On approximately fair cost allocation in Euclidean TSP games

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Abstract:

We consider the problem of fair cost allocation for Travelling Salesman games for which the triangle inequality holds. We give examples showing that the core of such games may be empty, even for the case of Euclidean distances. On the positive, we develop an LP-based allocation rule guaranteeing that no coalition pays more than α times its own cost, where α is the ratio between the optimal TSP-tour and the optimal value of its Held-Karp relaxation, which is also known as the solution over the "subtour polytope". A well known conjecture states that $\alpha \leq \frac{4}{3}$. We also exhibit examples showing that this ratio cannot be improved below $\frac{4}{3}$.

1 Introduction

Travelling Salesman games are well-studied examples of so-called "cooperative games". Formally, such games are defined as follows:

Definition 1.1 *Let N be a finite set, the set of* players*. A* cooperative game *is then defined by a function*

$$c : 2^N \longrightarrow \mathbb{R},$$

i.e. a function that associates a cost *$c(S)$ to every* coalition *$S \subseteq N$. The coalition N is also called the* grand coalition.

Several interesting games arise from combinatorial optimization problems.

Example 1.2 *Let $N = \{1, \ldots, n\}$, $N_0 := N \cup \{0\}$ and consider the complete undirected graph with vertex set N_0 and edges weighted with real numbers d_{ij}, $i, j \in N_0$. For every $S \subseteq N$, let*

$$c(S) = \text{minimum cost of a tree spanning all vertices in } S \cup \{0\}.$$

This game is known as the Minimum Spanning Tree game*, see [9].*

Example 1.3 *Same as in Example 1.2, except for*

$$c(S) = \text{minimum length of a travelling salesman tour visiting all vertices in } S \cup \{0\}.$$

This game is known as the Travelling Salesman game*, see [20].*

Example 1.2 can be interpreted as follows: Consider vertex 0 as a *supply node*, e.g. an electricity supply. The cost $c = c(N)$ of the grand coalition is the total cost of supplying electricity to all vertices or "players". The problem is to find a fair way to allocate this amount to all the players. More precisely, let x_i denote the amount of money player i is asked to pay, thus $\sum_{i \in N} x_i = c$. A coalition $S \subseteq N$ which is asked to pay more than its own cost, i.e. $\sum_{i \in N} x_i > c(S)$, will refuse to pay and prefer to split off the grand coalition by connecting to the supply on its own. Thus a *fair allocation* is one for which $\sum_{i \in S} x_i \leq c(S)$ holds for every coalition $S \subseteq N$.

Example 1.3 can be interpreted in a similar way. Consider vertex 0 as the home city of a speaker who has to give talks at the universities located in vertices $1, \ldots, n$. The total travel cost equals $c = c(N)$. Again the problem is to find a fair cost allocation, such that no coalition S will split off and invite the speaker to visit only the universities $i \in S$.

Definition 1.4 *Consider a cooperative game defined by $c : 2^N \longrightarrow \mathbb{R}$. An* allocation *is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = c(N)$. The* core *of the game is defined as the set of all allocation vectors $x \in \mathbb{R}^n$ satisfying*

$$\sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N.$$

Any vector x in the core is called a core allocation vector*, or simply a* fair allocation.

In Example 1.2 above, it is straightforward to verify that a fair allocation can be obtained as follows: Construct a minimum cost spanning tree T in K_{N_0}, and direct the edges of T such that node 0 becomes the root. Then for each $i \in N$, let x_i be the length of the (unique) edge of T entering node i.

As for the case of the Travelling Salesman games, it has been observed that the core of such games may be empty, see Tamir [24]. (Note that in that paper, a slightly more general setting is used, allowing non-hamiltonian graphs.) In the present paper, we will consider TSP games for Euclidean distances, which are among the most natural classes of TSP games, but have not received any special attention so far.

The rest of this paper is organized as follows: Section 2 presents a minimal example of a Euclidean TSP game with empty core. This motivates the search for "reasonably fair" allocations. One possible definition was given in [5], where the notion of "ε-approximately fair allocations" is introduced.

Definition 1.5 *Consider a game defined by $c : 2^N \longrightarrow R$. An ε-*approximately fair allocation *is an allocation vector $x \in \mathbb{R}^N$, such that*

$$x(N) \geq c(N) \text{ and } x(S) \leq (1+\varepsilon)c(S) \text{ for all } S \subseteq N.$$

The idea is that for small $\varepsilon > 0$, an ε-approximately fair allocation may still be acceptable, since the additional overhead costs for a coalition S after splitting off the grand coalition is likely to exceed a fraction of ε of its cost. Conversely, a limited deficit for the supplier may be acceptable.

Other ways of defining ε-approximate core allocations have been proposed in the literature, see [5] for more references.

In Section 3 we prove that TSP games with triangle inequality always have ε-approximately fair allocations for $\varepsilon = \frac{1}{2}$. If a well-known conjecture concerning the Held-Karp relaxation of the Euclidean TSP is true, our allocation rule presented in Section 3 would even achieve $\varepsilon = \frac{1}{3}$. Moreover, these allocations can be computed in polynomial time, even if the optimal tour is not known. In Section 4 we show that no allocation rule can guarantee $\varepsilon < \frac{1}{3}$. We end with some open problems and concluding remarks in Section 5.

2 A minimal example with empty core

Kuipers [14] showed that under quite general assumption (including the Euclidean case), every TSP-game with up to 5 players has a nonempty core. The following example shows that for $n = 6$ players, there exist Euclidean instances with empty core.

Example 2.1

See Figure 1. Consider an isosceles triangle of side length $l = \sqrt{3}$, centered at 0. Label the vertices by 1, 2, 3. Place three more nodes 4, 5, 6 at equal distance d from the center, such that node i lies on the line segment $\overline{0, i-3}$.

The distance d will be chosen appropriately. Let f denote the distance $d_{56} = d_{46} = d_{45}$; clearly, $f = \sqrt{3}d$. Let h denote the distance $d_{15} = d_{16} = d_{24} = d_{26} = d_{34} = d_{35}$. Applying Pythagoras' Theorem to the triangle $\Delta(5P3)$, we get

$$h = \sqrt{\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}d\right)^2 + \left(\frac{(1-d)}{2}\right)^2},$$

we conclude that $h = \sqrt{1+d+d^2}$. Finally, let $g := d_{36} = d_{25} = d_{14}$, i.e. $g = 1 - d$. As a consequence, the two obvious candidates for an optimal TSP tour of length L have the following lengths L_1 and L_2 (see Figure 2):

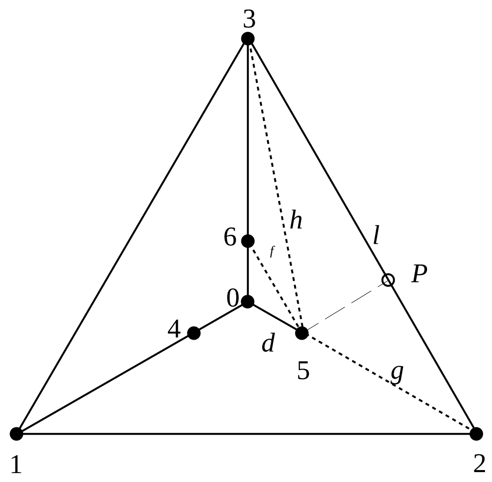

Figure 1: A minimum Euclidean example with empty core

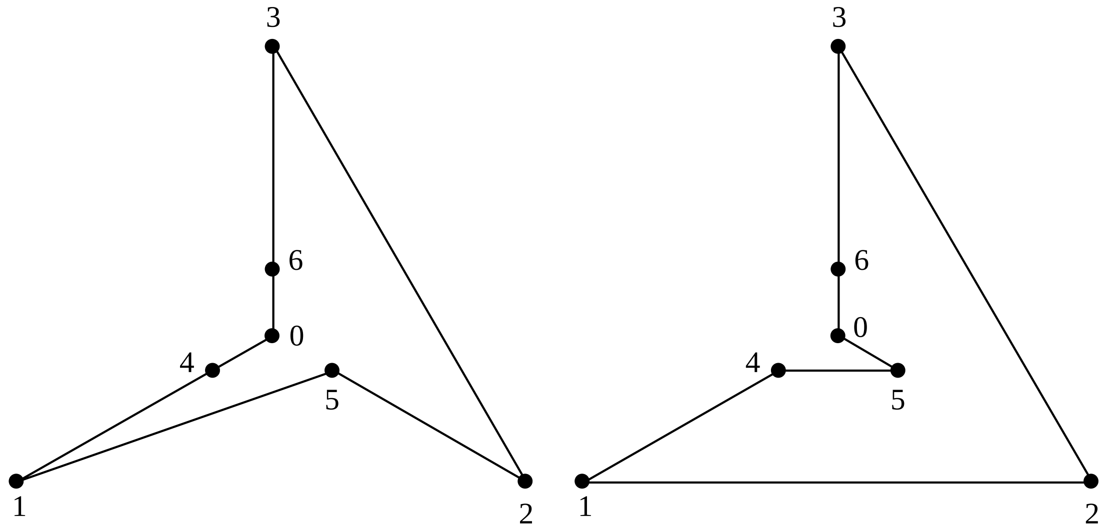

Figure 2: The two candidates for optimal TSP tours

$$L_1 = l + 3g + 2d + h = 3 - d + \sqrt{3} + \sqrt{1 + d + d^2},$$

$$L_2 = 2l + 2g + 2d + f = 2 + 2\sqrt{3} + \sqrt{3}d.$$

Now suppose $x \in \mathbb{R}^6$ were a core allocation. In particular, we must have

1. $\sum_{i=1}^{6} x_i = L$,

2. $x_i + x_{i+1} + x_{i+3} + x_{i+4} \leq 2d + 2g + l$ for $i = 1, 2$
   and $x_3 + x_1 + x_6 + x_4 \leq 2d + 2g + l$.

Adding up the three inequalities, we get

$$L = \sum_{i=1}^{6} x_i \leq \frac{3}{2}(2d + 2g + l) = 3 + \frac{3}{2}\sqrt{3}.$$

If the optimal tour has length L_1, we get

$$L_1 = 3 - d + \sqrt{3} + \sqrt{1 + d + d^2} \leq 3 + \frac{3}{2}\sqrt{3},$$

which is equivalent to

$$d \geq \frac{1}{4(\sqrt{3} - 1)} > \frac{1}{3}.$$

If the optimal tour has length L_2, we get

$$L_2 = 2 + 2\sqrt{3} + \sqrt{3}d \leq 3 + \frac{3}{2}\sqrt{3},$$

which is equivalent to

$$d \leq \frac{1}{\sqrt{3}} - \frac{1}{2} < \frac{1}{10}.$$

We conclude that there is no fair cost allocation if neither of these two conditions is satisfied, e.g. for $d = \frac{1}{4}$. □

3 Approximately fair allocations for Euclidean instances

We have seen in the previous section that not all TSP instances allow a fair cost allocation. This makes it desirable to examine *approximately fair cost allocations*, where the customers can be overcharged by a certain percentage, or the supplier is allowed to run a certain deficit. (See Definition 1.5.)

Interesting allocation rules (different from core allocations) have been studied in the context of TSP games, see [20]. In this section, we will approach the question from a geometric point of view and make use of linear programming duality.

There is another good reason for considering approximately fair cost allocations: Since computing the length L of an optimal TSP tour is NP-hard, we cannot expect to find an efficient way of computing cost allocations, whether they are fair or not. These computational difficulties make it desirable to consider performance bounds on cost allocations which can be computed in polynomial time. In section 3.2 below, we will introduce a modified game with cost function $c_{(HK)} \leq c$, for which we can efficiently compute core allocation vectors z. It is known that $c_{(HK)} \geq \frac{3}{2} c_{Chr}$,

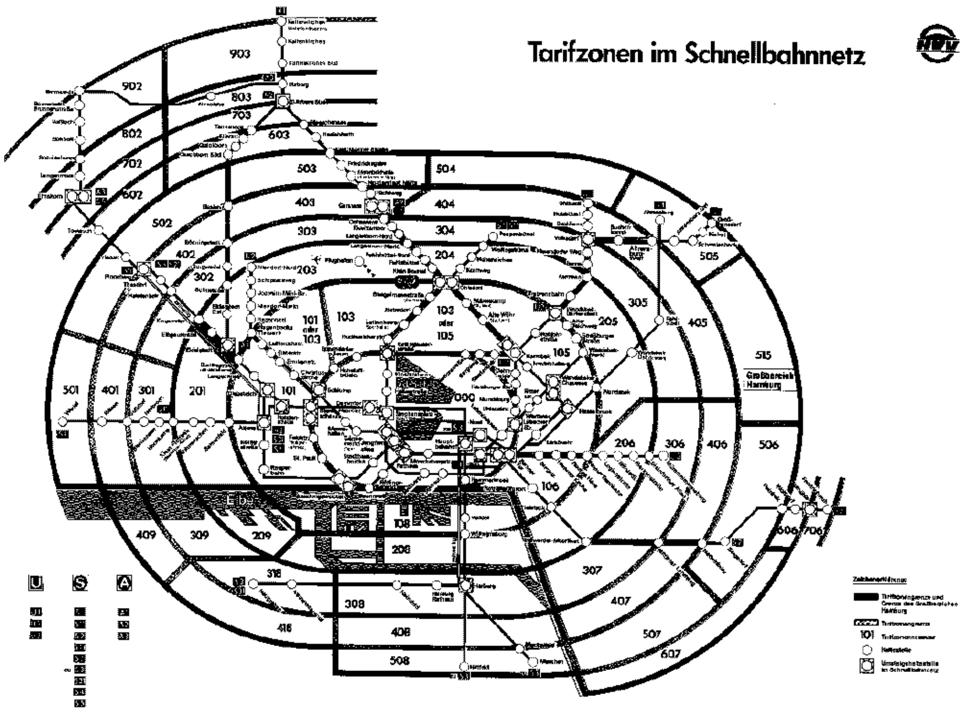

Figure 3: The fare zones of Hamburg, reproduced with kind permission by Hamburger Verkehrsverbund HVV

where c_{Chr} denotes the cost function corresponding to the well-known Christofides heuristic. Hence, by scaling the vector z by a factor $\alpha = \frac{c_{Chr}(N)}{c_{HK}(N)} \leq \frac{3}{2}$, we get an ε-approximately fair allocation for the original TSP game with $\varepsilon \leq \frac{1}{2}$ – see below for details. In case an optimal global tour of length L is known, we may of course scale the vector z by a factor of $\overline{\alpha} = \frac{L}{c_{HK}(N)} \leq \alpha$ to obtain an $\overline{\varepsilon} \leq \varepsilon \leq \frac{1}{2}$.

3.1 Geometric cost allocation

We encounter methods of allocating the cost of transportation in many instances in everyday life. Two of the easiest allocation rules are also the most common ones: A taxi charges by the distance that is traveled by an individual customer. On the other hand, public mass transportation typically charges a flat fee for anyone who uses it, regardless of the distance. The practicality of these two allocation rules relies on the fact that the number of players in a game is either extremely small, i.e. 1, or arbitrarily large. (Insufficiencies of the latter assumption are reflected by the deficits of most public transportation.)

In many cities, there are attempts to refine the fares by using a "zone structure": The region is subdivided into traffic zones, and a customer is charged a certain amount for crossing from one zone into the other. (See Figure 3 for a practical example.)

For our purposes, a subdivision of the plane into fixed zones (e.g., concentric circles around the depot) is much too crude. Instead, we have to take into account the relative position of the customers. In the following, we will describe a geometric cost allocation method that follows this idea.

Definition 3.1 *For a given set of vertices in the plane, a* moat *is a simply closed strip of constant width that separates two nonempty complementary subsets of the vertices. The* inside *of the moat is the region containing the depot, the other region we call the* outside*. A* moat packing *is a collection of moats with pairwise disjoint interior. The* cost *of a moat packing is twice the sum over all widths.*

Note that any tour has to cross every moat twice, hence the cost of a TSP tour is greater than or equal to the cost of a moat packing. Figure 4 shows a moat packing for an instance of cities in the American midwest, distances are taken from Nemhauser and Wolsey, [15], p. 530.

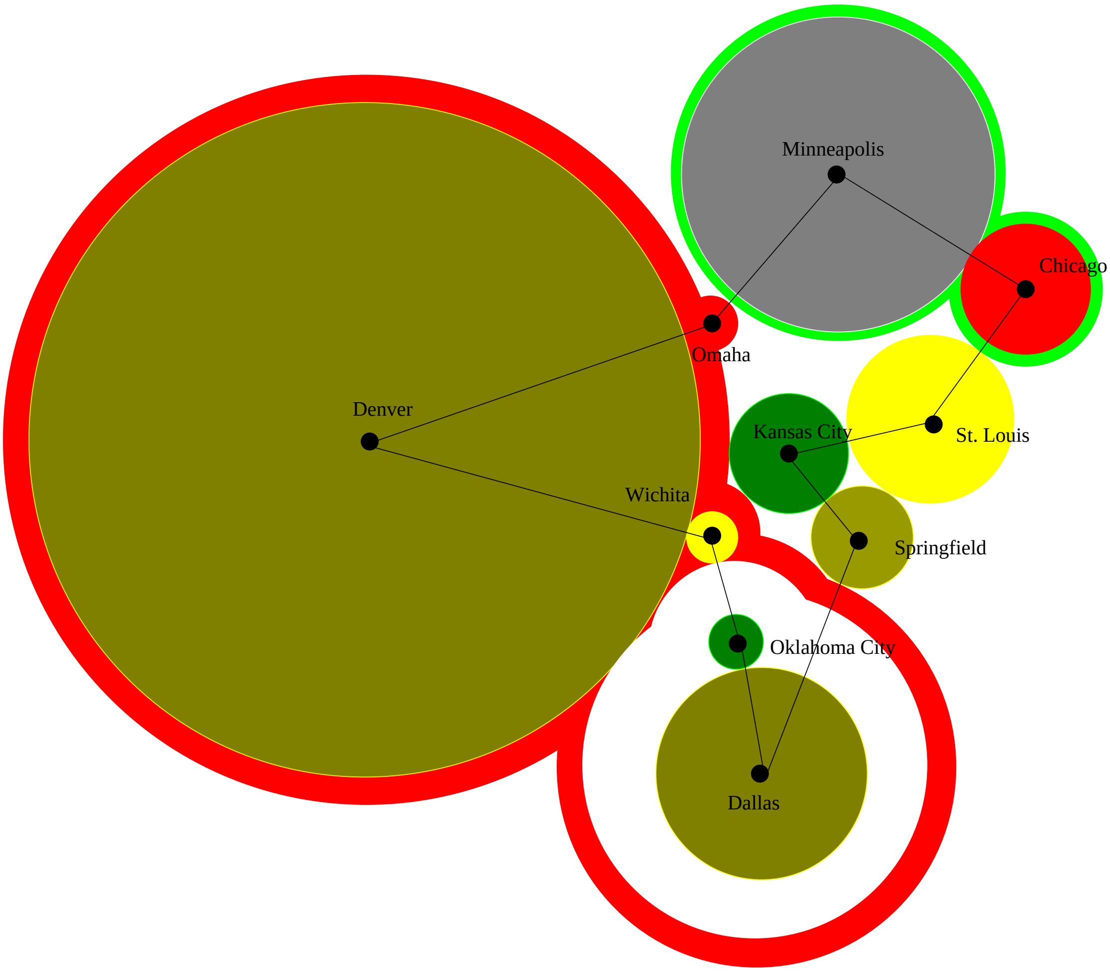

Figure 4: A moat packing for an instance of 10 cities in the midwestern United States

3.2 Moats and the subtour polytope

The cost of any moat packing can be allocated as follows: Distribute twice the width of any moat among the vertices on the outside (in an arbitrary way). It is intuitively clear that if the total cost of the moat packing is distributed this way, the resulting distribution is such that no coalition pays more than its TSP cost. (We will prove that formally later on.) Note, however that by distributing the cost of a moat packing we will in general not get an allocation vector of the TSP game, since the cost of a moat packing is in general strictly less than the cost of an optimal TSP tour. The situation where the two costs (maximum moat packing and minimum TSP tour) coincide occurs precisely if there is a moat packing and a tour such that:

1. no part of the given tour is left uncovered, and
2. each moat is crossed by the tour exactly twice.

In the general situation, i.e. when the maximum moat packing has a cost strictly less than the optimum TSP tour, we can still get an allocation by simply scaling the cost distribution above appropriately. This yields an approximately fair allocation for the TSP game, which will be discussed in detail below.

The allocated cost for any moat packing is twice the sum of the moat widths. Moat widths are required to be nonnegative and the sum of the widths of moats separating two vertices must not exceed their distance. This motivates the following linear program:

$$(M) \qquad \max\left(2 \sum_{S,\overline{S}\in\mathcal{M}} w_{\{S\overline{S}\}}\right)$$

subject to the constraints

$$\begin{array}{rcccc} w_{\{S\overline{S}\}} & \geq & 0 & \text{for all} & \{S,\overline{S}\}\in\mathcal{M}, \\ \sum_{(i,j)\in\delta(S)} w_{\{S\overline{S}\}} & \leq & d(i,j) & \text{for all} & i,j\in V. \end{array}$$

Here $\mathcal{M}$ denotes the set of all nontrivial partitions $(S,\overline{S})$ of V (assuming that the depot is contained in $\overline{S}$) and $\delta(S)$ is the set of all edges that join a vertex from S to a vertex from $\overline{S}$. Note that there are exponentially many variables. Furthermore, there may be solutions to the linear program which do not correspond to a moat packing, since there may be positive $w_{\{S_1\overline{S_1}\}}$, $w_{\{S_2\overline{S_2}\}}$ with $S_1\cap S_2\neq\emptyset$ and $S_1\not\subseteq S_2$ and $S_2\not\subseteq S_1$, in which case the two corresponding moats are forced to intersect. However, for any instance of (M), there is an optimal solution for which the sets with $w_{\{S\overline{S}\}}>0$ have the special structure of a *nested family*:

Definition 3.2 *A family of partitions* $(S_1,\overline{S_1}),\ldots,(S_k,\overline{S_k})$ *is called* nested, *if for any two partitions* $(S_i,\overline{S_i})$ *and* $(S_j,\overline{S_j})$, *we have* $S_i\cap S_j=\emptyset$ *or* $S_i\subseteq S_j$ *or* $S_j\subseteq S_i$.

For details on nested families, see Pulleyblank [21]; a proof for the above claim can be found in Cornuéjols, Fonlupt and Naddef [3].

It is not hard to see that a solution with the structure of a nested family allows a moat packing. The details of using a moat packing for allocating the optimal value of (M) will be discussed in the following section. We will assume that triangle inequality and (for the sake of simplicity) symmetry hold for the distance function. We assume the distance function to be defined for all $(i,j), i,j\in V$.

How can we solve (M) in polynomial time? Consider its dual, i.e. the following linear problem:

$$(D) \qquad \begin{array}{c} \min \sum_{(i,j)\in V^2} d(i,j)x_{ij} \\ \text{subject to the constraints} \\ \begin{array}{rcll} x_{ij} & \geq & 0 & \text{for all} \quad i,j, \\ \sum_{(i,j)\in\delta(S)} x_{ij} & \geq & 2 & \text{for all} \quad S,\overline{S}\in\mathcal{M}, \end{array} \end{array}$$

Consider an optimal solution to (D) and suppose there was a vertex i with $\sum_{j\neq i} x_{ij} > 2$. If there was only one vertex $k \neq i$ with $x_{ki} > 0$, we would have $x_{ik} > 2$ and we could lower x_{ik} and stay feasible. Since $d(i,k) \geq 0$, this would not increase the objective value. So assume there are two vertices k_1 and k_2 with $x_{ik_1} > 0$ and $x_{ik_2} > 0$. In that case we can lower both x_{ik_1} and x_{ik_2} and raise $x_{k_1k_2}$ by the same value and maintain feasibility. Since by triangle inequality $d(k_1,i)+d(i,k_2) \geq d(k_1,k_2)$, this does not increase the objective value. So we may assume that for any vertex i, we have $\sum_{j\neq i} x_{ij} = 2$. Furthermore, there can be no edge (k_1,k_2) with $x_{k_1k_2} > 1$, since otherwise we would get the violated constraint

$$\sum_{(i,j)\in\delta(\{k_1,k_2\})} x_{ij} = \sum_{i\neq k_1} x_{ik_1} + \sum_{i\neq k_2} x_{ik_2} - 2x_{k_1k_2} = 4 - 2x_{k_1k_2} < 2.$$

This means we may consider the following linear program (HK) instead of (D):

$$(HK) \qquad \begin{array}{c} \min \sum_{(i,j)\in V^2} d(i,j)x_{ij} \\ \text{subject to the constraints} \\ \begin{array}{rcll} x_{ij} & \geq & 0 & \text{for all} \quad i,j, \\ x_{ij} & \leq & 1 & \text{for all} \quad i,j, \\ \sum_{(i,j)\in\delta(S)} x_{ij} & \geq & 2 & \text{for all} \quad S,\overline{S}\in\mathcal{M}, \\ \sum_{j\neq i} x_{ij} & = & 2 & \text{for all} \quad i\in V. \end{array} \end{array}$$

The feasible region for this second linear program is known as the *subtour polytope* S^n, the program itself as *Held-Karp relaxation*: Any of the constraints corresponding to a moat variable is a so-called *subtour elimination constraint.*

These *subtour elimination constraints* were first introduced by Dantzig, Fulkerson and Johnson [4]. Grötschel and Padberg showed that they are facet-inducing for $n \geq 4$.

Grötschel, Lovász and Schrijver [10], and Karp and Papadimitriou [13], showed that a polynomial method for solving the separation problem for a polytope yields a polynomial method for optimization by means of the ellipsoid method. Padberg and Hong [17] (see also Padberg and Wolsey [19], and Padberg and Rao [18]) demonstrated how to solve separation for the subtour polytope in polynomial time by using the method of Gomory and Hu [9] for finding the minimum cost cut in a graph. Thus we know that optimization over the subtour polytope is possible in polynomial time by means of the ellipsoid algorithm.

For a comprehensive study of optimizing over the subtour polytope, see Boyd [1] and also Boyd and Pulleyblank [2].

Summarizing, we state:

Theorem 3.3 *We can determine an optimal moat packing in polynomial time.*

Since the feasible region Q^n of the Travelling Salesman Polytope is contained in S^n, any optimal solution to minimizing over S^n is a lower bound for the optimal value of TSP. It was proved by Wolsey [26] and by Shmoys and Williamson [22] that for any distance function d satisfying the triangle inequality, this bound can be at worst 2/3 of the optimum:

Theorem 3.4 ([26],[22]) *If the distances satisfy the triangle inequality then the optimum value of (HK) is at least $\frac{2}{3}$ of the length of a shortest tour.* □

It is a well-known open conjecture that the factor of 2/3 can be replaced by 3/4.

It should be noted that Shmoys and Williamson have shown an even stronger version of Theorem 3.4:

Theorem 3.5 ([22]) *If the distances satisfy the triangle inequality then the optimum value of (HK) is at least $\frac{2}{3}$ of the length of a tour obtained by the method of Christofides.* □

This means we can guarantee allocation of 2/3 of the cost of an *approximate tour* where both the allocation and the tour have been computed in polynomial time.

3.3 Tours and Moats

We now proceed to describe how to obtain allocation vectors from moat packings. The idea is to distribute twice the width of every moat in an arbitrary way (e.g. uniformly) among the vertices on its outside. Let w^* be an optimal solution of the linear program (M) and define

$$x_i := 2 \cdot \sum_{\substack{i \in S \\ 0 \in \bar{S}}} \frac{w^*_{S,\bar{S}}}{|S|}.$$

By linear programming duality this vector is in the core of the game associated with the linear program (HK) in a natural way:

Define the cost of a coalition $S \subseteq N$ by

$$c_{(HK)}(S) := \min \sum_{i,j} d(i,j) z_{ij}$$

subject to the constraints

$$\begin{array}{rcll} z_{ij} & \geq & 0 & \text{for all} \quad i,j, \\ \sum_{(i,j)\in\delta_S(T)} z_{ij} & \geq & 2 & \text{for all} \quad 0 \in T \subset S \cup \{0\}. \end{array}$$

(Here $\delta_S(T)$ denotes the set of edges joining T to $(S \cup \{0\} \setminus T.)$) The general idea to apply linear programming duality to combinatorial games goes back to [16].

The vector $x \in \mathbb{R}^N$ is easily seen to be a core vector of the game defined by the cost function $c_{(HK)}$: Indeed, by linear programming duality we have

$$\sum_{i \in N} x_i = 2 \cdot \sum_{i \in N} \sum_{\substack{i \in S \\ 0 \in \bar{S}}} \frac{w^*_{S,\bar{S}}}{|S|} = 2 \cdot \sum_{\{S,\overline{S}\} \in \mathcal{M}} \sum_{i \in S} \frac{1}{|S|} w^*_{S,\bar{S}} = 2 \cdot \sum_{\{S,\overline{S}\} \in \mathcal{M}} w^*_{S,\bar{S}} = c_{(HK)}(N).$$

To see that this is in fact a fair allocation vector, recall that for a coalition $S \subseteq N$, its cost $c_{(HK)}(S)$ is defined as the optimum of an LP. Its dual is given by

$$c_{(HK)}(S) = \max \quad 2 \cdot \sum_{0 \notin T \subset S} w_{T,(S\cup\{0\})\setminus T}$$

subject to the constraints

$$\begin{aligned} \sum_{\substack{0 \notin T \subset S \\ \delta_S(T) \ni (i,j)}} w_{T,(S\cup\{0\})\setminus T} &\leq d_{(i,j)} \quad \forall i,j \\ w_{T,(S\cup\{0\})\setminus T} &\geq 0. \end{aligned}$$

Now an optimum solution w^* of (M) induces a feasible solution of the LP above by $\tilde{w}_T := \sum_{S\cap U=T} w^*_U$. Hence we get

$$\begin{aligned} c_{(HK)}(S) &\geq 2 \cdot \sum_{0 \notin T \subset S} \tilde{w}_T \\ &= 2 \cdot \sum_{\substack{(i,T) \\ i \in T \subseteq S}} \frac{1}{|T|} \tilde{w}_T \\ &= \sum_{i \in S} 2 \cdot \sum_{i \in U} \frac{w^*_U}{|U \cap S|} \\ &\geq \sum_{i \in S} 2 \cdot \sum_{i \in U} \frac{w^*_U}{|U|} \\ &= \sum_{i \in S} x_i. \end{aligned}$$

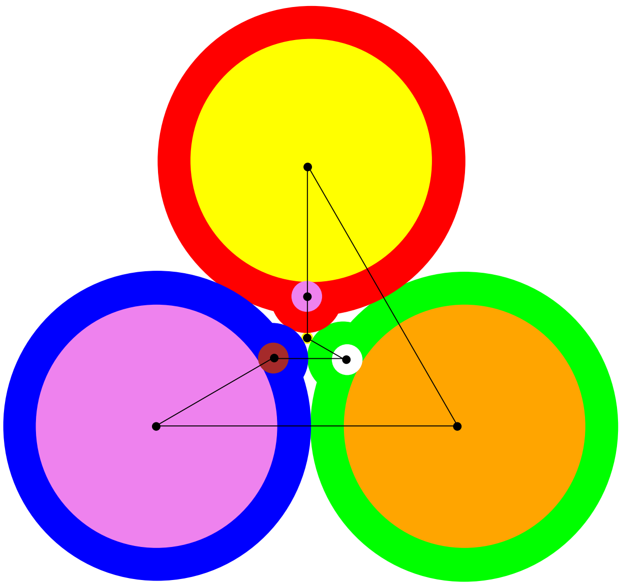

Figure 5: A moat packing

Note that we can choose any distribution of the cost of a moat among the outside vertices, i.e. instead of taking

$$x_i := 2 \cdot \sum_{\substack{i \in S \\ 0 \in \bar{S}}} \frac{w_{S,\bar{S}}}{|S|},$$

we could choose any

$$x_i := 2 \cdot \sum_{\substack{i \in S \\ 0 \in \bar{S}}} \lambda_{Si} w_{S,\bar{S}}$$

with $\sum_{i \in S} \lambda_{Si} = 1, 0 \leq \lambda_{Si} \leq 1$. without changing the validity of any of the above statements.

If the optimal tour is not an optimal solution of (HK) then for any moat packing either at least one moat is traversed more than two times (see Figure 5) or the optimal tour runs through territory which is not covered by any moat (see Figure 6 below).

Proposition 3.6 *We would like to note that even if $c(N) > c_{(HK)}(N)$, the core may be nonempty. The easiest example for this situation arises from Example 2.1, see Figures 1 and 5: It can be shown that an optimal moat packing has cost $3(\frac{f}{2}+g+\frac{l}{2})+2(d-\frac{f}{2}) = (3-d)+(3+d)\frac{\sqrt{3}}{2} =:$ L_3. Now consider the case where d satisfies one of the bounds given in Example 2.1 with equality; in that case, we have a tour of length $L = 3 + 3\frac{\sqrt{3}}{2} > L_3$, meaning that we cannot distribute the cost of the tour by the Held-Karp allocation rule. However, it is easy to see that allocating $\frac{L}{3} = 1 + \frac{\sqrt{3}}{2}$ to each of the players on the outside vertices is a fair allocation.*

3.4 Approximately fair allocations

Summarizing the results of the previous two sections, we state:

Theorem 3.7 *For TSP games with triangle inequality, there is a vector $x \in \mathbb{R}^N$, which can be computed in polynomial time and satisfies the following conditions:*

(i) x is an ε-approximative core allocation for $\varepsilon = \frac{1}{2}$.

(ii) If the "$\frac{4}{3}$-conjecture" on the Held-Karp bound is true, x is an ε-approximative core allocation for $\varepsilon = \frac{1}{3}$.

(iii) x ε-approximately allocates the cost of an approximative TSP tour obtained by the Christofides heuristic for $\varepsilon = \frac{1}{2}$.

Proof. Let x as above denote the fair allocation of the associated LP-game. Let $c^*_{(HK)} := c_{(HK)}(N)$ denote the optimum value of (HK) and let L be the length of a shortest tour. Obviously,

$$\hat{x} := \frac{L}{c^*} x$$

is a vector in $\mathbb{R}^N$ satisfying

1. $\hat{x}(N) = L$ and
2. $\hat{x}(S) = \frac{L}{c^*} x(S) \leq \frac{L}{c^*} L(S) \leq \frac{3}{2} L(S)$.

□

We will see in the following Section 4 that it is impossible to achieve a better general bound than $\varepsilon = \frac{1}{3}$, even in the case of Euclidean distances.

Remark 3.8 *Potter, Curiel and Tijs [20] have shown that in the case of a distance function induced by the Euclidean metric for a planar arrangement of points, any convex arrangement of players guarantees a fair cost allocation. As it was shown by Fekete and Pulleyblank [6], any such arrangement has a moat packing with cost equal to the length of the optimal tour, implying that this special case is covered by our above approach.*

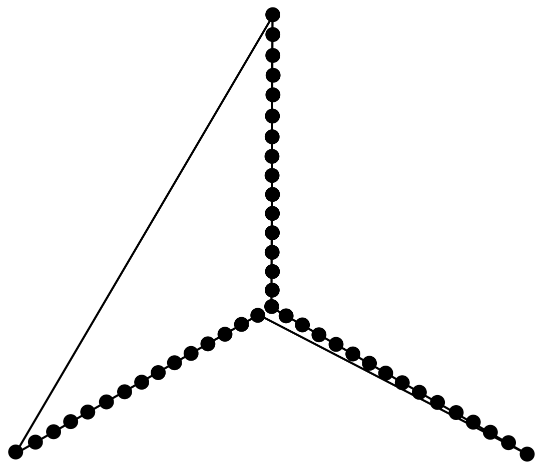

Figure 6: A generic example with empty core

Remark 3.9 *We would like to point out that the above worst case estimate for the ratio* $\frac{L}{c^*_{(HK)}}$ *is far from the average case ratio. It can be shown (cf. [23],[8])) that if* $p_1, \dots, p_n, \dots$ *are independently uniformly distributed in the unit square, and* $L(p_1, \dots, p_n)$ *denotes the length of a shortest tour through* $p_1, \dots, p_n$ *and* $c^*_{(HK)}(p_1, \dots, p_n)$ *denotes the optimum value of the Held-Karp relaxation, then there exist constants* β_{TSP} *and* $\beta_{(HK)}$*, such that*

$$\frac{L(p_1, \dots, p_n)}{\sqrt{n}} \overset{n\to\infty}{\longrightarrow} \beta_{TSP} \text{ a.s.}$$

$$\text{and} \quad \frac{c^*_{(HK)}(p_1, \dots, p_n)}{\sqrt{n}} \overset{n\to\infty}{\longrightarrow} \beta_{(HK)} \text{ a.s.}$$

Numerical experiments are reported [8] saying that $\beta_{TSP} \approx 0.709$ and $\beta_{(HK)} \approx 0.7$. This means that in the average one can expect ε-approximately fair allocations for $\varepsilon \approx 0.013$.

4 More examples with empty core and a lower bound on ε

In this section we present an infinite family of examples with empty core. We also show that the bound $\varepsilon \leq \frac{1}{3}$ which arises from the conjecture on the Held-Karp bound is tight by displaying a geometric instance where this bound is asymptotically met.

First, consider again an isoceles triangle with center 0. On each "spoke", we place a large number of points at small distance, see Figure 6.

Suppose that the length of a spoke equals 1, i.e. the triangle has side length $\sqrt{3}$. Then the length of an optimum tour comes arbitrarily close to $L = 4 + \sqrt{3}$. On the other hand, a simple argument, similar to that of Section 2, yields that every potential core vector $x \in \mathbb{R}^N$ must satisfy $x(N) \leq \frac{3}{2}(2 + \sqrt{3})$. This shows that such instances of TSP-games cannot have ε-approximate core allocations for $\varepsilon \leq \frac{2(4+\sqrt{3})}{3(2+\sqrt{3})} - 1 = 0.0239\dots$

This observation can be generalized to wheels with an arbitrary odd number $2k + 1$ of spokes of length 1, see Figure 7 for the case $k = 3$.

If s is the shortest distance between the endpoints of two adjacent spokes, the shortest tour

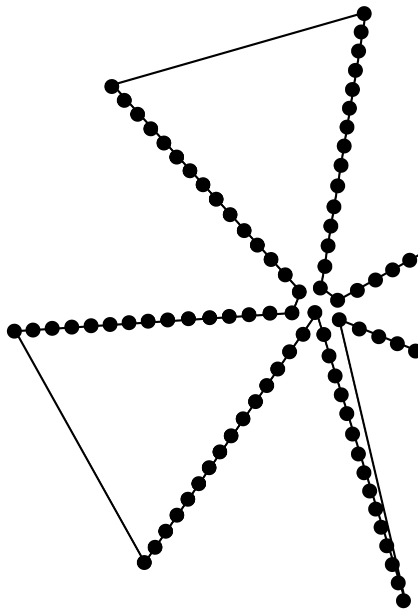

Figure 7: Odd stars have an empty core

has length close to $L = ks + 2k + 2$, while any potential core vector must satisfy

$$x(N) \leq \frac{(2k+1)(2+s)}{2} = ks + 2k + 1 + \frac{s}{2}.$$

Since $\frac{s}{2} < 1$, the core must be empty. Furthermore, there are no ε-approximately fair allocations for

$$\varepsilon \leq \frac{ks + 2k + 2}{ks + 2k + 1 + \frac{s}{2}} - 1 = \frac{1 - \frac{s}{2}}{ks + 2k + 1 + \frac{s}{2}}.$$

A natural question arising in this context is to determine the worst case bound for ε. We have seen in the previous section that (subject to the conjecture on the Held-Karp bound) there is a way to guarantee $\varepsilon \leq \frac{1}{3}$. In the following, we present an example with ε arbitrarily close to this bound of $\frac{1}{3}$.

Theorem 4.1 *The bound $\varepsilon \leq \frac{1}{3}$ resulting from the conjecture on the Held-Karp bound is best possible.*

Proof. Consider a set of n points, equally distributed on 3 columns, as shown in Figure 8.

Let h denote the height of the columns. Since the distance d between two columns can be made arbitrarily small (provided we place a sufficient number of points on each column), a shortest tour has length approximately equal to $4h$. On the other hand, any coalition consisting of two of the three columns is willing to pay at most $2h + 4d$. Hence any potential core vector must satisfy $x(N) \leq \frac{3}{2}(2h + 4d) = 3h + 6d$. This shows that there are no ε-approximately fair allocations for $\varepsilon \leq \dfrac{4h}{3h + 6d} - 1$. If we let d tend to 0, we get the desired result. □

Combined with our results in Section 3, this bound implies a previous result due to Goemans [8], stating that there are TSP instances for which the ratio between the length of a shortest tour and the optimal value of the Held-Karp relaxation is arbitrarily close to $\frac{4}{3}$.

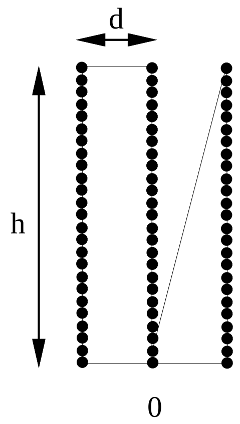

Figure 8: An example with $\varepsilon \longrightarrow \frac{1}{3}$

5 Concluding remarks

A proof of the "$\frac{3}{4}$-conjecture" on the Held-Karp relaxation would completely settle the question about the worst case analysis of the ε-approximability problem. On the other hand, it may be possible to find completely different methods of constructing approximately fair allocations with guaranteed worst case approximation error $\varepsilon = \frac{1}{3}$. For example, Faigle and Kern [5] propose some allocation rules based on the minimum spanning tree allocation and the distance from the supply node. An empirical study of randomly generated small instances ($n \leq 10$) seems to indicate that the Held-Karp allocation rule is by far preferable to other heuristic rules based on minimum spanning trees and relative distance functions, cf. Hunting [11].

Another interesting question concerns the average case behavior. We conjecture that if the points are independently and uniformly generated, say, in the unit square, then the probability of the core being empty tends to zero as $n \longrightarrow \infty$, where n is the number of players. One way of approaching this problem may be using the so-called *zero-or-one law* from probability theory, see Feller [7], Theorem 3, Chapter IV.7, volume II. This is easily seen to imply the following: Let X_0, X_1, $X_2, \ldots$ be independent random variables that are uniformly distributed in the unit square. Let $A \subseteq ([0,1]^2)^\infty$ be defined as

$$A := \{(a_0, a_1, a_2, \ldots) \mid \text{ the TSP game has nonempty core for almost all begin sequences}\}.$$

Then either $P(A) = 0$ or $P(A) = 1$, where P is the infinite product measure on $([0,1]^2)^\infty$.

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