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NOTE ON THE COMPUTATIONAL COMPLEXITY OF j-RADII OF POLYTOPES IN $\mathbb{R}^n$

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Abstract. We show that, for fixed dimension n, the approximation of inner and outer j-radii of polytopes in $\mathbb{R}^n$, endowed with the Euclidean norm, is in $\mathbb{P}$.

1. Introduction

In this note we assume that n is fixed and consider the complexity of computing inner and outer j-radii of polytopes in $\mathbb{R}^n$, endowed with the usual Euclidean norm. We show, in Section 2, that these problems amount to the determination of the minimum or maximum of a univariate linear objective function, with as side condition the solvability of a certain system of multivariate polynomial (in)equalities. Then, in Section 3, we use a known result [1] that —in fixed dimension— the solvability of such a system can be decided in NC. This enables us to show that for polytopes the computation of outer and inner j-radii is in $\mathbb{P}$. These results answer an open problem posed by Gritzmann and Klee in [4], who also considered the complexity of computing j-radii in more general spaces (cf. [3]).

We do not intend to provide algorithms that are of practical interest. Streng and Wetterling [7] show that the width of a polytope given by its vertices can be computed by Lipschitz optimization techniques. The computation of outer j-radii has been considered from the viewpoint of nonlinear optimization theory by Streng [6] and by Jonker, Streng and Twilt [5], who also considered stability aspects.

2. Formulation of the problem

Throughout this note the space $\mathbb{R}^n$ will be endowed with the usual Euclidean norm. First we define the inner and outer j-radii of a polytope. For that purpose we need some preliminaries. The phrase "j-dimensional affine subspace" will be abbreviated by the term *j-flat.* The unit ball in $\mathbb{R}^n$ is denoted by S. A *j-ball* of radius r in $\mathbb{R}^n$ is a set of the form

$$(q + rS) \cap F_j = \{x \in F_j \mid \|x - q\| \le r\},$$

for some j-flat $F_j \subset \mathbb{R}^n$ and some point $q \in F_j$.

Definition 2.1. Let $P \subset \mathbb{R}^n$ be a polytope.

(1) The outer j-radius $R_j(P)$ is the minimum of all positive numbers r such that there is an $(n-j)$-flat F_{n-j} with $P \subset F_{n-j} + rS$.
(2) The inner j-radius $r_j(P)$ $(1 \leq j \leq n)$ is the maximum radius of the j-balls contained in P.

The existence of these radii as minima or maxima is guaranteed by a standard compactness argument. For convenience we will distinguish between polytopes given by their vertices and those given as intersection of finitely many closed halfspaces.

Definition 2.2. Let $P \subset \mathbb{R}^n$ be a polytope.

(1) A $\mathcal{V}$-presentation of P consists of integers m, n $(m > n \geq 1)$ and an m-tuple of points $\{y_1, \dots, y_m\}$ with rational coordinates in $\mathbb{R}^n$ such that P is the convex hull of these points.
(2) An $\mathcal{H}$-presentation of P consists of integers m, n $(m > n \geq 1)$, m rational column vectors $c_k \in \mathbb{R}^n$, $(k = 1, \dots, m)$, and m rational numbers γ_k, $(k = 1, \dots, m)$ such that $P = \{x \in \mathbb{R}^n \mid c_k^T x \leq \gamma_k, \quad k = 1, \dots, m\}$.

Note that for fixed dimension, these two presentations are polynomially equivalent. Hence we may choose whichever presentation seems more adequate. We first formulate the problem of computing the outer j-radius of a $\mathcal{V}$-presented polytope. An elementary observation is that this radius is the minimal number r such that there is a j-flat F_j such that

$$d^2(y_k, F_j) - r^2 \leq 0, \quad k = 1, \dots, m,$$

where $d^2(y_k, F_j)$ denotes the squared Euclidean distance between y_k and F_j.

We will represent a j-flat F_j by a pair (a, B), with $a \in \mathbb{R}^n$ and B an $n \times j$ matrix. Then

$$F_j = \{x \in \mathbb{R}^n \mid x = a + Bs, \quad s \in \mathbb{R}^j\}.$$

Without loss of generality we may assume $B^T B = I_j$ and $B^T a = 0$, where I_j denotes the $j \times j$ unit matrix. Then

$$d^2(y_k, F_j) = \min_{s \in \mathbb{R}^j} \|y_k - a - Bs\|^2.$$

It is easily seen that this minimum is attained at $s = s_k := B^T y_k$, which leads to

$$d^2(y_k, F_j) = (y_k - a)^T (I - BB^T)(y_k - a),$$

and we obtain

Lemma 2.1. *Let P be a $\mathcal{V}$-presented polytope with vertices $\{y_1, \dots, y_m\}$. The outer j-radius is the least number r such that*

$$\exists a, B : (y_1 - a)^T (I - BB^T)(y_1 - a) - r^2 \leq 0 \wedge \dots$$
$$\dots \wedge (y_m - a)^T (I - BB^T)(y_m - a) - r^2 \leq 0 \wedge B^T B = I_j \wedge B^T a = 0.$$

Next we turn to the inner j-radius for $\mathcal{H}$-presented polytopes. A j-ball S_j can be represented by a tuple (r, a, B), where r is the radius of S_j, and $a \in \mathbb{R}^n$ is its center, which, together with the $n \times j$ matrix B can represent the j-flat F_j containing S_j. We will again assume $B^T B = I_j$, but because now a denotes the center of S_j, we cannot take $B^T a = 0$, as we did in the case of outer j-radii.

The condition that an arbitrary point $x \in \mathbb{R}^n$ lies in F_j can now be translated into $(x - a) = BB^T (x - a)$, so our j-ball S_j is in fact the set

$$S_j = \{x \in \mathbb{R}^n \mid (x - a)^T (x - a) \leq r^2 \wedge (x - a) = BB^T (x - a)\}.$$

This j-ball is contained in P if every point x on S_j satisfies $c_k^T x \leq \gamma_k$ for $k = 1, \ldots, m$. Therefore we have

Lemma 2.2. *Let P be an $\mathcal{H}$-presented polytope given by*

$$P = \{x \in \mathbb{R}^n \mid c_k^T x \leq \gamma_k, \quad k = 1, \ldots, m\}.$$

The inner j-radius is the maximal number r such that

$$\exists a, B \neg \exists x : (x-a)^T(x-a) \leq r^2 \wedge (x-a) = BB^T(x-a) \wedge (c_1^T x > \gamma_1 \vee \ldots \vee c_m^T x > \gamma_m).$$

In Section 3 we shall see that, using a complexity theoretic result from Ben-Or et al. [1], the decision problems formulated in lemmas 2.1 and 2.2 are in NC, hence can be solved in polynomial time.

3. Complexity

In [1] it is shown that the theory of real closed fields in fixed dimension can be decided in NC. This strengthens a previous result by Collins [2] on polynomial decidability. More precisely, the following holds.

Theorem 3.1. *(Ben-Or et al., [1], Section 4)*
For fixed k, the following decision problem is in NC (hence can be solved in polynomial time): Given polynomials $p_1(x_1, \ldots, x_k), \ldots, p_s(x_1, \ldots, x_k)$, a boolean formula $\phi(x_1, \ldots, x_k)$ which is a boolean combination of polynomial equations and inequalities, i.e. $p_i(x_1, \ldots, x_k) = 0$ or $p_i(x_1, \ldots, x_k) < 0$, and quantifiers $Q_1, \ldots, Q_k$, decide the truth of the statement

$$Q_1(x_1 \in \mathbb{R}) \ldots Q_k(x_k \in \mathbb{R})\phi(x_1, \ldots, x_k).$$

The term "polynomial time" in Theorem 3.1 refers to the size of the boolean formula ϕ, which equals $k + s +$ number of boolean operations (i.e. $\wedge, \vee, \neg$) occurring in ϕ + number of bits needed to represent the polynomials $p_1, \ldots, p_s$. (We assume that all these polynomials have rational coefficients whose denominators and enumerators are encoded in binary.)

From Theorem 3.1 it is immediate that the decision problems of Lemmas 2.1 and 2.2 are in NC, hence polynomially time solvable w.r.t. the size of the input given by $y_1, \ldots, y_m$, provided the squared radius r^2 has polynomial size. From this fact we may further conclude that straightforward binary search yields a polynomial time approximation algorithm for (approximately) solving the problems mentioned in Section 2. More precisely, we get

Proposition 3.1. *For fixed dimension n, there exist fully polynomial approximation schemes for solving the outer- and inner j-radius problem, i.e., given $\epsilon > 0$, one can compute an approximate solution $\hat{a}, \hat{B}$ such that the corresponding outer (inner) j-radius $\hat{r}$ differs at most ϵ from the optimum $\overline{r}$, and the computation is polynomially bounded in the input size and* $\log(\frac{1}{\epsilon})$.

Proof. Consider, for example, the problem of computing the outer j-radius for a $\mathcal{V}$-presented polytope P given by $y_1, \ldots, y_m \in \mathbb{R}^n$. Given r, we will denote the decision problem occurring in Lemma 2.1 by $E(r)$. Let $\epsilon > 0$ be given. We first compute $\hat{r} \in \mathbb{R}$ such that $|\hat{r} - \overline{r}| < \frac{\epsilon}{2}$, where $\overline{r}$ is the outer j-radius of P. This can be achieved by straightforward binary search starting with the interval $[r_0, r_1]$, where $r_0 = 0$, $r_1 = \max_i ||y_i||$.

Thus we end up with some $\hat{r}$ such that $E(\hat{r})$ is true and $E(\hat{r} - \frac{\epsilon}{2})$ is false. Next we perform binary search on the components of a and B to determine these within an error of $\delta > 0$ (to be specified below). Note that $\|B\|_\infty \leq 1$ and that we may restrict ourselves to $\|a\|_\infty \leq a_+ := \max_i \|y_i\|_\infty$. Now we first perform binary search on $[-a_+, a_+]$ until we have found an $\hat{a}_1 \in \mathbb{R}$ such that

$$E(\hat{r}) \wedge (\hat{a}_1 - \delta \leq a_1 \leq \hat{a}_1 + \delta)$$

is true. We then perform again binary search to compute $\hat{a}_2 \in \mathbb{R}$ such that

$$E(\hat{r}) \wedge (\hat{a}_1 - \delta \leq a_1 \leq \hat{a}_1 + \delta) \wedge (\hat{a}_2 - \delta \leq a_2 \leq \hat{a}_2 + \delta)$$

is true, and so on. The computation is polynomially bounded in the input size and $\log(\frac{1}{\delta})$. the approximate solution $\hat{a}, \hat{B}$ is such that $\|(\hat{a}, \hat{B}) - (a, B)\|_\infty \leq \delta$ for some (not necessarily optimal) solution (a, B) of $E(\hat{r})$. Note that, of course, the computed $\hat{a}$ and $\hat{B}$ in general will not satisfy $\hat{B}^T\hat{B} = I_j$ or $\hat{B}^T\hat{a} = 0$. Yet they define a j-flat $\hat{F}$ which is a good approximation to a solution of the outer j-radius problem. More precisely, let F be the j-flat defined by a and B, and let r be the corresponding radius, i.e. $r = \max_k d(y_k, F)$. For $k = 1, \ldots, m$ let $s_k \in \mathbb{R}^j$ such that $d(y_k, F) = \|y_k - a - Bs_k\|$. Then we get

$$\begin{aligned} d(y_k, \hat{F}) &= \min_{s \in \mathbb{R}^j} \|y_k - \hat{a} - \hat{B}s\| \\ &\leq \|y_k - \hat{a} - \hat{B}s_k\| \\ &\leq \|y_k - a - Bs_k\| + \|a - \hat{a}\| + \|B - \hat{B}\|\,\|s_k\| \\ &\leq d(y_k, F) + n\delta + nj\delta^2\|y_k\| \\ &\leq \hat{r} + n^2\delta(1 + \|y_k\|). \end{aligned}$$

Thus by choosing

$$\delta \leq \frac{\epsilon/2}{n^2(1 + \max_k \|y_k\|)},$$

we get

$$d(y_k, \hat{F}) \leq d(y_k, F) + \frac{\epsilon}{2} \leq \hat{r} + \frac{\epsilon}{2} \leq \overline{r} + \epsilon.$$

The proof for the inner j-radius problem is similar. □

References

1. M. Ben-Or, D. Kozen, and J. Reif, *The complexity of elementary algebra and geometry*, J. Comp. Syst. Sc. **32** (1986), 251–264.
2. G.E. Collins, *Quantifier elimination for real closed fields by cylindric algebraic decomposition*, Lecture Notes in Computer Science (Berlin), vol. 33, Springer Verlag, Berlin, 1975, pp. 134–183.
3. P. Gritzmann and V. Klee, *Inner and outer j-radii of convex bodies in finite-dimensional normed spaces*, Discrete computational geometry **7** (1992), 255–280.
4. ———, *Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces*, Mathematical Programming **59** (1993), 163–215.
5. P. Jonker, M. Streng, and F. Twilt, *Chebyshev line approximation: a sensitivity analysis*, *in:* parametric optimization and related topics III (*J. Guddat, H. Th. Jongen, B. Kommer, F. Nožička, eds.), Approximation and Optimization, Peter Lang Verlag, Frankfurt am Main, Berlin* (1993), 303–322.
6. M. Streng, *Chebyshev approximation of finite sets by curves and linear manifolds*, Ph.D. thesis, University of Twente, 1993.
7. M. Streng and W. Wetterling, *Chebyshev approximation of a point set by a straight line*, to appear in Constructive Approximation (1993).