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Corollary 23 *If A is covered by a martingale $d(w)$, and a there is a function $\hat{d}_r(w)$ such that $|d(w) - \hat{d}_r(w)| \leq 1/r^2$ and the i^{th} bit of $\hat{d}_r(w)$ can be computed on input i, then A has an exactly-computed null cover whose value is output in binary.*

Proof. Such a machine can output the polylog bits in the cover of Theorem 22. ∎

Corollary 24 *Fix a base p. If A has a martingale d with a computation $\hat{d}_r$ such that $|d - \hat{d}_r| \leq 1/r^2$ and the i^{th} base-p digit of $\hat{d}_r$ can be computed on input i, then A has a base-two cover.*

Proof. For each of polylog positions i in Theorem 22, we can compute the i^{th} bit of the binary representation of $d(w)$. ∎

The situation is similar for density systems, but the density system notation makes the situation *appear* worse. Since martingales are normalized in the sense that $d(\lambda) = 1$, when we claim that $|d(w) - \hat{d}_r(w)| \leq f(r)$ we are really giving a *relative* error. To get comparable results for density systems, it seems we need a computation $\hat{d}_{k,r}(w)$ with $|d_k(w) - \hat{d}_{k,r}(w)| \leq 2^{-k}/r^2$ (exponential in k but a power in r). But relaxing the precision of a computation of a density system from $1/2^r$ to $1/2^k r^2$ is no big feat, since given a computation satisfying the latter, it's easy to get a computation satsifying the former: $\hat{d}_{k+r,r}$ satisfies $|d_k(w) - \hat{d}_{k,r}(w)| \leq 2^{-r}$. Next, while a martingale can easily output 1 in binary, a density function d_k cannot output 2^{-k} in binary in polylog time, which would be analogous. Therefore density systems must settle for scientific notation. We omit the density-system analogs of the above corollaries.

6 Conclusions

The study of resource-bounded measure is still new, and it is useful to note that the definitions presented in [L92] have evolved slightly over time. Still, a large and growing body of results have shown that resource-bounded measure is a useful tool providing surprising connections to other questions in computer science [RSC].

The extension of this notion to small classes such as P is a much newer notion, and although the results of [AS] have shown that interesting results can be obtained using one definition of a measure on P, we should not be surprised if this notion evolves as further experience is gained.

This paper represents the next step of such an evolution. Although we were able to show here that the measure of [AS] is robust under many changes to the details of the definition, we have learned the surprising fact that one can obtain strictly more measurable sets by considering betting strategies that throw away information periodically.

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covers at least the same set as d. Also, E and $\hat{d}$ and therefore d' can be computed by a $\Gamma'(\mathrm{P})$ machine. (At first, the division might make it seem as though a $\Gamma'(\mathrm{P})$ machine would have trouble computing this. However, without loss of generality $\hat{d}_0(\lambda) = 2$ (since this approximates $d(\lambda) = 1$ within $1/(0^2+1)$). Thus this is division by 2^3, which is easy in this model.) Finally,

$$\begin{aligned}
d'(w) &= \hat{d}_{|w|}(w) + E(|w|) \\
&\geq d(w) - \frac{1}{|w|^2+1} + E(|w|) \\
&\geq \frac{d(w0)+d(w1)}{2} - \frac{1}{|w|^2+1} + E(|w|) \\
&\geq \frac{\hat{d}_{|w0|}(w0) - 1/(|w0|^2+1)}{2} \\
&\quad + \frac{\hat{d}_{|w1|}(w1) - 1/(|w1|^2+1)}{2} \\
&\quad - \frac{1}{|w|^2+1} + E(|w|) \\
&\geq \frac{\hat{d}_{|w0|}(w0) + \hat{d}_{|w1|}(w1)}{2} + E(|w|+1) \\
&= \frac{d'(w0)+d'(w1)}{2}
\end{aligned}$$

so d' obeys the average law (without conservation). ∎

As in Sect. 3, we have to treat conservative martingales specially:

Theorem 20 *Under the hypotheses of Theorem 19, if d is conservative then d' of the conclusion can be made conservative.*

Proof. Note the d' constructed in Theorem 19 is generally not conservative.

We will parallel Theorem 4, by constructing a slothful two-sided quasipolynomially precise cover. The construction of a conservative cover from a slothful cover preserves two-sided quasipolynomially precision.

We assume powers of two are in the dependency sets. Paralleling Theorem 4, put $i = \max G_{\hat{d},\lceil|w|\rceil,\lceil|w|\rceil} \cap \{0,\ldots,|w|\}$. We note that except on a set of polylog size, $\hat{d}_{|w|}(w)$ makes changes of size at most $1/(|w|^2+1)$. In Theorem 4 these small changes were absorbed by adding $3\cdot 2^{-i}$ to $\hat{d}_{\lceil|w|\rceil}(w[0..i])$; here we add $E(i)$. ∎

Next we observe that these martingales can't grow too quickly:

Theorem 21 *Let d be a computable martingale. Then $d(w) \leq 2^{\mathrm{polylog}(|w|)}$.*

Proof. We give the proof for exactly-computed d; the general case is similar. Let $i < j$ be consecutive elements of $G_{d,|w|}$. Then by definition of dependency set, $d(z) = d(z')$ if $|z| = |z'| = j \leq |w|$, and z agrees with z' on bits $0,\ldots,i$ and on bit j. By the average law, $d(w[0,\ldots i])$ is at least the average of $d(z)$ over all z of length j and extending $w[0,\ldots i]$. There are only two values these z's can take, depending on $z[j]$. Since we can't have half the z's be greater than twice the average, we conclude that $d(w[0,\ldots,j]) \leq 2d(w[0,\ldots,i])$, so d changes, and at most doubles, on a dependency set. ∎

Combining the last two theorems, we get

Theorem 22 *If A is covered by a (conservative) martingale d, then A is covered by a (conservative) martingale with at most polylog-many nonzero bits on either side of the radix point.*

Proof. By Theorem 21, d has at most polylog-many bits to the left of the radix point.

As for the right of the radix point, we must consider conservative and nonconservative martingales separately. If d is nonconservative, we assume by Theorem 19 that d is exactly computed, truncate d after bit $2\log r$, getting $\hat{d}$ within $1/r^2$ of d, then add $E(|w|) \approx 4/|w|$ (from the proof of Theorem 19) to restore the average inequality.

Finally, if d is conservative, we make a slothful-but-not-necessarily-conservative martingale as above. At *this* stage we truncate to polynomial precision, and add a slothful variant of $E(|w|)$ to restore the (nonconservative) average inequality. Finally, construct a conservative martingale from the slothful martingale by the construction of Theorem 5, preserving the number of significant bits. ∎

approaches fail, and a new model of computation *would* be needed. Such a model would likely be quite complicated, however, in regards to giving the machine the length of its input: $\mu_{\Gamma'(\mathrm{PSPACE})}$ machines get the exact length of their input and make good use of this information, whereas $\mu_{\Phi(\mathrm{PSPACE})}$ machines *must not* be given their exact input length. (This is so that $M(z)$ and $M(w)$ are initially the same computation for $z \sqsubseteq w$ and approximately the same length, which guarantees that a PSPACE machine can diagonalize against M. See [M].)

5 Quasipolynomial Precision

A cover d is required to have a computation $\hat{d}_r$ such that for all w we have $|d(w) - \hat{d}_r(w)| \leq 2^{-r}$, and in Theorem 2, we showed that a set A has such a cover iff A has an exactly computable cover. In this section, we first show how to weaken the hypothesis to $|d(w) - \hat{d}_r(w)| \leq \frac{1}{r^2}$ (one could substitute any reasonable function $f(r)$ such that $\sum_{r=1}^{\infty} f(r)$ is finite). Next, we observe that $d_k(w) \leq 2^{\mathrm{polylog}(k+|w|)} d_k(\lambda)$, since $d_k(w[0..j]) \leq 2d_k(w[0..i])$, where i and j are consecutive elements of $G_{d,|w|,k}$. From this it follows that two-sided quasipolynomial precision suffices for our machines, i.e., our machines need only output a polylog number of significant bits to either side of the radix point. We draw two important corollaries:

- The usual way that sublinear-time machines compute functions is to output the i^{th} bit as a function of i. The most natural way is to output the value in binary. In [AS] functions output "differences of formal sums of powers of two." Now, since two-sided quasipolyonmial precision suffices, we see that all three conventions are equivalent.

- If a set A has a cover d with approximation $\hat{d}$, such that $|d - \hat{d}_r| \leq \frac{1}{r^2}$ and $\hat{d}$ is computable if output is expressed as a "differences of formal sums of powers of p" for $p \neq 2$, then A also has a base-two cover. (Essentially, this is because two-sided quasipolynomial precision is a concept independent of base.)

The results of this section hold for both $\Gamma(\mathrm{P})$- and $\Gamma'(\mathrm{P})$- martingales.

The following is similar to Theorem 2, but exponentially better, in the sense that the assumption about the goodness of approximation has been relaxed:

Theorem 19 *Let A be a set for which there exists a martingale d and a computable function $\hat{d}_r(w)$ such that for all w we have $|d(w) - \hat{d}_r(w)| \leq \frac{1}{r^2+1}$. Then A has an exactly computed martingale d'.*

Proof. We reprove theorem 2, using an approximation to $4/r \approx \sum \frac{1}{r^2}$ instead of $2^{-r} \approx \sum 2^{-r}$. The martingale d obeys the average inequality, so $\hat{d}_{|w|}(w)$ is close to obeying the average inequality. We add to $\hat{d}_{|w|}(w)$ a decreasing function $E(|w|)$, which obeys the average inequality "with room to spare:" enough room to absorb the average inequality error $(\hat{d}_{|w0|}(w0) + \hat{d}_{|w1|}(w1))/2 - \hat{d}_{|w|}(w)$.

We define

$$E(r) = \begin{cases} 4\frac{3\lceil r\rceil - 2r}{\lceil r\rceil^2} & r > 0 \\ 6 & r = 0 \end{cases}$$

Note that $E(r) = 4/r$ if r is a power of 2, and $E(r)$ is linear between powers of 2, so $E(r) > 4/r$ there. Intuitively,

$$\begin{aligned} E(r) - E(r+1) &\sim 4/r - 4/(r+1) \\ &\sim 4/r^2 \\ &\geq 2|d - \hat{d}_r|. \end{aligned}$$

Formally, one easily verifies that $E(r) = E(r+1) + 2/\lfloor r\rfloor^2$ for $r > 0$, and in any case

$$E(r) - \frac{1}{r^2+1} - \frac{1}{(r+1)^2+1} \geq E(r+1).$$

Put

$$d'(w) = \frac{\hat{d}_{|w|}(w) + E(|w|)}{\hat{d}_0(\lambda) + E(0)}.$$

Note that the normalizing denominator $\hat{d}_0(\lambda) + E(0)$ is constant, and $E(|w|)$ is positive, so d'

falls by at least $2^{2+2|n|-n}d(\omega[0..2^n-1])$. For these j's we give up on a match but make sure d decreases. But there are other j's such that no y makes d drop much, and so by Markov's inequality most y's make d drop or rise by very little. Combining this with the last paragraph, we've found a j and a y such that $\omega[R^0_{j,n}] = y$ allows C and $\omega[R^1_{j,n}] = y$ makes d not rise much. We've found our match; fill in the other bits according to the path of decreasing d.

Now more formally and quantitatively:

Let D be the value of $d(\omega[0..2^n-1])$. (That is, D is the value of d after treating the previous n.)

Consider only the C's that are reached by at least $1/n^2$ of their fair share of w's (e.g., if there are 2^{n^k} C's, only the C's reached by at least $2^{2^{n-1}}/(2^{n^k}n^2)$ of the $2^{2^{n-1}}$ w's). Note this leaves at least one C. Also, it leaves at least $(1-1/n^2)$ of the w's, so the average, over remaining w's, of $d(w)$ is at most $(1+2/n^2)D$. Fix one of the remaining C's with $d(C) \le (1+2/n^2)D$. Let $W = \{w : C \text{ is reached by } w\}$; note that $|W| \ge 2^{2^{n-1}}/(2^{n^k}n^2)$.

Initialize j to 0, and initialize S to the empty string. (In general, as j changes, S will contain the bits in positions $R^1_{0,n} \cup \cdots \cup R^1_{j-1,n}$). We will talk about $d(CS)$, and mean $d(zS)$ where z is any string that takes the machine computing d to configuration C.

For half of the $2^{n-2|n|-1}$ j's, at least the fraction $3/4$ of the y's at j allow C. Otherwise, if $a \le 2^{n-2|n|-2}$ of the j's have this property, then

$$\begin{aligned}|W| &\le (\tfrac{3}{4}2^{n^2})^{-a+2^{n-2|n|-1}}(2^{n^2})^a \\ &= (3/4)^{-a+2^{n-2|n|-1}}(2^{2^{n-1}}) \\ &< (3/4)^{2^{n-2|n|-2}}(2^{2^{n-1}}) \\ &< 2^{2^{n-1}}/(2^{n^k}n^2),\end{aligned}$$

a contradiction. Let A be the set of j's such that $3/4$ of the y's at j allow C.

Consider the j's in increasing order. If $j \notin A$ then reset $S \leftarrow Sy$, for y along the path of decreasing d. If $j \in A$, then if there's a setting j of $\omega[R^1_{j,n}]$ with $d(CSy) < d(CS) - 2^{3+2|n|-n}D$, reset $S \leftarrow Sy$. After considering $2^{n-2|n|-3}$ of the j's in A (i.e., less than half the j's in A), either the value of $d(CS)$ has decreased to zero or we've found a j such that for all y we have $d(CSy) \ge d(CS) - 2^{3+2|n|-n}D$. Then, by Markov, at least $3/4$ of the y's make $d(CSy) < d(CS) + 2^{5+2|n|-n}D$. Since $j \in A$, $3/4$ of the y's at j allow C, so $1/2$ of the y's satisfy both

- $\omega[R^0_{j,n}] = y$ allows C
- $d(CSy) < d(CS) + 2^{5+2|n|-n}D$.

Fix one of these y's, extend $S \leftarrow Sy$, find a setting of $\omega[\Phi^0_n]$ witnessing that y at j allows C, and finally set $\omega[R^1_{>j,n}]$ along the path of decreasing d.

We have $d(\omega[0..2^{n+1}-1]) \le d(C) + 2^{5+2|n|-n}D \le (1+O(1)/n^2)D$, which was our goal.

It remains to modify the above proof to produce a language in PSPACE.

Instead of finding C with $d(C) < (1+2/n^2)D$ reached by $1-1/n^2$ of the w's, find C with $d(C) < (1+2/n^2)D$ such that for half of the j's as least $3/4$ of the y's at j allow C (such a C exists by the previous argument). This can be done by cycling through all j's (counting as we go), for each j cycling through the y's, and for each (j,y) using a Savitch divide-and-conquer technique to determine if y at j allows C. Later, as we consider the j's in turn, instead of maintaining S, maintain only the configuration of S. The rest of the proof is similar. ∎

To cover ODD a martingale needs to look at all its input, whereas to cover DOUBLE a martingale needs to be able to look at input in a dynamically-determined order. In this regard these examples are complementary, and we see that the two measures are very different.

We've shown our notion of measure on PSPACE is incomparable to that of [M]. One might ask about a join, a measure on PSPACE strictly richer than both, and one might hope that a join can be constructed without defining a new model of computation (say by adding a Γ'(PSPACE) martingale to a Φ(PSPACE) martingale.) It seems, however, that "clean-hands"

Definition 15 *Let* MATCH *be the set of sequences ω such that for almost all n there exists j with $\omega[R^0_{j,n}] = \omega[R^1_{j,n}]$.*

Note even the infinitely-many-n version of MATCH has Lebesgue measure zero, which is shown by using the Borel-Cantelli lemma: MATCH is the limsup of n-sections having measure

$$1 - \left(1 - \frac{1}{2^{n^2}}\right)^{2^n - 2|n| - 1} \approx 1 - e^{-2^{n-n^2}}.$$

Since 2^{n-n^2} is small we have $e^{-2^{n-n^2}} \approx 1 - 2^{n-n^2}$, and so $(1 - e^{-2^{n-n^2}}) \approx 2^{-n^2}$ is exponentially small. Since $\sum(1-e^{2^{n-n^2}}) < \infty$, we can apply the Borel-Cantelli lemma and conclude that MATCH has measure zero.

Theorem 16 *The set* MATCH *has* Γ'(PSPACE)*-measure zero.*

Proof. Similar to Theorem 11 above (this actually covers the infinitely-many-n version).

The desired martingale bets evenhandedly through first phases. Let R'_k enumerate all the $R^1_{j,n}$'s, so $k < 2^n$. On input w, if $s_{|w|} \in R^1_{j,n} = R'_k$ then d bets all $1/k^2$ of its capital that $w[R^1_{j,n}] = w[R^0_{n,j}]$. Since $w[R^1_{j,n}] = w[R^0_{n,j}]$ with probability $2^{-n^2} < 1/k^3$, when this event occurs d wins k^3 per unit bet, i.e., $k^3/k^2 = k \nearrow \infty$. Since $\sum 1/k^2 < \infty$, d never runs out of money. ∎

Theorem 17 *The set* MATCH $\cap$ PSPACE *does not have* $\mu_{\Phi(\text{PSPACE})}$*-measure zero.*

Proof. First, notation and an overview:

We will concentrate on one n at a time for the bulk of the proof. The variable $y, |y| = n^2$, will denote a setting of some $R^i_{j,n}$. The variable $w, |w| = 2^{n-1}$, will denote a setting of Φ^0_n. A *configuration* will mean a configuration of the machine after reading through bit $2^n + 2^{n-1}$ (i.e., just after reading the last bit of Φ^0_n and before reading the first bit of Φ^1_n).

We are given a conservative martingale d computed by a $\log^k(n)$-space-bounded online Turing machine, which we may assume works in the limit (see [M]): ω will is covered by d if $\lim_{n\to\infty} \omega[1..n]$ exists and is infinite. Therefore, for a counterexample it suffices to construct a sequence ω with $\{d(\omega[2^n]) : n \in \mathbf{N}\}$ bounded. We will let d increase by a factor of $1 + O(1/n^2)$ at the n^{th} stage, and, since $\prod(1 + c/n^2) < \infty$, d will be bounded.

We will use the following form of Markov's inequality:

Lemma 18 *If the average over a multiset A of reals is at most 1, and all elements of A are greater than $1 - \epsilon$, then for all $a > 1$ at least $1 - 1/a$ of the elements are $\leq 1 + \epsilon a$.*

Proof. Otherwise, if more than $1/a$ of the elements are more than $1+\epsilon a$, then even if the other $(1 - 1/a)$ elements are all the minimum value of $1 - \epsilon$, that gives an average value of

$$\begin{aligned}
&(1/a)(1 + \epsilon a) + (1 - 1/a)(1 - \epsilon) \\
&= 1 + (1/a)\epsilon a - (1 - 1/a)\epsilon \\
&= 1 + \epsilon - \epsilon + \epsilon/a \\
&> 1.
\end{aligned}$$

♣

Now, the overview. There may be a configuration reached by only one string w of length 2^{n-1}. From that configuration, a martingale is prepared to make many successful bets on the second phase. So we will begin by excluding from consideration for our language the configurations reached by too few w's. This leaves at least one configuration, since some configuration is reached by the average number of w's. Also, the remaining configurations are reached by most of the w's, so the average of d over the remaining configurations is not too large. Thus there is some configuration C reached by a large number of w's and with $d(C)$ not rising much. (These comments will be made quantitative below.)

If C is reached by many w's, then for many j there are many settings y of $\omega[R^0_{j,n}]$ that are consistent with C ("$\omega[R^0_{j,n}] = y$ allows C").

For at most half of the j's (i.e., $2^{n-2|n|-2}$ of the j's) can there be a setting y of $\omega[R^1_{j,n}]$ in which d

polynomial in n, and with dependency set R_j of size n^2.

In a setting having at least exponential resources, our desired martingale would be $\sum d_j/j^2$. In our subexponential setting we do not have enough time to compute an approximation to $\sum d_j/j^2$, so instead we do the following:

- Make sure (inductively) we have $1/(j-1)$ capital available before starting to bet on R_j.
- Bet on R_j using strategy d_j/j^2, risking just $1/j^2$ of our capital but winning $j^3 \cdot 1/j^2 = j$ for infinitely many j's.
- Before starting to bet on R_{j+1}, "throw away" the potential winnings of j, and assume we have only $1/(j-1) - 1/j^2 \geq 1/j$, enough to continue inductively.

Continuing in this way our winnings will be unbounded, yet we will be able to keep dependency sets small.

Define $d(w)$ as follows. Determine $j \approx |w|/\log|w|$ such that $s_{|w|} \in R_j$. Put

$$d(w) = \frac{1}{j} + \frac{d_j(w)}{j^2}.$$

We verify that this is a (nonconservative) martingale by checking that if $s_{|w|}$ is the last word of R_j then $s_{|wb|} \in R_{j+1}$, so

$$d(w) \geq 1/j \geq 1/(j+1) + 1/(j+1)^2 = \operatorname{avg}(d(wb)).$$

The dependency set $G_{d,|w|} = R_j$, and $|R_j| = n^2 \approx \log^2|w|$. Finally, let L be a language of density less than ϵ. Then the density of L is less than ϵ on R_j for infinitely-many j's, and for such j d climbs to $d_j/j^2 = j$ along L. ∎

In [AS] it is noted that the set of languages with density n^k (fixed k) is $\Gamma(\mathrm{P})$-null, whereas all of SPARSE is not. Thus [AS] presents a threshold density for $\Gamma(\mathrm{P})$-measure. Theorem 11 shows that $\Gamma'(\mathrm{P})$-measure is significantly stronger in this regard.

4.2 Space

Now we compare our PSPACE measure to that of [M]. We will exhibit two sets, ODD and MATCH, such that ODD is measurable by [M] but not by our measure, and MATCH is measurable by our measure by not by [M].

Definition 12 *Let* ODD *denote the set of languages L such that for each n, L has an odd number of words of length n.*

Note that ODD has Lebesgue measure zero.

Theorem 13 *The set* ODD *has $\mu_{\Phi(\mathrm{PSPACE})}$-measure zero.*

Proof. Immediate; also see [M]. ∎

Theorem 14 *The set* ODD $\cap$ PSPACE *does not have $\mu_{\Gamma'(\mathrm{PSPACE})}$-measure zero.*

Proof. Let d be a $\Gamma'(\mathrm{PSPACE})$-martingale. We will construct a language $L \in$ ODD $\cap$ PSPACE such that d is bounded along L.

We will define L for all words of length n at once. Suppose L is defined for all words of length less than n. First extend according to the path of decreasing d, as in the proof of Theorem 4 of [AS]; call this extended language L'. Thus d' is bounded along L'. For all large n, let x be the lexicographically largest string of length n such that $\operatorname{pos}(x) \notin G_{\operatorname{pos}(1^n)}$. Note that (by transitivity) for all $m \in (\operatorname{pos}(x), \operatorname{pos}(1^n)]$ we have $\operatorname{pos}(x) \notin G_m$. Define $L = L'$ except that $x \in L$ when that makes L odd. Then $d(w) \leq 2d(w')$ whenever w and w' are equal-length prefixes of L and L' respectively. Also, the possible final parity fix leaves $d(L[1..m]) = d(L'[1..m])$ for all other $m \leq \operatorname{pos}(1^n)$, so d obeys twice the bound on L that it obeys on L'. ∎

We now present a set measurable in the sense of [AS] but not in the sense of [M].

Partition Σ^* as follows. Write Σ^n as two phases $\Phi_n^0 \cup \Phi_n^1$, where Φ_n^i consists of words of length n beginning with i (so Φ_n^0 consists of the words in positions 2^n through $2^n + 2^{n-1}$). Partition each Φ_n^i into $2^{n-2|n|-1}$ regions $R_{j,n}^i$ of $2^{2|n|} > n^2$ lexicographically consecutive words.

Given the monotone, exactly computed cover d, on input w put $i = \max G_{d,\lceil |w| \rceil} \cap \{0, \ldots, |w|\}$, and put $d'(w) = d(w[0, \ldots, i])$. Since d' changes only on a dependency set, it is slothful. ∎

In summary:

Theorem 9 *All sets with covers have exactly computed covers. A set is covered by a density system iff it is covered by a martingale. If a set is covered by a conservative, monotone, regular,* or *limit cover, then it is covered by an exactly-computed, conservative, monotone, regular density system and covered in the limit by an exactly-computed, conservative, monotone martingale.*

∎

The reader is invited to convert a conservative density system into a martingale of the desired form by tracing the entire Fig. 1. We know of no way to do this other than using *all* of theorems 4 through 8.

3.3 Space

The notion of measure of [AS] can be defined for space as well as time. We conclude this section by comparing our conservative measure on PSPACE, denoted $\mu_{\Gamma(\mathrm{PSPACE})}$, with the measure of [M], here denoted $\mu_{\Phi(\mathrm{PSPACE})}$.

Theorem 10 *If a set has a $\mu_{\Gamma(\mathrm{PSPACE})}$ cover, than it also has a $\mu_{\Phi(\mathrm{PSPACE})}$ cover.*

Proof. It is sufficient to give a polylogspace online algorithm for the given cover d which, by the results of the previous section, can be assumed to be monotone.

On input w, a polylogspace online machine is furnished with the available workspace. From this it can compute $\log |w|$ and $\lceil |w| \rceil$ even before it sees input. The machine will first compute $G_{d,\lceil |w| \rceil}$. As the input is read, the machine can cache the bits in positions in $G_{d,\lceil |w| \rceil}$, then compute $d(w)$ from the cached bits alone. ∎

In [AS] it is shown that SPARSE does not have $\mu_{\Gamma(\mathrm{PSPACE})}$ measure zero. It is easy to see that SPARSE does have $\mu_{\Phi(\mathrm{PSPACE})}$ measure zero, so the measure of [M] is strictly richer.

4 Inequivalence

Henceforth we will denote the conservative measure by $\mu_{\Gamma(\mathcal{C})}$ and the nonconservative measure by $\mu_{\Gamma'(\mathcal{C})}$. In this section we show that the two measures differ almost exponentially in the largest f such that $\{L : L \text{ has density at most } f\}$ has measure zero. In the previous section we showed $\mu_{\Gamma(\mathrm{PSPACE})}$ is strictly weaker than $\mu_{\Phi(\mathrm{PSPACE})}$; here we show that $\mu_{\Gamma'(\mathrm{PSPACE})}$ is incomparable with $\mu_{\Phi(\mathrm{PSPACE})}$.

4.1 Conservative versus Nonconservative measure

In [AS] it was shown that SPARSE, the set of languages with at most polynomially many words of a given length, does not have an exactly-computable conservative cover. The theorems of the last section show that SPARSE has no conservative cover at all. In this section we show that SPARSE does have a nonconservative cover (which is not monotone, not regular if a density system, and not limit if a martingale). We also show that there are sets A and B such that $\mu_{\Gamma'(\mathrm{PSPACE})}(A) = 0$ but $\mu_{\Phi(\mathrm{PSPACE})}(A) \neq 0$ and $\mu_{\Gamma'(\mathrm{PSPACE})}(B) \neq 0$ but $\mu_{\Phi(\mathrm{PSPACE})}(B) = 0$, so the nonconservative version of our measure on PSPACE is incomparable with that of [M].

Theorem 11 *The set of languages with density less than $\epsilon < 1/2$ is $\mu_{\Gamma'(\mathrm{P})}$-null.*

Proof.

Partition Σ^* into consecutive regions $R_0, R_1, \ldots$ as follows. $R_0 = \{w : |w| < 16\}$, and for $n \geq 16$ the 2^n strings of length n are divided up into $2^{n-2|n|}$ blocks of $2^{2|n|} > n^2$ lexicographically consecutive words. Thus R_j consists of words of length n, for some $n > \log j$.

Let X_j denote the languages with density less than ϵ on the j^{th} region. By the Chernoff inequality, for some c that depends on ϵ,

$$\mu(X_j) \leq e^{-cn^2} \leq 2^{-3n} \leq \frac{1}{j^3}.$$

It is straightforward to construct a martingale d_j that climbs from 1 to j^3 on X_j, works in time as a conservative cover. Similarly, one can fix regularity, because if there is some prefix $z \sqsubseteq w$ such that $d_k(z) \geq 1$, we simply find the first such z and set $d_k(w) = d_k(z)$. Our sloth condition insures there are only polylog-many z's to check, so we can do this using the allowable resources. ∎

Theorems 4 and 5 together give us an exact computation lemma for conservative covers: Given a conservative cover make a slothful cover by theorem 4, then make a conservative, slothful (and hence exactly computed) cover by theorem 5.

Next, continuing around the square of Fig. 1, we mention a key relationship between density systems and martingales, mostly unchanged from the setting of Lutz:

Theorem 6 ([JLM]) *If a set is covered by a regular density system, then it is covered in the limit by a martingale.*

Proof. We have $d'(w) = \sum_{k=0}^{\infty} d_{2^k}(w)$ is a limit martingale with computation $\sum_{k=0}^{\log(|w|+r)} \hat{d}_{2^k,r}(w)$. ∎

Next, the up-arrow:

Theorem 7 *If a set is covered in the limit by a martingale, then it has a limit, monotone cover. This construction preserves sloth and conservation.*

Proof. Let d be a limit martingale covering a set A. We will construct a martingale d' and monotone dependency sets $G'_{d',n}$. By Theorem 2, assume that d is exactly computed and that the powers of 2 are in the dependency sets $G_{d,n}$. Put

$$G'_{d',n} = \{0 \ldots, n\} \cap \bigcup_{j \leq \log \lceil n \rceil} G_{d,2^j},$$

and put $d'(w) = d(w[1..\max G'_{d',|w|}])$. Note that the G''s are monotonic.

Since d is limit, we conclude that for $L \in A$ we have $\lim_{w \to \chi_L} d'(w) = \infty$.

We need to check that d' satisfies the average law. We need only check where d' changes, i.e., for $|w| \in G'_{d',|w|}$, we need to check that $d'(w_-) \geq \frac{d'(w)+d'(w')}{2}$, where w_- denotes w with the last bit dropped, and w' denotes w with the last bit flipped.

Suppose $|w| \in G'_{d',|w|}$. Then $d'(w) = d(w)$ and $d'(w') = d(w')$. If $|w_-|$ is a power of 2, then $|w_-| \in G_{d,|w_-|}$, so d' agrees with d on $\{w_-, w, w'\}$, and the average law is satisfied. Therefore we may assume $|w_-|$ is not a power of 2, so that $\lceil |w_-| \rceil = \lceil |w| \rceil$. Let $k = \max\{1..|w_-|\} \cap G_{d,\lceil |w_-| \rceil}$, so $d'(w_-) = d(w[1..k])$. Since $|w| \in G'_{d,|w|}$, by definition of G' we have $|w| \in G_{d,\lceil |w| \rceil}$, and by transitivity of dependency sets, $G_{d,|w|} \subseteq G_{d,\lceil |w| \rceil} = G_{d,\lceil |w_-| \rceil}$. We conclude that there is exactly one element of $G_{d,|w|}$ that is strictly bigger than k, and that is $|w|$. It follows that $d(z) = d(w)$ for any z with $|z| = |w|$ and z differing from w only in bits after the k^{th} and before the last. We conclude that

$$\begin{aligned} \frac{d'(w)+d'(w')}{2} &= \frac{d(w)+d(w')}{2} \\ &= \operatorname{avg} d(z) \\ &\leq d(w[1..k]) \\ &= d'(w_-), \end{aligned}$$

where the average is over z of length $|w|$ extending $w[1..k]$. (If d is conservative, then $\operatorname{avg} d(z) = d(w[1..k])$ and thus d' is conservative as well.) ∎

Finally, the left-arrow:

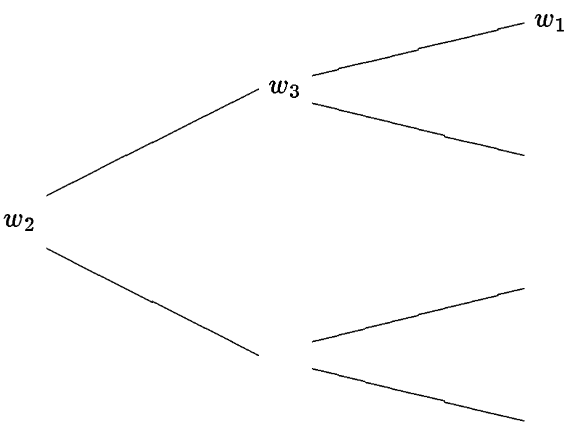
Theorem 8 *If a set has a monotone cover, then it has a slothful, monotone cover. This construction preserves all other properties.*

Proof. Using Theorem 2, we may assume the cover is monotone and exactly computed.

Fix w, let i be the largest element of $G_{d,|w|}$, let j satisfy $i \leq j \leq |w|$, and let w' extend $w[0,\ldots,i]$ with $|w'| = |w|$. By definition of dependency set, $d(w) = d(w')$. Since the dependency sets are monotone, i is also the largest element of $G_{d,j}$, so $d(w[0,\ldots,j]) = d(w'[0,\ldots,j])$. By the average law, $d(w[0,\ldots,j]) \geq d(w[0,\ldots,j'])$ for $i \leq j \leq j' \leq |w|$. (That is, d is constant on levels, and nonincreasing from level to level.)

Without loss of generality, assume all small enough powers of 2 are in the dependency set.

Figure 2: Tree in Theorem 4



Fix $w = w_1$, and let $w_2 = w[0..i]$ (see Fig. 2). Let T be the complete tree at w_2 of height $|w_1| - i$; we will show that d is constant on T. By definition of dependency set, d is constant on the set of leaves of T, and for any $w_3 \sqsupseteq w_2$ with $|w_3| \leq |w_1|$, by the average law at w_3 and succeeding nodes, we conclude that $d(w_3)$ equals the common leaf value of d.

Since $G_{d,i} \subseteq G_{d,|w|}$, it is now easy to see that for all $j \notin G_{d,|w|}$, $d(w[0..j-1]) = d(w[0..j])$.

♣

Now we turn to not-necessarily-exactly-computed martingales. The idea is that an approximation $\hat{d}_{|w|}(w)$ is close to the martingale, so that the above proof shows the martingale is close to slothful, i.e., except on a set of size polylog, the martingale makes changes of size at most $2^{-|w|}$. These changes are small enough that by increasing the martingale slightly we can absorb them.

Fix w, and consider the computable function $\hat{d}_{|w|}(w)$. As above, assume $\max(G_{\hat{d},|w|}) < |w|$, and note that by definition of dependency set, for all w' with $|w'| = |w|$ such that $w[0..i] = w'[0..i]$, we have $\hat{d}_{|w'|}(w') = \hat{d}_{|w|}(w)$. Since $\hat{d}$ is a computation, $|d(w') - \hat{d}_{|w|}(w)| \leq 2^{-|w|}$; i.e., d varies by at most $2 \cdot 2^{-|w|}$ on a level. Let i be the maximal element of $G_{d,|w|}$. By the conservation law, $|d(w[0\ldots i]) - \hat{d}_{|w|}(w)| \leq 2^{-|w|}$, and as above the complete tree at $w[0\ldots i]$ of height $|w| - i$ is within $2^{-|w|}$ of $\hat{d}_{|w|}(w)$.

Without loss of generality, assume that all powers of 2 at most n are in $G_{d,n}$. This means take the original dependency set, throw in the appropriate powers of 2, *and* take the transitive closure.

Given a conservative martingale d, we construct a slothful martingale d' as follows. On input w, put $i = \max G_{\hat{d},\lceil|w|\rceil,\lceil|w|\rceil} \cap \{0, \ldots, |w|\}$ (recall that $\hat{d}_r$ has a subscript, so the dependency set for $\hat{d}_r(z)$ has the form $G_{\hat{d},|z|,r}$). Put

$$d'(w) = \frac{1}{4}\left(\hat{d}_{\lceil|w|\rceil}(w[0,\ldots,i]) + 3 \cdot 2^{-i}\right).$$

Then clearly d' is computable. As in Theorem 2, d' satisfies the average law (but without equality), and $d' \geq d/4$ so d' covers the same set as d. Finally, note that both $\hat{d}_{\lceil|w|\rceil}(w[0,\ldots,i])$ and $3 \cdot 2^{-i}$ are slothful, so their sum is slothful as well. The factor of $\frac{1}{4}$ merely maintains the property that $d'(\lambda) = d(\lambda) = 1$.

∎

Next we assume sloth and prove conservation and regularity at once.

Theorem 5 *If a set is covered by a slothful null density system, then it is covered by a conservative, slothful, regular null density system. If a set is covered by a slothful martingale, then it is covered by a conservative, slothful martingale. This construction preserves monotonicity and limit coverage (for martingales).*

Proof. The proof is the same for martingales and density systems.

Note that a slothful density system d is exactly computed, by definition. We produce a density system d' of the desired form.

We handle conservation and regularity at once. In the setting of Lutz et al, the covers have at least linear time. It has been shown [JLM] that one can take an exactly-computed but not-necessarily-conservative cover $d(w)$, check the linearly many prefixes $z \sqsubseteq w$ where d may fail to be conservative, and fix each one, that is, output

$$d(w) + \sum_{z \sqsubseteq w}\left(d(z) - \frac{d(z0) + d(z1)}{2}\right)$$

so

$$\begin{aligned} d'(w) &\geq d(w) + 2 \cdot 2^{-|w|} \\ &\geq \frac{d(w0)+d(w1)}{2} + 2 \cdot 2^{-|w|} \\ &\geq \frac{d'(w0) - 4 \cdot 2^{-|w0|}}{2} \\ &\quad + \frac{d'(w1) - 4 \cdot 2^{-|w0|}}{2} \\ &\quad + 2 \cdot 2^{-|w|} \\ &= \frac{d'(w0)+d'(w1)}{2} \end{aligned}$$

It is clear that coverage in the limit is preserved; it remains to show that monotonicity is preserved. So assume $G_{d,m,r} \subseteq G_{d,n,r}$ for all r and for $m \leq n$. Then, for all $m \leq n$ we have

$$\begin{aligned} G_{d',m} &= G_{d,m,m} \\ &\subseteq G_{d,m,n} \text{ by Lemma 1} \\ &\subseteq G_{d,n,n} \text{ by hypothesis} \\ &= G_{d',n} \end{aligned}$$

∎

Finally we show that for nonconservative covers the density system and martingale formulations coincide.

Theorem 3 *A set covered by a density system is covered by a nonconservative martingale.*

Proof. Given a density system d_k, note that $2^k d_{2k}$ is a density system unbounded on any set covered by d_k, so the "sparse sum" $d = \sum 2^{2^k} d_{2 \cdot 2^k}$ is a martingale covering the same set. If $\hat{d}_{k,r}$ is a computation for d_k, then, as in [AS], the martingale d has computation

$$\hat{d}_r(w) = \sum_{k=0}^{\log(r+|w|)} 2^{2^k} \hat{d}_{2 \cdot 2^k, r+2 \cdot 2^k}(w).$$

∎

This construction destroys conservation, so one cannot use it to convert a conservative density system into a conservative martingale. The entire next subsection is devoted to this conversion.

Figure 1: Order of constructions

$$\begin{array}{ccccc} \text{conservative} & \leftrightarrow & \text{slothful} & \leftarrow & \text{monotone} \\ & & \downarrow & & \uparrow \\ & & \text{regular} & \rightarrow & \text{limit} \end{array}$$

3.2 Equivalence of Niceness conditions

In this subsection we show the niceness conditions are equivalent. Requiring any one of the niceness conditions places a severe restriction on the measure, as is discussed in the next section.

We prove the equivalence according to Fig. 1. The reader is cautioned that some of the constructions destroy some properties. Fig. 1 is used as follows:

- Given a martingale or density system covering a set A and satisfying any of the five properties in Fig. 1, one can construct a limit martingale and a monotone density system covering A.
- Given a limit martingale one can make the martingale in turn monotone, slothful and conservative, and these properties accumulate.
- Given a monotone density system one can make the density system in turn slothful, regular and conservative, and these properties accumulate.

We recall that regularity is predicated of density systems and limit coverage is predicated of martingales.

Theorem 4 *If a set is covered by a conservative cover, then it is covered by a slothful cover.*

Proof. We will give the proof for increasingly general cases. First we review an argument in [AS] for exactly computed martingales, then give the proof for martingales that are not exactly computed. Density systems are handled similarly.

We show that for any n, if i is the maximum element of $G_{d,n}$ and $|w| = n$, then $d(w[0..i]) = d(w[0..j])$ for any j in the range $i \leq j \leq n$.

- d is *conservative* if $d(w) = \frac{d(w0)+d(w1)}{2}$. (A function satisfying

$$d(w) \geq \frac{d(w0) + d(w1)}{2}$$

but not necessarily with equality is sometimes called a supermartingale.)

- d is *monotone* if $G_{d,m} \subseteq G_{d,n}$ for $m \leq n$.

- d is *slothful* if (1) d is exactly computed, and (2) given input (n, k), we can compute in time polylog in $n + k$ a list of indices from $\{0, \ldots, n\}$ such that, if i is *not* in this list, then for any string w of length n, $d_k(w[0..i-1]) = d_k(w[0..i])$. (If d is a martingale disregard k.) The condition of sloth is merely a technical notion that is useful in our proofs.

In subsequent sections we will prove that without loss of generality one may assume that the density systems come from martingales, i.e., $d_k = 2^{-k}d$ for a martingale d, and that d and d_k are exactly computable. We will also show, curiously, that if one assumes *any* of the other niceness conditions above, then *all* the others follow.

3 Equivalence

First we discuss several simplifications that are possible without assuming the niceness conditions. Later we discuss the equivalence of the niceness conditions.

3.1 General Equivalence

Lemma 1 *Let $G_{\hat{d},|w|,r}$ be dependency sets for a computation $\hat{d}_r(w)$. Without loss of generality we may assume $G_{\hat{d},|w|,r} \subseteq G_{\hat{d},|w|,s}$ for $r \leq s$.*

Proof. We note that without loss of generality we can replace $\hat{d}_r$ with $\hat{d}_{\lceil r \rceil}$, since the latter is just as computable and gives a better approximation. Next, we can replace $G_{\hat{d},|w|,r}$ with

$$\bigcup_{j \leq \log r} G_{\hat{d},|w|,2^j},$$

since, as required, this set is transitively closed under input querying, and is polylog-sized. Thus, without loss of generality, we can assume $G_{\hat{d},|w|,r} \subseteq G_{\hat{d},|w|,s}$ for $r \leq s$. ∎

While the observation of Lemma 1 is immediate, it is important for the constructions in this paper, as well as for Theorem 4 in [AS]. ([AS] was somewhat unclear on this point.) The reader should not confuse monotonicity in r (which is always possible and henceforth always assumed) with the niceness property "monotonicity" (i.e., monotonicity in $|w|$).

Next we give an exact computation lemma for martingales and density systems in our setting. The reader should also see Theorem 19 for an exponentially better exact-computation lemma for martingales in both our setting and Lutz's setting.

Theorem 2 *Given any cover d, one can construct an exactly-computed cover d' covering the same sets. This construction preserves coverage in the limit and monotonicity, (but may destroy conservation).*

Proof. We give the proof for martingales; a similar proof works for density systems. Given a martingale $d(w)$ with computation $\hat{d}_r(w)$, define $d'(w) = \hat{d}_{|w|}(w) + 3 \cdot 2^{-|w|}$.

First, we show that d' has dependency set $G_{d',n} = G_{d,n,n}$: One can compute $d'(z)$ using only the bits in $G_{d,|z|,|z|}$. Suppose $m \in G_{d,|z|,|z|} \setminus \{|z|\}$; then $G_{d,m,|z|} \subseteq G_{d,|z|,|z|}$ by definition of dependency set, and we have

$$\begin{aligned} G_{d',m} &= G_{d,m,m} \\ &\subseteq G_{d,m,|z|} \text{ by Lemma 1} \\ &\subseteq G_{d,|z|,|z|} \\ &= G_{d',n} \end{aligned}$$

Also, note that d' is dyadic, d' is computable using the allowable resources, and since $d' \geq d$ we have that d' covers the same set as d. An easy induction shows

$$d(w) + 2 \cdot 2^{-|w|} \leq d'(w) \leq d(w) + 4 \cdot 2^{-|w|},$$

the delicate definition in [AS] was *not* done in vain. In order that the notion of measure be a "reasonable" notion of big and small, certain axioms have to be satisfied, and this is most easily done in the delicate formulation of [AS], which involves output as differences of formal sums of powers of two, and martingales whose values are approximated rather than computed exactly.

In nearly all cases, the arguments here also apply to the measures of [L92] and [M]. Thus this paper gives a unified treatment of the robustness theorems.

2 Definitions

First we review the measure defined in [AS]. The definitions given there were designed to facilitate proving basic properties of the measure, and so the definitions were in some cases more restrictive and in other cases more general than what is "natural." In this paper we show that in almost all regards (with one important exception), the alternate formulations are all equivalent.

We state the results here for measure on P but note that the results hold for measure on any class DTIME($\mathcal{C}$) (or DSPACE($\mathcal{C}$)) with $\mathcal{C}$ closed under squaring. We equate a language L with its characteristic sequence χ_L. Given a string w (or a sequence ω) we use $w[i..j]$ ($\omega[i..j]$) to denote the string occupying positions i through j of $w(\omega)$. We write $w \sqsubseteq z$ to denote that w is a prefix of z.

Now we present a key notion that is essential to defining the measure in [AS]. Given a natural number n and a Turing machine M having random access to its input, define a *dependency set* $G_{M,n} \subseteq \{0, \ldots, n\}$ to be a set such that for each $i \in G_{M,n} \cup \{n\}$, and each word w of length n, M can compute $M(w[0..i])$ querying only input bits in $G_{M,n} \cap \{0, 1, 2, \ldots i\}$. Note that for all M and n, there is a unique minimal dependency set for M and n, which can easily be computed by expanding the tree of queries that one obtains by assuming both possible values for each queried bit.

A Γ(P) machine (Γ(PSPACE) machine) M is such that M runs in time (space) $\log^{O(1)} n$ and has dependency sets $G_{M,|w|} \subseteq \{0, \ldots, n\}$ with size bounded by $\log^{O(1)} n$. The machine M is given the length of its input. The output of M is a rational number represented as the difference of formal sums of powers of 2. By convention, numerical arguments are passed to M in both unary and binary (so M has time polylog in the *value* of such arguments, which is enough time to read the binary form of the argument). If M computes a function with numerical arguments then the dependency sets have subscripts to match: If M computes $d_r(w)$ then M has dependency sets $G_{M,|w|,r}$ that must be polylog in $|w| + r$. We will sometimes write $G_{d,|w|,r}$ for $G_{M,|w|,r}$ when M computes d.

A set is said to be $\Gamma(\mathcal{C})$-null if it is covered by a density system d, with $d_k(\lambda) \leq 2^{-k}$, and d is approximated by a $\Gamma(\mathcal{C})$ machine M in the sense that $|d_k(w) - M_{k,r}(w)| \leq 2^{-r}$. If in fact $d_k(w)$ is *equal* to $M_k(w)$ (instead of merely being approximated) then we say that d is *exactly computed.* The approximation to d computed by M is denoted by $\hat{d}$. *A priori* we require only $d_k(w) \geq \frac{d_k(w0)+d_k(w1)}{2}$, not equality.

2.1 Niceness Properties

A *cover* will refer to either a density system or a martingale. Unless specified otherwise, a result for a "cover" is claimed to hold for either martingales or density systems. Note one can build a density system from a martingale by defining $d_k(w)$ to be $2^{-k}d(w)$, and this preserves all the properties discussed below that hold for both notions of cover.

We have already defined what it means for a cover to be exactly computed. Below, we define the other "niceness" properties that will be considered.

Definitions: Consider a cover d with dependency set G_d.

- A martingale d covering set A is a *limit* martingale if d has a limit of infinity (and not just a lim sup) along languages in A.
- A density system d is *regular* if $d_k(z) \geq 1$ and $z \sqsubseteq w$ imply $d_k(w) \geq 1$ for all k.

obtained by putting limits on the complexity of the function d. (Details may be found in Section 2.)

In later work [L93], Lutz defines his measure equivalently using the notion of a "martingale". This is a "betting strategy" that starts with some fixed amount of money and, for each input sequence w, "bets" a fraction of the money it currently has on what the next bit of the sequence will be. It is known that a set S has measure zero if and only if there is a martingale that "succeeds" (i.e., is unbounded) on all sequences in S. (In the setting of resource-unbounded measure, see [Schn]; for the resource-bounded case, see [JL, M].)

In Lutz's setting, and in Mayordomo's setting (as in Lebesgue measure), the class of measure-zero sets one obtains is the same, regardless of which of the following choices one picks in making the definitions:

1. The martingale succeeds on sequence ω if (1) it is unbounded on ω (i.e., the lim sup is infinite), or (2) it succeeds only if it has a limit of infinity on ω.

2. The martingale either (1) must be "conservative" in the sense that the amounts of money after $w0$ and $w1$ exactly average to the amount of money after w, or (2) it can "throw money away" by having the average after $w0$ and $w1$ be less than the amount after w.

(Depending on what one is trying to prove, it can be more convenient to choose more stringent or more lenient conditions.)

In this paper, we show that the class of measure zero sets one obtains using the definition of [AS] is the same as that one obtains by formulating the definition in terms of conservative martingales (both in the lim sup sense and in the limit sense) and it is also equivalent to being covered in the limit sense by a martingale that is not assumed to be conservative.

However, to our surprise (and in contrast to the case for Lebesgue measure and for Lutz's or Mayordomo's notions of measure), one obtains strictly more measurable sets in our setting, if one considers covering sets in the lim sup sense by non-conservative martingales. Furthermore, some of the classes one is able to cover in this sense are fairly interesting and natural. For instance, using the definition presented in [AS], we showed that the class of sparse sets does not have measure zero in P. However, using the more generous notion of measure (non-conservative martingales) we show here that the class of all sets that are not exponentially dense has measure zero in P.

Our previous paper [AS] provided a notion of measure not only for P (and other time-bounded classes), but also for PSPACE (and other space-bounded classes). The measure on PSPACE presented in [AS] provides strictly fewer measurable sets than the measure of [M]. However, if one defines a measure on PSPACE using the non-conservative, lim sup notion, we show here that one obtains a notion that is incomparable with that of [M].

Thus we now have two notions of measure on P: a conservative measure and a non-conservative one. One might now worry that any of a number of other slight modifications to the technical aspects of our definitions could lead to other distinct notions of measure. One goal of the current paper is to demonstrate that this is not the case. In fact, as the following paragraph tries to explain, we find it surprising that the class of sets that are covered by martingales is so robust to the details of how the martingales are computed.

The reason this strikes us as surprising is that one of the technical difficulties that had to be overcome in defining the measure in [AS] was that an appropriate representation for numerical outputs had to be found that could be represented in a small number of bits, but still would allow efficient arithmetic operations. (In [AS], we represented numbers as the difference of two sums of powers of two. Thus we can write $2^n + 2^{-r} - 2^m$ in $\log nrm$ bits.) However, we have been able to show that, at least in the martingale formulation, one can use the more natural binary notation, and obtain the same class of measure zero sets.

Note that the work performed in formulating

Measure on P: Robustness of the Notion*

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Abstract

In [AS], we defined a notion of measure on the complexity class P (in the spirit of the work of Lutz [L92] that provides a notion of measure on complexity classes at least as large as E, and the work of Mayordomo [M] that provides a measure on PSPACE). In this paper, we show that several other ways of defining measure in terms of covers and martingales yield precisely the same notion as in [AS]. (Similar "robustness" results have been obtained previously for the notions of measure defined by [L92] and [M], but – for reasons that will become apparent below – different proofs are required in our setting.)

To our surprise, and in contrast to the measures of Lutz [L92] and Mayordomo [M], one obtains strictly *more* measurable sets if one considers "nonconservative" martingales that succeed merely in the lim sup rather than having a limit of infinity. For example, it is shown in [AS] that the class of sparse sets does *not* have measure zero in P, whereas here we show that using the "nonconservative" measure, the class of sparse sets (and in fact the class of sets with density $\epsilon < 1/2$) *does* have measure zero. We also show that our "nonconservative" measure on PSPACE is incomparable with that of [M].

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1 Introduction

Our purpose in this paper is to prove additional basic properties of the notions of measure on P and PSPACE that were defined in our earlier paper [AS], and to clarify the relationship between our measure on PSPACE and the measure that was presented by Mayordomo in [M]. (Both the notion of measure presented in [AS] and the notion presented in [M] coincide with that of [L92] whenever Lutz's measure is defined. There is by now a large body of interesting work demonstrating the utility and importance of resource-bounded measure; we refer the reader to [AS, L93] for pointers to this material.)

Our definition of a measure on P in [AS] shares the following aspects of the definition of [L92]:

> A null cover of a set S of languages is a function $d : \mathbf{N} \times \Sigma^* \to \mathbf{R}$ (where $d(k, w)$ is denoted by $d_k(w)$) such that
>
> - $d_k(\lambda) \leq 1/2^k$
> - $d_k(w) = \text{avg}\{d_k(w0), d_k(w1)\}$
> - For every sequence $\omega \in S$ there is some prefix w of ω such that $d_k(w) \geq 1$.
>
> A function of this sort is called a "density system." Note that any such d_k corresponds to covering S by a sequence of intervals whose sizes sum to $1/2^k$. Measures on specific complexity classes are