1] it follows that

$$x_v = x_v \sum_{J \ni v} w_J,$$

where J runs through the maximal independent sets in H. We construct a fractional clique x^* in $G \circ H$ as follows: Let $x^*_{(s,u)} = \frac{x_u}{\omega_f(H)}$, where $s \in V(G), u \in V(H)$. Let us denote $x_S = \sum_{s \in S} x_s$ for any subset S of vertices from $V(G)$ and similarly $x^*_T = \sum_{(s,u) \in T} x^*_{(s,u)}$ for any subset T of vertices from $V(G \circ H)$. To prove that x^* is a fractional clique, we have to show that for every independent set $I \subseteq G \circ H$, $x^*_I \le 1$. For an independent set $I = \bigcup_{i=1}^k s_i \times A_i$, let us write $B = \bigcup_{i \ne j} A_i \cap A_j$, where $i,j = 1, \ldots, k$ and let $B_i = A_i - B$. Then $I \subseteq \bigcup_{i=1}^k s_i \times (B_i \cup B)$, and therefore

$$x^*_I \le \sum_{i=1}^k x^*_{s_i \times B_i} + x^*_{\{s_1,\ldots,s_k\} \times B} = \frac{1}{\omega_f(H)} \sum_{i=1}^k x_{B_i} + \frac{k}{\omega_f(H)} x_B.$$

By our assumption $k \le \omega_f(H)$, therefore the following inequality holds:

$$x^*_I \le \frac{1}{\omega_f(H)} \sum_{i=1}^k x_{B_i} + x_B = \frac{1}{\omega_f(H)} \sum_{i=1}^k \sum_{v \in B_i} x_v + \frac{1}{\omega_f(H)} \sum_J w_J x_B =$$

$$= \frac{1}{\omega_f(H)} \sum_{i=1}^{k} \sum_{v \in B_i} x_v \sum_{J \ni v} w_J + \frac{1}{\omega_f(H)} \sum_J w_J x_B =$$

$$= \frac{1}{\omega_f(H)} \sum_{i=1}^{k} \sum_J w_J \sum_{v \in J \cap B_i} x_v + \frac{1}{\omega_f(H)} \sum_J w_J x_B =$$

$$= \frac{1}{\omega_f(H)} \sum_{i=1}^{k} \sum_J w_J x_{J \cap B_i} + \frac{1}{\omega_f(H)} \sum_J w_J x_B =$$

$$= \frac{1}{\omega_f(H)} \sum_J w_J (\sum_{i=1}^{k} x_{J \cap B_i} + x_B),$$

where J runs through the maximal independent sets of H. One can observe that the set $B \cup \bigcup_{i=1}^{k}(J \cap B_i)$ is an independent set in H and therefore $\sum_{i=1}^{k} x_{J \cap B_i} + x_B \leq 1$. Hence

$$x_I^* \leq \frac{1}{\omega_f(H)} \sum_J w_J = 1.$$

It remains to show that the size of this fractional clique is $|V(G)|$.

$$\sum_{(s,u) \in V(G \circ H)} x^*_{(s,u)} = \sum_{(s,u) \in V(G) \times V(H)} \frac{x_u}{\omega_f(H)} = |V(G)| \sum_{u \in V(H)} \frac{x_u}{\omega_f(H)} = |V(G)|.$$

□

This theorem shows that although the graph homomorphism problem is NP-complete, for any fixed non-bipartite graph H [9], the fractional graph homomorphism problem is polynomial, for every fixed graph H. The theorem also shows that if H is part of the input, then fractional homomorphism is *co*-NP-complete. Another interesting consequence of this theorem is that for a given k and a graph G, it is NP-hard to determine if G has fractional chromatic number at least k (or equal to k). This result was originally proved in [5] by the ellipsoid method.

Corollary 8 *The following problem is NP hard:*
Instance:A graph G and a number n
Question:Is $\omega_f(G)$ at least (equal to) n?

Proof: We give a reduction from the clique problem, i.e., given a graph G and an integer n, it is known to be NP complete to decide whether $\omega(G) > n$. But $n = \omega_f(K_n)$ and therefore by our theorem this is equivalent to the question whether $G \not\to_f K_n$ which is equivalent to $\omega_f(G \circ K_n) < |V(G)|$.

□

A longstanding conjecture in graph theory states that whenever $G \not\to K_n$ and $H \not\to K_n$, then also $G \times H \not\to K_n$, where $G \times H$ is the categorical product of G and H, defined by $V(G \times H) = V(G) \times V(H)$ and $((s,u),(t,w)) \in E(G \times H)$ iff $(s,t) \in E(G) \wedge (u,w) \in E(H)$ c.f. [6].

Corollary 9 *$G \not\to_f W$ and $H \not\to_f W$ imply $G \times H \not\to_f W$*

Proof: By our theorem $G \not\to_f W$ and $H \not\to_f W$ imply $\omega(G) > \omega_f(W)$ and $\omega(H) > \omega_f(W)$. One can easily observe that $\omega(G \times H) = \min\{\omega(G), \omega(H)\}$ and therefore also $\omega(G \times H) > \omega_f(W)$ which means $G \times H \not\to_f W$.

□

The fractional version of the product conjecture follows by taking $W = K_n$.

6 Pseudo-homomorphisms

In this section we will relate pseudo-homomorphism to the concept of a hoax introduced by Feige and Lovász [3].First we describe the notion of a hoax, as defined by Feige and Lovász. It is based on the following two-prover interactive proof system:

A verifier is trying to determine if a graph G is homomorphic to a graph H. The verifier chooses randomly and independently vertices s and t from G and sends them, respectively to the provers P_1 and P_2. P_1 replies with a vertex u from H as the claimed image of s, and P_2 with a vertex w from H as the claimed image of t. The verifier accepts just in the following situations:
1) If $s = t$ then $u = w$
2) If s is adjacent to t in G, then u is adjacent to w in H.
For each input (G, H), consider the 0-1 matrix V (with rows and columns labeled by su where the s's run through the vertices of G and the u's runs through the vertices of H) where $V_{su,tw} = 1$ iff the verifier accepts the answer u and w to the requests s and t respectively. Note that V is the incidence matrix of $G \circ H$.

Let p_{su} be the probability that P_1 or P_2 (we can assume that P_1 and P_2 use the same strategies as the game is symmetric) answers u on s. The provers P_1 and P_2 want to maximize the probability that the verifier accepts. This probability is given by $|V(G)|^{-2} \sum_{su,tw} V_{su,tw} p_{su} p_{tw}$ subject to the condition that for all s : $\sum_u p_{su} = 1$, and for all s, u: $p_{su} \geq 0$. The optimum of this quadratic program is 1 iff G is homomorphic to H because only in this case is the probability equal to 1. By rewriting $Q_{su,tw} = p_{su} p_{tw}$, and $C = |V(G)|^{-2} V$, we can convert this quadratic program to the following form:

maximize $\sum_{s,u,t,w} C_{su,tw} Q_{su,tw}$
s.t.
Q is a rank 1 matrix
Q is symmetric
$\forall s, t : \ \sum_{u,w} Q_{su,tw} = 1$
$\forall s, t, u, w : \ Q_{su,tw} \geq 0$.

To make the above program convex, Feige and Lovász replace the rank 1 constraint with the condition 'Q is positive semidefinite'. Let us denote this modified program by $(*)_{G,H}$. The ellipsoid algorithm can be used to solve the new program c.f. [3].

The optimal solution of the modified program with objective value 1 is called a *hoax* (c.f. [3]). If we replace both the rank 1 constraint and the nonnegativity constraint with the condition 'Q is positive semidefinite', we obtain a modified program $(**)_{G,H}$; any optimum solution with objective value 1 of this modified program is called a *semi-hoax*. Feige and Lovász [3] proved the following necessary and sufficient conditions for a given instance to admit a hoax.

Lemma 10 *[3] The system $(*)_{G,H}$ has a hoax iff there exists a system of vectors $v_{su}, s \in V(G), u \in V(H)$ satisfying the following conditions:*

$$\tilde{v} = \sum_u v_{su} \tag{1}$$

is independent of s,

$$\tilde{v} v_{su}^T = |v_{su}|^2 \tag{2}$$

$$|\tilde{v}| = 1 \tag{3}$$

$$\forall s, u, t, w : v_{su} v_{tw}^T \geq 0 \tag{4}$$

and

$$v_{su} v_{tw}^T = 0 \text{ whenever } V_{su,tw} = 0. \tag{5}$$

One can similarly prove the following necessary and sufficient conditions for a semi-hoax.

Lemma 11 *The system* $(**)_{G,H}$ *has a semi-hoax iff there exists a system of vectors* $v_{su}, s \in V(G), u \in V(H)$ *satisfying the conditions (1),(2),(3), and (5).*

We now prove that our notions of pseudo$_{1/2}$-homomorphism and pseudo-homomorphism coincide with the notions of hoax and semi-hoax respectively.

Theorem 12 *The system* $(*)_{G,H}$ *has a hoax iff* $G \rightarrow_{p/2} H$*.*

Proof: Let Q be a hoax of the system $(*)_{G,H}$. By Lemma 10 there exists a system of vectors v_{su} satisfying (1)-(5). One can observe that $Q_{su,tw} = v_{su}v_{tw}^T = 0$ whenever (s,u) is adjacent to (t,w) in $\overline{G \circ H}$. Also

$$\sum_{s\in V(G),u\in V(H)} Q_{su,su} = \sum_{s,u} v_{su}v_{su}^T = \sum_{s,u} \tilde{v}v_{su}^T = \tilde{v}(\sum_s \sum_u v_{su})^T = \tilde{v}(\sum_s \tilde{v})^T = |V(G)|.$$

Moreover

$$\sum_{s,t\in V(G),u,w\in V(H)} Q_{su,tw} = \sum_{s,t,u,w} v_{su}v_{tw}^T = \sum_{s,t} \tilde{v}\tilde{v}^T = |V(G)|^2.$$

Taking $B = \frac{1}{|V(G)|}Q$, we have proved that $\vartheta_{1/2}(\overline{G \circ H}) \geq |V(G)|$, and from Theorem 2 it follows that $G \rightarrow H$.

Let $\vartheta_{1/2}(\overline{G \circ H}) = |V(G)|$. Let $B = AA^T$ be the matrix for which

$$\vartheta_{1/2}(\overline{G \circ H}) = |V(G)| = \sum_s a_s a_t^T.$$

We notice that $B_{su,sw} = 0$ for $u \neq w$ and therefore the a_{su}'s (the rows of matrix A) are orthogonal when s is fixed. If we denote $a_s = \sum_u a_{su}$, then from the condition $B * I = 1$ we have

$$1 = \sum_{s\in V(G),u\in V(H)} B_{su,su} = \sum_{s,u} a_{su}a_{su}^T = \sum_{s,u} |a_{su}|^2 = \sum_s |\sum_u a_{su}|^2 = \sum_s |a_s|^2,$$

and therefore from $B * J = \vartheta_{1/2}(\overline{G \circ H})$, it follows that

$$\vartheta_{1/2}(\overline{G \circ H}) = \sum_{s,t\in V(G),u,w\in V(H)} B_{su,tw} = \sum_{s,t,u,w} a_{su}a_{tw}^T = \sum_{s,t} a_s a_t^T \leq \sum_{s,t} |a_s||a_t| =$$

$$= (\sum_s |a_s|)^2 = (\sum_s 1.|a_s|)^2 \leq \sum_s 1^2 \sum_s |a_s|^2 = |V(G)|.$$

The last two inferences are a consequence of Cauchy's inequality. Because $\vartheta_{1/2}(\overline{G \circ H}) = |V(G)|$, it follows that both inequalities have to be equalities. That is,

$$(\sum_s 1.|a_s|)^2 = \sum_s 1^2 \sum_s |a_s|^2,$$

and this happens iff all a_s's have the same lengths. From $B * I = 1$ it follows that for all $s \in V(G)$

$$|a_s| = |V(G)|^{-\frac{1}{2}}.$$

Also the inequality

$$\sum_{s,t} a_s a_t^T \leq \sum_{s,t} |a_s||a_t|$$

must be an equality, i.e., the angle between a_s and a_t must be zero for each $s, t \in V(G)$. Therefore all a_s's are equal. Taking $Q = |V(G)|B$, we proved the conditions (1) and (3). The conditions (4) and (5) follow from the definition of B. From the orthogonality of the a_{su}'s for a fixed s, we have

$$a_s a_{su}^T = \sum_w a_{sw} a_{su}^T = a_{su} a_{su}^T = |a_{su}|^2,$$

implying (2). Therefore we can deduce from Lemma 10 that Q is a hoax of the system $(*)_{G,H}$.

□

We can similarly prove the following.

Theorem 13 *The system $(**)_{G,H}$ has a semi-hoax iff $G \to_p H$.*

Although we do not have a complete characterization of pseudo-homomorphisms, we can prove a necessary condition for existence of a pseudo-homomorphism which will be enough for our applications. First let us recall some definitions and results from [13].

If G and H are two graphs, then their *strong product* $G \cdot H$ is defined as the graph with $V(G \cdot H) = V(G) \times V(H)$, in which (s, u) is incident with (t, w) iff s is incident to t in G and u is incident to w in H.

The *tensor product* of two vectors $u = (u_1, \ldots, u_n)$ and $w = (w_1, \ldots, w_m)$ is the vector $u \otimes w = (u_1 w_1, \ldots, u_1 w_m, u_2 w_1, \ldots, u_n w_m)$ of length nm. One can easily verify that

$$(s \otimes t)(u \otimes w)^T = (su^T)(tw^T). \tag{6}$$

Let G be a graph with $V(G) = \{1, \ldots, n\}$. An *orthonormal representation* of G is a system $\{v_1, \ldots, v_n\}$ of unit vectors in a vector space $\mathcal{R}^r$ such that if $(i \neq j) \wedge (i,j) \notin E(G)$ then $v_i v_j^T = 0$. Every graph has an orthonormal representation, for instance an orthonormal basis of $\mathcal{R}^n$.

Lemma 14 *[13] Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be orthonormal representations of G and H respectively. Then the vectors $u_i \otimes v_j$ form an orthonormal representation of $G \cdot H$.*

Lemma 15 *Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_m\}$ be orthonormal representations of G and $\overline{H}$ respectively. Then the vectors $u_i \otimes v_j$ form an orthonormal representation of $\overline{G \circ H}$.*

Proof: The proof follows from the previous Lemma 14 and the fact that $G \cdot \overline{H} \subseteq \overline{G \circ H}$.

□

Theorem 16 *[13] Let $(u_1, \ldots, u_n)$ range over all orthonormal representations of G and c over all unit vectors. Then*

$$\vartheta(G) = \min \max_{1 \le i \le n} \frac{1}{(cu_i^T)^2}.$$

Theorem 17

$$\vartheta(\overline{G \circ H}) \le \vartheta(G)\vartheta(\overline{H}).$$

Proof: Let $(u_1, \ldots, u_n)$ be an orthonormal representation of G, $(v_1, \ldots, v_m)$ be an orthonormal representation of $\overline{H}$ and c, d be unit vectors such that $\vartheta(G) = \max_{1 \le i \le n} \frac{1}{(cu_i^T)^2}$ and $\vartheta(\overline{H}) = \max_{1 \le j \le m} \frac{1}{(dv_j^T)^2}$. Then $c \otimes d$ is a unit vector by (6), and $u_i \otimes v_j$ is an orthonormal representation of $\overline{G \circ H}$ by Lemma 15. Therefore by Theorem 16

$$\vartheta(\overline{G \circ H}) \le \max_{i,j} \frac{1}{((c \otimes d)(u_i \otimes v_j)^T)^2} = \max_{i,j} \frac{1}{(cu_i^T)^2} \frac{1}{(dv_j^T)^2} = \vartheta(G)\vartheta(\overline{H}).$$

□

Corollary 18 *If* $G \to_p H$ *then* $\vartheta(G)\vartheta(\overline{H}) \geq |V(G)|$.

In the last part of this section we prove that in general we cannot reverse implications in the Corollary 6. We need the following results (see [13, 12]):
for odd n

$$\vartheta(C_n) = \frac{n cos(\pi/n)}{1 + cos(\pi/n)},$$

$$\vartheta(\overline{C_n}) = \frac{1 + cos(\pi/n)}{cos(\pi/n)},$$

and for every n

$$\vartheta(K_n) = 1,$$

$$\vartheta(\overline{K_n}) = n.$$

Theorem 19 *For odd* n, m *such that* $3 \leq n < m$, $C_m \not\to_p K_2$ *and* $C_n \not\to_p C_m$.

Proof: The proof follows from the fact that $\vartheta(C_n)$ is increasing function of $n \geq 3$ while $\vartheta(\overline{C_n})$ is decreasing and their product is n. Also we have $\vartheta(C_5) = \sqrt{5}$. Therefore $\vartheta(C_n)\vartheta(\overline{C_m}) < \vartheta(C_n)\vartheta(\overline{C_n}) = n$ and also $\vartheta(C_m)\vartheta(\overline{K_2}) = 2\vartheta(\overline{C_m}) < \sqrt{5}\vartheta(\overline{C_m}) = \vartheta(C_5)\vartheta(\overline{C_m}) \leq \vartheta(C_m)\vartheta(\overline{C_m}) = m$. From Corollary 18 we can conclude the assertion of our theorem.

□

As $C_5 \to_f K_2$, and the above theorem shows that $C_5 \not\to_p K_2$, pseudo-homomorphsim is strictly stronger than fractional homomorphism.

The next Theorem shows that homomorphism is strictly stronger than pseudo$_{1/2}$-homomorphism.

Let G be the graph on 11 vertices with the following adjacency matrix:

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

and $H = K_3$.

Theorem 20 $G \not\to H$ *but* $G \to_{p/2} H$.

Proof: The first assertion of the theorem follows from the fact that G is not three colorable. The odd cycle $1, 2, 3, 4, 5$ requires at least three colors and the vertices $6, 7, 8, 9, 10$ are forced to use all three colours. Hence the vertex 11 needs a fourth colour. The second assertion follows from the fact that the following matrix Q is the hoax. We will write its s,t blocks:

$$Q_{s,s} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix},$$

$$Q_{s,t} = \begin{pmatrix} 0 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \end{pmatrix},$$

for $s \sim t$, and

$$Q_{s,t} = \begin{pmatrix} 1/6 & 1/12 & 1/12 \\ 1/12 & 1/6 & 1/12 \\ 1/12 & 1/12 & 1/6 \end{pmatrix},$$

for $s \not\sim t$.
One can check that all conditions for Q to be a hoax hold.

□

7 Discussion

As alluded to earlier, the theory presented here applies to any NP problem, although in this paper we have concentrated on graph homomorphism. We now briefly describe how the theory applies to other problems. The *hoax set* is a set of instances for which there is a hoax according to [3].

7.1 Graph Homomorphism

To prove that graph homomorphism is polynomial for some classes of graphs, we will need a slight strengthening of the Theorem 8.1 in [3].

Theorem 21 *Let $\mathcal{C}$ be a family of graphs such that $\mathcal{C} \not\to_p H$, i.e. $\forall G \in \mathcal{C} : G \not\to_p H$. Then the class $(\to_p H) = \{G | G \to_p H\}$ is in P, and separates $(\mathcal{C}\to) = \{H | \exists G \in \mathcal{C} : G \to_p H\}$ and $(\to H) = \{G | G \to H\}$.*

Proof: Obviously, $(\to H) \subseteq (\to_p H)$. Assume, there is a graph $G \in (\mathcal{C}\to) \cap (\to_p H)$. Then there is a graph $G' \in \mathcal{C}$ such that $G' \to G$, and hence also $G' \to_p G$. We have also $G \to_p H$ and from the transitivity of $\to_p$ (see [3]), $G' \to_p H$ and hence $\mathcal{C} \to_p G$ which is a contradiction.

□

The class of all odd cycles will be denoted by $\mathcal{C}_{odd}$.

Theorem 22 *For H bipartite, $G \to H$ iff $G \to_p H$.*

Proof: The graph H is bipartite iff it does not contain any odd cycle and it is iff H is 2-colorable. Therefore we have $(\mathcal{C}_{odd}\not\to) = (\to K_2)$. By the Theorem 19 $\mathcal{C}_{odd} \not\to_p K_2$ and hence from the Theorem 21 it follows that the class $(\to_p K_2)$ is in P, and separates $(\to K_2)$ (all bipartite graphs) from $(\mathcal{C}_{odd}\to)$ (graphs which contain an odd cycle, i.e. nonbipartite graphs). Therefore $(\to_p K_2) = (\to K_2)$. But for bipartite graph H, one can easily observe that $G \to H \; iff \; G \to K_2$ and hence the theorem follows.

□

Note that this also confirm the well known results that 2-colorability is polynomial and also it follows from Theorem 19 that finding the odd girth of a graph (size of the smallest odd cycle) is polynomial.

7.2 Digraph Homomorphism

A directed graph G is said to be *homomorphic* to a directed graph H, if there exists a function $f : V(G) \to H$ called a *homomorphism*, such that whenever (s,t) is an edge in G, $(f(s), f(t))$ is an edge in H. We can now similarly define the hom-product of two directed graphs G, H as the undirected graph $G \circ H$ with $V(G \circ H) = V(G) \times V(H)$ and $(s,u) \sim (t,w)$ iff $(s \neq t) \wedge ((s,t) \in E(G) \Rightarrow (u,w) \in E(H)) \wedge ((t,s) \in E(G) \Rightarrow (w,u) \in E(H))$.

Hell, Nešetřil, and Zhu found polynomial algorithms for classes of graphs which have a property referred to as *tree-duality* in [7, 10, 8]. They showed that the following labeling algorithm works for H-coloring (the problem whether $G \to H$ for a fixed graph H) for H with tree-duality. We will later refer to this algorithm as the 1-*consistency test* (cf. [7]).

Instance: A digraph G;
Question: Is $G \to H$?
Define labels $L_k : V(G) \to 2^{V(H)}, k \geq 0$ by induction as follows:
$L_0(s) = V(H)$ for all $s \in V(G)$
$$L_{k+1}(s) = \{u \in L_k(s) \mid \forall t \in V(G)\ \exists w \in L_k(t) : ((s,t) \in E(G) \Rightarrow (u,w) \in E(H)) \wedge \\ \wedge ((t,s) \in E(G) \Rightarrow (w,u) \in E(H))\}$$

For any vertex s, the set $L_{k+1}(s) \subseteq L_k(s), k \geq 0$, and $L_0(s) = V(H)$. Therefore there is some i such that $L_i(s) = L_{i+1}(s)$.
Output: G is H-colorable if for all $s \in V(G)$: $L_i(s) \neq \emptyset$.

We now prove that the class of graphs with tree duality also admits a polynomial time solution by the theory described in this paper.

Theorem 23 *For any H-coloring problem which can be solved by the* 1-*consistency test we have $G \to H$ iff $G \to_f H$.*

Proof: We will show that if there is an $s \in V(G) : L_i(s) = \emptyset$, then also $G \not\to_f H$.
Let us assume that $G \to_f H$, and a fractional clique x satisfies

$$\omega_f(G \circ H) = \sum_{(s,u) \in V(G \circ H)} x_{(s,u)} = |V(G)|. \tag{7}$$

First we will show by induction on $k \geq 1$ that if $u \notin L_k(s)$, then the corresponding weight $x_{(s,u)} = 0$.
If $k = 1$, and $u \notin L_1(s)$, then there exists $t \in V(G)$ such that for all $w \in L_0(t) = V(H) : (s \sim t) \wedge (u \not\sim w)$. Therefore for all $w \in V(H) : (s,u) \not\sim (t,w)$. Hence (s,u) together with vertices $(t,w), w \in V(H)$, form an independent set, and therefore

$$x_{(s,u)} + \sum_{w \in V(H)} x_{(t,w)} \leq 1.$$

The equality (7) implies that $\sum_{w \in V(H)} x_{(t,w)} = 1$ and hence $x_{(s,u)} = 0$.
Assume that $u \notin L_k(s)$ implies $x_{(s,u)} = 0$. For $u \notin L_{k+1}(s)$ we have t such that for all $w \in L_k(t)$: $(s \sim t) \wedge (u \not\sim w)$, which means also that vertices (s,u), and $(t,w), w \in L_k(t)$ form an independent set. For $w \notin L_k(t)$ we can use our induction hypothesis that $x_{(t,w)} = 0$, and therefore we can apply the equations from the basic case.
If $L_i(s) = \emptyset$, then for all $u \in V(H) : x_{(s,u)} = 0$, which contradicts (7).

□

According to the latest results of Hell, Zhu, and Nešetřil [7] , and independently of Feder and Vardi [2], the case of digraphs H with tree duality is extended to digraphs H with bounded tree width duality, where 1-consistency check is replaced by a more general k-consistency check. We can also modify our game so that the verifier gives each prover k vertices. It then shows out that as in Theorem 21, any H-coloring problem which can be solved by a k-consistency test has $G \to H$ iff $\omega_f(G \circ_k H) = |V(G)|$ where $G \circ_k H$ is the graph whose incidence matrix equals to the verifier's matrix. This provides another proof of the polynomiality of these problems.

7.3 SAT

For the satisfiability problem, the two-prover game is as follows. The two provers claim a formula in the conjunctive normal form is satisfiable. The verifier randomly picks two clauses and gives one to prover P_1 and the other to P_2. The provers are supposed to give truth assignments to the variables occurring in their respective clauses. The verifier accepts iff the corresponding truth assignments make the respective clauses true, the two truth assignments are compatible and whenever the verifier gives the same clause to both the provers, the provers return the same truth assignment. We can then show that if all clauses are of size at most 2, then the hoax set contains exactly the satisfiable clauses, and hence we have polynomial time algorithm for 2-SAT. We can also characterize certain families of instances (for 3-SAT) for which the hoax set is the same as the set of satisfiable clauses thus providing polynomial algorithms for these families.

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