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On Extracting Randomness From Weak Random Sources *
Extended Abstract

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Abstract

We deal with the problem of extracting as much randomness as possible from a defective random source. We devise a new tool, a "merger", which is a function that accepts d strings, one of which is uniformly distributed, and outputs a single string that is guaranteed to be uniformly distributed. We show how to build good explicit mergers, and how mergers can be used to build better extractors.

Previous work has succeeded in extracting "some" of the randomness from sources with "large" min-entropy. We improve on this in two respects. First, we build extractors for any source, whatever its min-entropy is, and second, we extract all the randomness in the given source.

Efficient extractors have many applications, and we show that using our extractor we get better results in many of these applications, e.g., we achieve the first explicit N–superconcentrators of linear size and $polyloglog(N)$ depth.

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1 Introduction

1.1 The Problem

We deal with the problem of "extracting" as much randomness as possible from a "defective" source of randomness. There are various ways to define what a "defective" source is, the most general one [Zuc90] assumes nothing about the nature of the source, except that no string has too high a probability in the given distribution:

DEFINITION 1.1 *The min-entropy of a distribution D is $H_\infty(D) = min_x(-log(D(x)))$.*

An extractor [NZ93] is a function $E(x,y)$ s.t. the distribution of $E(x,y)$ is very close to uniform, for any source X with large min-entropy, when x is taken from X and y is a short truly random string. I.e., the extractor uses the short random string y to extract the randomness present in X.

DEFINITION 1.2 (A variation on [NZ93][1]) *$E : \{0,1\}^n \times \{0,1\}^t \mapsto \{0,1\}^{m'}$ is an (n, m, t, m', ϵ)-extractor, if for any distribution X on $\{0,1\}^n$ whose min-entropy is at least m, the distribution of $E(x,y)$ when choosing x according to the distribution X and y uniformly in $\{0,1\}^t$, is within statistical distance of ϵ from uniform.*

1.2 Previous Work and Our New Extractor

Extractors have been studied extensively in many papers [Zuc90, Zuc91, NZ93, SZ94, Zuc93, Zuc]. Much research has also been done on a related structure, called disperser, in which the output bits are not necessarily close to uniform, but have a weaker random property that suffices ,e.g., for "RP simulations" [WZ93, SSZ95]. In the following table we list some of the currently known explicitly constructible extractors:

required min-entropy	no. of truly random bits	number of output bits	reference
$m = \Omega(n)$	$t = O(log^2 n \cdot log(\frac{1}{\epsilon}))$	$m' = \Omega(m)$	[NZ93]
$m = \Omega(n)$	$t = O(log(n) + log(\frac{1}{\epsilon}))$	$m' = \Omega(m)$	[Zuc]
$m = \Omega(n^{1/2+\gamma})$	$t = O(log^2 n \cdot log(\frac{1}{\epsilon}))$	$m' = n^\delta$, $\delta \leq \gamma$	[SZ94]
$m = \Omega(n^\gamma)$	$t = O(log n)$	$m' = n^\delta$, $\delta \leq \gamma$	[SSZ95] Disperser

We improve upon previous results in two ways. First, our extractor works for any min-entropy, *small or large*. Second, we extract *all* the randomness present in the source:

Theorem 1 *For every $\epsilon = \epsilon(n)$ and $m = m(n) \leq n$, there is an explicit family of $(n, m, t = poly(log(n), log(\frac{1}{\epsilon})), m' = m, \epsilon)$–extractors E_n, that can be constructed and run in polynomial time in n.*

In addition we use our new technique to construct a different extractor:

Theorem 2 *For every constant k and $\gamma > 0$ there is some constant $\delta > 0$ and an $(n, n^\gamma, O(log(n)log^{(k)}n), \Omega(n^\delta), \frac{1}{n})$ extractor, where $log^{(k)}n = \underbrace{loglog\ldots log}_{k}\ n$.*

[1] see remark 3.1.

Non-constructive extractors require only $O(log(n)+log(\frac{1}{\epsilon}))$ truly random bits, matching the current lower bound of [NZ93], while our construction requires $poly(log(n), log(\frac{1}{\epsilon}))$ truly random bits. Thus, we can add to the above table:

required min-entropy	number of truly random bits	number of output bits	
any m	$t = poly(log(n)+log(\frac{1}{\epsilon}))$	$m' = m$	Theorem 1
$m = \Omega(n^\gamma)$	$t = O(log(n) \cdot log^{(k)}n)$	$m' = \Omega(n^\delta), \delta < \gamma$	$\epsilon = \frac{1}{n}$, Thm 2
any m	$t = O(log(n)+log(\frac{1}{\epsilon}))$	$m' = m$	Lower bound

Reducing t to the optimal remains the major open problem.

1.3 Our Technique

The main new tool we use is the notion of a "merger". This is a function that accepts d strings, one of which is uniformlly distributed, and outputs a single string which is guaranteed to be uniformly distributed. The difficulty of designing such a merger is that we do not know beforehand which of the strings is the uniformly distributed one, and that the other strings may be correlated with it in an arbitrary manner. In fact, we use the merger under more general conditions: the index of the "uniformly distributed" string may be a random variable itself, and may be correlated with the input strings.

Mergers, as extractors, extract randomness from a defective source of randomness. However, mergers deal with random sources having "simple" structure, which simplifies the task of constructing good mergers.

We briefly illustrate how mergers can be used to build extractors. Suppose for any random source X there is some hint $Y = Y(x)$ s.t. it is easy to extract randomness from $(X \mid Y)$, and suppose we do not know what this value $Y = Y(X)$ is. Using mergers, we can build an extractor by trying to extract randomness from $(X \mid Y = y)$ for all possible values y of Y, and then *merging* all these possibilities into a single quasi-random string using a merger.

1.4 Applications

Our new construction improves many of the known applications of extractors. We list only three applications here, the rest (along with the proofs) appear in appendix E.

Corollary 1.1 *(following [WZ93]) For any N and $1 \leq a \leq N$ there is an explicitly constructible a–expanding graph with N vertices, and maximum degree $O(\frac{N}{a}2^{polyloglog(N)})$* [2] *.*

Corollary 1.2 *(following [WZ93]) For any N there is an explicitly constructible superconcentrator over N vertices, with linear size and $polyloglog(N)$ depth* [3]*.*

Corollary 1.3 *For any $\delta > 0$ and $k > 0$, BPP can be simulated in time $n^{O(log^{(k)}n)}$ using a weak random source X with minentropy at least n^δ* [4] *.*

[2] The obvious lower bound is $\frac{N}{a}$. The previous upper bound [WZ93, SZ94] was $O(\frac{N}{a} \cdot 2^{log(N)^{1/2+o(1)}})$. A proof of this corollary appears in appendix E.2.

[3] This improves the current upper bound of $O(log(N)^{1/2+o(1)})$. A proof of this corollary appears in appendix E.3.

[4] This improves the [SZ94] $n^{O(log(n))}$ bound. For RP, [SSZ95] showed this can be done in polynomial time. The corollary immediately follows Theorem 2. A sketch of the proof of Theorem 2 appears in Appendix F.

1.5 Paper Organization

In section 2 we describe our notation, and in section 3 we give some preliminary defresults and results. In section 4 we prove Theorem 1, assuming two theorems on mergers and composition of extractors which we prove in section 5 and section 6 respectively.

Due to lack of space, many of the technical proofs appear in the appendix, where full and detailed proofs are given. We also sketch the proof of Theorem 2 in Appendix F.

2 Notation

We use standard notation for random variables and distributions. We distinguish between random variables and distributions: the value of a random variable is the result of some trial done in a probability space, and thus random variables may be correlated, as opposed to distributions that are merely functions assigning probabilities to events. Given a random variable A we denote by $\overline{A}$ the distribution A induces.

We use capital letters to denote random variables and distributions, and small letters to denote elements. If $\overline{X}$ is a distribution, $x \in \overline{X}$ denotes picking x according to the distribution $\overline{X}$. If A and B are (possibly correlated) random variables then $(A \mid B = b)$ is the conditional distribution of A given that $B = b$. We denote by U_t the uniform distribution over $\{0,1\}^t$.

For a random variable $X = X_1 \circ \ldots \circ X_n$ over $\{0,1\}^n$ we write $X_{[i,j]}$ as an abbreviation to the random variable $X_i \circ X_{[i+1]} \ldots \circ X_j$, where $\circ$ stands for concatenation. The same applies for instances $x_{[i,j]}$.

We define the variation distance between two distributions $\overline{X}$ and $\overline{Y}$ defined over the same space, as $d(\overline{X},\overline{Y}) = \frac{1}{2}|\overline{X} - \overline{Y}|_1 = \frac{1}{2}\Sigma_a|\overline{X}(a) - \overline{Y}(a)|$. We say $\overline{X}$ is ϵ-close to $\overline{Y}$ if $d(\overline{X},\overline{Y}) \leq \epsilon$. We say two random variables A and B are ϵ–close if $d(\overline{A},\overline{B}) \leq \epsilon$. We say $\overline{X}$ is ϵ quasi-random if it is ϵ-close to uniform. In appendix A we list some of the well known properties of variation distance.

3 Preliminaries

3.1 Extractors

DEFINITION 3.1 *The min-entropy of a distribution $\overline{D}$ is $H_\infty(\overline{D}) = min_x(-log(\overline{D}(x)))$.*

DEFINITION 3.2 *$E : \{0,1\}^n \times \{0,1\}^t \mapsto \{0,1\}^{m'}$ is an (n,m,t,m',ϵ)-extractor, if for any distribution $\overline{X}$ on $\{0,1\}^n$ with $H_\infty(\overline{X}) \geq m$, the distribution of $E(x,y)$ when choosing $x \in \overline{X}$ and $y \in U_t$, is ϵ close to $U_{m'}$.*

REMARK 3.1 *This definition is slightly different from the one in [NZ93, SZ94], where E is called an extractor if $E(x,y) \circ y$ is close to uniform, while we only demand that $E(x,y)$ is close to uniform.*

3.2 Explicit Extractors

DEFINITION 3.3 *We say E is an explicit (n,m,t,m',ϵ) –extractor, if there is a Turing machine that given n, outputs an $(n, m = m(n), t = t(n), m' = m'(n), \epsilon = \epsilon(n))$-extractor E_n, in polynomial time in n.*

As mentioned in the introduction, various explicit extractors have been built so far. Of those we use the following two extractors:

Lemma 3.1 *[SZ94] There is some constant $c > 1$ s.t. for any $m = \Omega(log(n))$ there is an explicit $(n, 2m, m, cm, 2^{-m/2})$– extractor A_m. We denote this constant c by c_{sz}.*

Lemma 3.2 *[SZ94]* [5] *Let $m(n) \geq n^{1/2+\gamma}$ for some constant $\gamma > 0$, then for any ϵ there is an explicit $(n, m(n), O(log^2 n \cdot log(\frac{1}{\epsilon})), \frac{m^2(n)}{n}, \epsilon)$– extractor.*

The first extractor is actually a restatement of the existence of explicit tiny families of hash functions [SZ94, GW94], and can easily be achieved using small ϵ–biased sample spaces [SZ94].

We also need the following simple lemma from [NZ93]:

Lemma 3.3 *[NZ93] Let X and Y be two correlated random variables. Let $\overline{B}$ be a distribution, and call an x "bad" if $(Y \mid X = x)$ is not ϵ close to $\overline{B}$. If $Prob_{x \in \overline{X}}(x \text{ is bad}) \leq \eta$ then $\overline{X \circ Y}$ is $\epsilon + \eta$ close to $\overline{X} \times \overline{B}$.*

3.3 More of The Same

Suppose we have an extractor E that extracts randomness from any source having at least m min-entropy, how much randomness can we extract from sources having M min-entropy when $M >> m$?

The following algorithm is implicit in [WZ93]: use the same extractor E many times over the same string x, each time with a fresh truly random string r_i, until you get $M - m$ output bits. The idea is that as long as the length of $E(x, r_1) \circ \ldots \circ E(x, r_k)$ is less then $M - m$, then with high probability $(X \mid E(x, r_1) \circ \ldots \circ E(x, r_k))$ still contains m min-entropy, and therefore we can use the extractor E to further extract randomness from it. Thus, we have the following two lemmas, that are proven in detail in appendix A:

Lemma 3.4 *Suppose that for some $m = m(n)$ there is an explicit (n, m, t, m', ϵ)–extractor E_m. Then, for any $M = M(n) \geq m$, and any safety parameter $s > 0$, there is an explicit $(n, M, kt, min\{km', M - m - s\}, k(\epsilon + 2^{-s}))$–extractor.*

Lemma 3.5 *Suppose that for any $m \geq \bar{m}$ there is an explicit $(n, m, t(n), \frac{m}{f(n)}, \epsilon(n))$–extractor E_m. Then, for any m, there is an explicit $(n, m, O(f(n)log(n)t(n)), m - \bar{m}, log(n)(\epsilon + 2^{-t(n)}))$–extractor.*

4 Main Theorem

In this section we prove Theorem 1, using our new tool of mergers. In subsection 4.1 we define what a "merger" is, and in subsection 4.2 we explicitly build "mergers". In subsection 4.3 we use mergers to efficiently compose extractors, and in subsection 4.4 we show how good somewhere random mergers imply good extractors. Finally, in subsection 4.5 we use this to prove our main theorem.

[5] The parameters here are simplified. The real parameters appearing in [SZ94] are somewhat better.

4.1 Somewhere Random Mergers

DEFINITION 4.1 *$X = X_1 \circ \ldots \circ X_d$ is a d–block (m, ϵ, η) somewhere random source, if each X_i is a random variable over $\{0,1\}^m$, and there is a random variable Y over $[0..d]$ s.t. for any $i \in [1..d]$: $d((X_i|Y=i), U_m) \leq \epsilon$ and $Prob(Y=0) \leq \eta$. We also say that Y is an (m, ϵ, η) selector for X.*

A somewhere random merger is a function extracting randomness from a somewhere random source:

DEFINITION 4.2 *$M : \{0,1\}^{dm} \times \{0,1\}^t \mapsto \{0,1\}^{m'}$ is a (d, m, t, m', ϵ)–somewhere random merger, if for any d–block $(m, 0, 0)$ somewhere random source X, the distribution of $E(x, y)$ when choosing $x \in \overline{X}$ and $y \in U_t$, is ϵ close to $U_{m'}$.*

4.2 Explicit Somewhere Random Mergers

DEFINITION 4.3 *We say M is an explicit (d, m, t, m', ϵ) –somewhere random merger, if there is a Turing machine that given d, m outputs a $(d = d(n), m = m(n), t = t(n), m' = m'(n), \epsilon = \epsilon(n))$-somewhere random merger M, in polynomial time in dm.*

It turns out that any $(2m, m, t, m', \epsilon)$–extractor is a $(2, m, t, m', \epsilon)$–somewhere random merger. Having a 2–block somewhere random merger, we build a d–block merger by merging the d blocks in pairs in a tree wise fashion. Thus, we have the following theorem, which we prove in detail in section 5:

Theorem 3 *Assume for every m there is an explicit $(2m, m, t(m), m - k(m), \epsilon(m))$ extractor E_m, for some monotone functions t and k. Then there is an explicit $(2^l, m, l \cdot t(m), m - l \cdot k(m), l \cdot \epsilon(m))$ somewhere random merger.*

The [SZ94] extractor of lemma 3.2 works for any source with $H_\infty(\overline{X}) \geq n^{1/2+\gamma}$. Thus, using lemma 3.4, by repeatedly using the [SZ94] extractor, we can extract from a source having $H_\infty(\overline{X}) \geq \frac{n}{2}$ at least $\frac{n}{2} - n^{1/2+\gamma}$ quasi-random bits. Thus, we have a 2–merger that does not lose much randomness in the merging process. Applying Theorem 3 we get a good n–merger. Thus we have the following lemma, that is proven in detail in Appendix B :

Lemma 4.1 *Let $b > 1$ be any constant and suppose $f = f(m) = f(m(n))$ is a function s.t. $f(m) \leq \sqrt[3]{m}$ and for every $m \geq m_0(n)$: $f(m) \geq b \cdot log(n)$. Then for every $m \geq m_0$ there is an explicit $(n, m, log(n) \cdot polylog(m) \cdot f^2(m) \cdot log(\frac{1}{\epsilon}), m - \frac{m}{b}, log(n) \cdot poly(m) \cdot \epsilon)$ somewhere random merger.*

Corollary 4.2 *For every $m \geq 2^{\sqrt{log(n)}}$, there is an $(n, m, polylog(n) \cdot log(\frac{1}{\epsilon}), \Omega(m), poly(n) \cdot \epsilon)$–somewhere random merger $M = M_m$.*

Proof: Take $f(m) = log^d m$ for some constant $d > 2$. For any constant b, $m \geq 2^{\sqrt{log(n)}}$ and n large enough, $log^d m \geq b \cdot log(n)$, and the corollary follows lemma 4.1. □

Notice that Theorem 3 and corollary 4.2 take advantage of the simple structure of somewhere random sources, giving us an explicit somewhere random merger that works even for sources with very small min-entropy that can not use the [SZ94] extractor of lemma 3.2.

4.3 Composing Extractors

Having good somewhere random mergers, we can efficiently compose extractors. The idea is to split the input string x into two consecutive strings $x_1 \circ x_2$, s.t. X_1 and $(X_2 \mid X_1 = x_1)$ contain a lot of randomness. Given such a splitting point, we can use E_1 to extract randomness from x_2, and use E_2 with the *extracted randomness* to further extract randomness from x_1. We show that for most strings such a splitting point exists. An obvious difficulty is that given a string x we do not know what is the right splitting point. We solve this by trying all $|x| = n$ possible splitting points. This gives us a somewhere random source with n blocks, that can be merged into a single quasi-random string if we have a good somewhere random merger. We call this new extractor the *composed* extractor.

ALGORITHM 4.1 *Suppose E_1 is an $(n, m_1, t_1, t_2, \zeta_1)$–extractor, E_2 is an $(n, m_2, t_2, t_3, \zeta_2)$–extractor, and M is an $(n, t_3, \mu_1, o_1, \zeta_3)$–somewhere random merger. Define the function $E_2 \overset{M}{\odot} E_1$ as follows: Given $a \in \{0,1\}^n$, toss $r_1 \in \{0,1\}^{t_1}$, and $r_2 \in \{0,1\}^{\mu_1}$.*

- *Let $q_i = E_1(a_{[i,n]}, r_1)$ and $z_i = E_2(a_{[1,i-1]}, q_i)$, for $i = 1, \ldots, n$.*
- *Let $E_2 \ominus E_1 = z_1 \circ \ldots \circ z_n$, and $E_2 \overset{M}{\odot} E_1 = M(E_2 \ominus E_1, r_2)$.*

Theorem 4 *Suppose E_1 is an $(n, m_1, t_1, t_2, \zeta_1)$–extractor, E_2 is an $(n, m_2, t_2, t_3, \zeta_2)$–extractor, and M is an $(n, t_3, \mu_1, o_1, \zeta_3)$–somewhere random merger. Then for every safety parameter $s > 0$, $E_1 \overset{M}{\odot} E_2$ is an $(n, m_1 + m_2 + s, t_1 + \mu_1, o_1, \zeta_1 + \zeta_2 + \zeta_3 + 8n2^{-s/3})$–extractor.*

Now we define composure of many extractors by:

DEFINITION 4.4 *Suppose E_i is an $(n, m_i, t_i, t_{i+1} + s_{i+1}, \zeta_i)$–extractor, for $i = 1, \ldots, k$, $s_i \geq 0$ and $s_2 = 0$. Suppose M_i is an $(n, t_{i+2} + s_{i+2}, \mu_i, t_{i+2}, \bar{\zeta_i})$ –somewhere random merger, for any $i = 1 \ldots k-1$. We define by induction the function $E_k \overset{M_{k-1}}{\odot} E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2 \overset{M_1}{\odot} E_1$ to equal $E_k \overset{M_{k-1}}{\odot} (E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2 \overset{M_1}{\odot} E_1)$.*

REMARK 4.1 *In appendix D.3 we explain why we choose right associativity rather than left associativity.*

Theorem 5 *Suppose E_i, M_i are as above, then for any safety parameter $s > 0$, $E = E_k \overset{M_{k-1}}{\odot} E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2 \overset{M_1}{\odot} E_1$ is an $(n, \Sigma_{i=1}^k m_i + (k-1)s, t_1 + \Sigma_{i=1}^{k-1}\mu_i, t_{k+1}, \Sigma_{i=1}^k \zeta_i + \Sigma_{i=1}^{k-1}\bar{\zeta_i} + (k-1)n2^{-s/3+3})$ –extractor. If E_i, M_i are explicit, then so is E.*

4.4 Good Somewhere Random Mergers Imply good Extractors

Assume for every $m \geq \bar{m}$ we have a good somewhere random merger M. Then we can let $E = A_m \overset{M}{\odot} \ldots A_{b^2\bar{m}} \overset{M}{\odot} A_{b\bar{m}} \overset{M}{\odot} A_{\bar{m}}$, where A_i is the extractor of lemma 3.1 and b is some constant, $1 < b < c_{sz}$, to get an extractor that extracts $\Omega(m)$ bits from sources having m min-entropy. Using lemma 3.5, we get an extractor that extracts m bits. Thus, we see that good somewhere random mergers imply good extractors. We prove the following lemma in detail in Appendix B :

Lemma 4.3 *Suppose $\bar{m} = \bar{m}(n)$ is a function s.t. for every $m \geq \bar{m}(n)$ there is an explicit $(n, m, t, m - \frac{m}{\bar{c}}, \epsilon)$ somewhere random merger, where $\bar{c}$ is the constant $\frac{2c_{sz}}{c_{sz}-1}$, and c_{sz} is the constant in lemma 3.1. Then for any m there is an explicit $(n, m, O(\bar{m} \cdot log(n) \cdot log(\frac{1}{\epsilon}) + log^2(n) \cdot t), m, poly(n) \cdot \epsilon)$– extractor.*

Unfortunately, corollary 4.2 asserts the existence of good mergers only for $m \geq 2^{\sqrt{log(n)}}$, and therefore plugging this into lemma 4.3 we get:

Corollary 4.4 *For every m there is an $(n, m, O(2^{\sqrt{log(n)}} \cdot polylog(n) \cdot log(\frac{1}{\epsilon})), m, poly(n) \cdot \epsilon)$–extractor B_m.*

4.5 Putting It Together

The extractor B in corollary 4.4 uses $O(2^{\sqrt{log(n)}} \cdot polylog(n) \cdot log(\frac{1}{\epsilon}))$ truly random bits to extract all of the randomness in the given source. Although $O(2^{\sqrt{log(n)}} \cdot polylog(n) \cdot log(\frac{1}{\epsilon}))$ is quite a large amount of truly random bits, we can use the [SZ94] extractor, to extract $n^{1/3}$ bits from $n^{2/3}$ min-entropy, and then use these $n^{1/3} >> O(2^{\sqrt{log(n)}} \cdot polylog(n) \cdot log(\frac{1}{\epsilon}))$ bits, to further extract all of the remaining min-entropy. More precisely, if B is the extractor in corollary 4.4, E_{sz} the extractor from lemma 3.2 and M the merger from corollary 4.2, then $E = B \overset{M}{\odot} E_{sz}$ extracts $\Omega(m)$ bits from sources having $m \geq n^{2/3}$ min-entropy, using only $polylog(n)$ truly random bits! That is, we get the following lemma (whose proof appears in detail in Appendix B) :

Lemma 4.5 *Let $\epsilon \geq 2^{-n^\gamma}$ for some constant $\gamma < 1$. There is some constant $\beta < 1$ s.t. for every $m \geq n^\beta$ there is an explicit $(n, m, polylog(n) \cdot log(\frac{1}{\epsilon}), \Omega(m), poly(n) \cdot \epsilon)$ extractor.*

Now that we know how to extract all the randomness from sources having $\Omega(n^\beta)$ min-entropy with only $polylog(n)$ truly random bits, by lemmas 3.5 and Theorem 3 we have good somewhere random mergers, for every m. Thus by lemma 4.3 we have good extractors for every m. We prove in detail in Appendix B that:

Theorem: *For every constant $\gamma < 1$, $\epsilon \geq 2^{-n^\gamma}$, and every $m = m(n)$ there is an explicit $(n, m, polylog(n) \cdot log(\frac{1}{\epsilon}), m, \epsilon)$–extractor.*

5 Explicit Somewhere Random Mergers

In this section we build explicit somewhere random mergers. We observe that a 2–block merger can be obtained from the previously designed extractors of [NZ93, SZ94]. Once such a merger is obtained, any number of blocks can be merged in a binary-tree fashion.

5.1 A 2-block somewhere random merger

A d–block (m, ϵ, η)–somewhere random source X, can be viewed intuitively as a source composed of d strings of length m, with a selector that for all but an η fraction of the inputs, can find a block that is ϵ quasi-random. Thus, it is natural to suspect that $\overline{X}$ is $\epsilon + \eta$ close to a distribution with m min-entropy. The following lemma (proved in appendix C.1) states this precisely:

Lemma 5.1 *(1) Any (m, ϵ, η) somewhere random source X is $\epsilon + \eta$ close to an $(m, 0, 0)$–somewhere random source X'. (2) For any $(m, 0, 0)$ somewhere random source X, $H_\infty(\overline{X}) \geq m$.*

Since a $(2m, m, t, m', \epsilon)$–extractor E extracts randomness from any source X with $H_\infty(\overline{X}) \geq m$, in particular it extracts randomness from any $(m, 0, 0)$ somewhere random source, and therefore by definition E is a $(2, m, t, m', \epsilon)$–somewhere random merger. Thus:

Corollary 5.2 *Any $(2m, m, t, m', \epsilon)$–extractor is a $(2, m, t, m', \epsilon)$–somewhere random merger.*

5.2 A d–block somewhere random merger

Given a d–block somewhere random source, we merge the blocks in pairs in a tree like fashion, resulting in a single block. We show that after each level of merges we still have a somewhere random source, and thus the resulting single block is necessarily quasi-random.

ALGORITHM 5.1 *Assume we can build a $(2, m, t(m), m - k(m), \epsilon(m))$–somewhere random merger E, for some monotone functions t and k. We build $M_l : \{0,1\}^{2^l m} \times \{0,1\}^{l \cdot t(m)} \mapsto \{0,1\}^{m - l \cdot k(m)}$, by induction on l:*

Input *: $x^l = x_1^l \circ \ldots x_{2^l}^l$, where each $x_i^l \in \{0,1\}^m$.*

Output *: Toss $t = t_1 \circ \ldots t_l$, where $t_j \in \{0,1\}^{t(m)}$. If $l = 0$ output x^l, otherwise:*

- *Let $x_i^{l-1} = E(x_{2i-1}^l \circ x_{2i}^l \ , \ t_l)$, for $i = 1, \ldots, 2^{l-1}$.*
- *Let the output be $M_{l-1}(x_1^{l-1} \circ \ldots x_{2^{l-1}}^{l-1} \ , \ t_1 \circ \ldots t_{l-1})$.*

Theorem: *[Same as Theorem 3] M_l is a $(2^l, m, l \cdot t(m), m - l \cdot k(m), l \cdot \epsilon(m))$ somewhere random merger.*

Proof: For $j = l, \ldots, 0$ denote by Z^j the random variable whose value is $x^j = x_1^j \circ \ldots x_{2^j}^j$, where the input x is chosen according to $\overline{X}$, and t is uniform. Notice that $\overline{Z^l}$ is the distribution $\overline{X}$, and $\overline{Z^0}$ is the distribution of the output.

The theorem follows immediately from the following claim:

Claim 5.1 *Denote $m_j = m - (l - j)k(m)$. If X is an $(m_l, 0, 0)$ somewhere random source, then for any $1 \leq i \leq 2^j$, $d(\ (Z_i^j \mid Y \in [2^{l-j}(i-1) + 1, 2^{l-j} i])\ ,\ U_{m_j}\ ) \leq (l - j) \cdot \epsilon(m)$*

□

The proof of the claim is by downward induction on j. The basis $j = l$ simply says that for any i , $d((X_i \mid Y = i)\ ,\ U_m\ ) = 0$, which is exactly the hypothesis. Suppose it is true for j. Informally, the induction hypothesis says that if Y "points" to Z_{2i-1}^j then Z_{2i-1}^j is uniform, and if Y "points" to Z_{2i}^j then Z_{2i}^j is uniform. Thus, it is natural to suspect that if Y "points" to Z_{2i}^j or to Z_{2i}^j, then $Z_{2i-1}^j \circ Z_{2i}^j$ is a 2–block somewhere random source. Indeed, we show that $Z_{2i-1}^j \circ Z_{2i}^j$ with the right conditioning on Y is $(l - j) \cdot \epsilon(m)$ close to some $\overline{W}$ with $H_\infty(\overline{W}) \geq m_j$. Since $Z_i^{j-1} = E(Z_{2i-1}^j \circ Z_{2i}^j \ , \ t_j)$ and E is an extractor the claim follows. A full proof appears in Appendix C.

REMARK 5.1 *Notice that we use the same random string t_j for all merges occuring in the j'th layer, and that this is possible because in a somewhere random source we do not care about dependencies between different blocks. Also notice that the error is additive in the depth of the tree of merges (i.e. in l), rather than in the size of the tree (2^l).*

6 Composing Extractors

In subsection 6.1 we show how to efficiently compose two extractors, and in subsection 6.2 we compose many extractors.

6.1 Composing Two Extractors

The algorithm computing the composed extractor was presented before as algorithm 4.1, along with Theorem 4, that we prove here.

Proof: [Of Theorem 4]

Obviously, it is enough to show that $E_1 \ominus E_2$ is an $(t_3, \zeta_1+\zeta_2, 8n2^{-s/3})$–somewhere random source. To prove this, assume $H_\infty(\overline{X}) \geq m_1 + m_2 + s$. Denote by Q_i and Z_i the random variables with values q_i and z_i respectively. Also, let $\epsilon_3 = 2^{-s/3}$, $\epsilon_2 = 2\epsilon_3$, and $\epsilon_1 = 2\epsilon_2$. We define a selector for $Z = Z_1 \circ \ldots \circ Z_n = E_1 \ominus E_2$ in two phases: first we define a function f which is almost the selector but has few "bad" values, then we correct f to obtain the selector Y.

DEFINITION 6.1 *Define $f(w)$ to be the last i s.t $Prob(X_{[i,n]} = w_{[i,n]} \mid X_{[1,i-1]} = w_{[1,i-1]}) \leq (\epsilon_2 - \epsilon_3) \cdot 2^{-m_1}$.*

DEFINITION 6.2 *Define w to be "bad" if $f(w) = i$ and:*

1. *$Prob_{x \in X}(f(x) = i) \leq \epsilon_1$, or,*
2. *$Prob_{x \in X}(f(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_2$, or,*
3. *$Prob_{x \in X}(X_i = w_i \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_3$*

We denote by B the set of all bad w. We denote by B_i $(i = 1,2,3)$ the set of all w satiisfing condition (i).

DEFINITION 6.3 *Let Y be the random variable, obtained by taking the input a and letting $Y = Y(a)$ where $Y(w) = 0$ if w is bad, and $f(w)$ otherwise.*

It holds that $Prob(w \text{ is bad}) \leq n(\epsilon_1 + \epsilon_2 + \epsilon_3) \leq 8n \cdot 2^{-s/3}$ (the proof is easy and appears in appendix D.3). We complete the proof by showing that $(Z_i \mid Y = i)$ is $\zeta_1 + \zeta_2$–close to uniform.

Claim 6.1 *If $Prob(Y = i \mid X_{[1,i-1]} = w_{[1,i-1]}) > 0$ then $H_\infty(X_{[i,n]} \mid Y = i \text{ and } X_{[1,i-1]} = w_{[1,i-1]}) \geq m_1$*

Therefore, for any such $w_{[1,i-1]}$, $(Q_i \mid Y = i \text{ and } X_{[1,i-1]} = w_{[1,i-1]})$ is ζ_1–close to random (since E_1 is an extractor). Hence by lemma 3.3, the distribution $(X_{[1,i-1]} \mid Y = i) \times (Q_i \mid Y = i \text{ and } X_{[1,i-1]} = w_{[1,i-1]})$ is ζ_1–close to the distribution $(X_{[1,i-1]} \mid Y = i) \times U$. But,

Claim 6.2 *$H_\infty(X_{[1,i-1]} \mid Y = i) \geq m_2$.*

Therefore, using the extractor E_2 we get that $(Z_i \mid Y = i)$ is $\zeta_1 + \zeta_2$–close to uniform.

□

The proofs of claims 6.1 and 6.2 appear in appendix D.

6.2 Composing Many Extractors

In this subsection we prove Theorem 5, which claims that $E_k \overset{M}{\odot} \ldots \overset{M}{\odot} E_1$ can be efficiently calculated:

Proof: [of Theorem 5]

Correctness :

By induction on k. For $k = 2$ this follows from theorem 4. For larger k's this is a straight forward combination of the induction hypothesis and Theorem 4.

Running time :

We compute $E_k \overset{M_{k-1}}{\odot} E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2 \overset{M_1}{\odot} E_1$ using a dynamic programming procedure:

- Given $x \in \overline{X}$, we toss $y \in \{0,1\}^{t_1}$ and $y_j \in \{0,1\}^{\mu_j}$ for $j = 1, \ldots, k$.
- Next, we compute the matrix M where $M[i,j] = (E_j \overset{M_{j-1}}{\odot} E_{j-1} \overset{M_{j-2}}{\odot} \ldots E_2 \overset{M_1}{\odot} E_1)(x_{[i,n]}, y \circ y_1 \circ \ldots \circ y_j)$, for $1 \leq i \leq n$ and $1 \leq j \leq k$.

  The entries of the first row of M, $M[1,i]$ can be filled by calculating $E_1(x_{[i,n]}, y)$. Suppose we know how to fill the j'th row of M. We show how to fill the $j+1$'th row.

  - Denote $q_l = M[j,l]$ for $l = i, \ldots, n$, and let $z_l = E_{j+1}(x_{[i,l-1]}, q_l)$.
  - Let $M[j+1,i] = M_j(z_i \circ \ldots z_n, \mu_j)$.

  By the composition definition $M[j,i]$ has the correct value, and clearly, the computation takes polynomial time in n.

□

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A Basic definitions and Preliminaries

A.1 Distance between Distributions

DEFINITION A.1 *Let $\overline{D_1}, \overline{D_2}$ be two distributions over the same space Λ. We define the "distance" between the distributions by: $d(\overline{D_1}, \overline{D_2}) = \frac{1}{2}|\overline{D_1} - \overline{D_2}|_1 = \frac{1}{2}\Sigma_{x\in\Lambda}|\overline{D_1}(x) - \overline{D_2}(x)| = max_{Y\subseteq\Lambda}|\overline{D_1}(Y) - \overline{D_2}(Y)|$. It is easy to check that d is a metrics.*

Fact A.1 *Let $\overline{D_1}, \overline{D_2}$ be two distributions on Λ_1, and let $f : \Lambda_1 \mapsto \Lambda_2$ be any (deterministic or probabilistic) function, then $d(f(\overline{D_1}), f(\overline{D_2})) \leq d(\overline{D_1}, \overline{D_2})$. I.e., distance between distributions can not be created out of nowhere.*

Fact A.2 *Let A, B, C and D be any random variables, then $d(\overline{A}, \overline{B}) \leq d(\overline{A \circ C}, \overline{B \circ D})$.*

Fact A.3 *Let A, B and C be any random variables, then $d(\overline{A \circ B}, \overline{A \circ C}) = \Sigma_{a\in\Lambda_A} Pr(A = a) \cdot d((B \mid A = a), (C \mid A = a))$.*

A.2 Proof of lemma 3.4

Proof: [Of lemma 3.4]

Denote by A_i the random variable with the value $E(X, R_i)$. Denote by $A_{[1,i]} = A_1 \circ \ldots A_i$ the random variable whose value is $E(X, R_1) \circ \ldots \circ E(X, R_i)$, and let $l_i = |A_{[1,i]}|$.

DEFINITION A.2 *We say that $a_{[1,i]}$ is "s-tiny" if $Prob(A_{[1,i]} = a_{[1,i]}) \leq 2^{-l_i - s}$*

Claim: For any $1 \leq i \leq k$, $Prob(a_{[1,i]}$ *is* s$-tiny) \leq 2^{-s}$.

Proof: $A_{[1,i]}$ can have at most 2^{l_i} possible values, and each tiny value has probability at most $2^{-l_i - s}$. □

Claim: For any prefix $a_{[1,i]}$ that is not s–tiny, $H_\infty(X \mid A_{[1,i]} = a_{[1,i]}) \geq M - l_i - s$

Proof: For any x, $Prob(X = x \mid A_{[1,i]} = a_{[1,i]}) \leq \frac{Prob(X=x)}{Prob(A_{[1,i]}=a_{[1,i]})} \leq \frac{2^{-M}}{2^{-l_i - s}} = 2^{-M+l_i+s}$. □

Claim: If $l_{i-1} \leq M - m - s$, then $\overline{A_{[1,i]}}$ is $i(2^{-s} + \epsilon)$ quasi-random.

Proof: By induction on i. For $i = 1$ this follows from the properties of E. Assume for i, and let us prove for $i + 1$.

Since $l_i \leq M - m - s$, then for any prefix $a_{[1,i]}$ that is not s–tiny, $H_\infty(X \mid A_{[1,i]} = a_{[1,i]}) \geq M - l_i - s \geq m$. Therefore, for any non-tiny prefix $a_{[1,i]}$, $(A_{i+1} \mid A_{[1,i]} = a_{[1,i]})$ is ϵ quasi-random. Therefore by lemma 3.3, $\overline{A_{[1,i+1]}}$ is $2^{-s} + \epsilon$ close to the distribution $\overline{A_{[1,i]}} \times U$, and by induction $\overline{A_{[1,i+1]}}$ is $(i + 1)(2^{-s} + \epsilon)$ quasi-random. □

Therefore, if we take k s.t. $l_k \leq M - m - s$, we invest kt random bits, and we get km' bits that are $k(2^{-s} + \epsilon)$ quasi-random, as required.

□

A.3 Proof of lemma 3.5

Proof: [of lemma 3.5]

Define $E(x, r_1 \circ \ldots \circ r_k) = E_{m_1}(x, r_1) \circ \ldots \circ E_{m_k}(x, r_k)$, where $s = t(n)$, $l_0 = 0$, $m_i = m - l_{i-1} - s$, and $l_i = l_{i-1} + \frac{m_i}{f(n)}$. Denote by A_i the random variable $E_{m_i}(X, R_i)$, and $A_{[1,i]} = A_1 \circ \ldots A_i$. Intuitively, $l_i = |A_{[1,i]}|$, and m_i is how much min-entropy is left in $(X \mid A_{[1,i]} = a_{[1,i]})$ with the safety parameter $s = t(n)$.

Claim: If $m_i \geq \bar{m}$ then $\overline{A_{[1,i]}}$ is $i(2^{-s} + \epsilon)$ quasi-random.

Proof: By induction on i. For $i = 1$ this follows from the properties of E. Assume for i, and let us prove for $i + 1$.

For any prefix $a_{[1,i]}$ that is not s-tiny, $H_\infty(X \mid A_{[1,i]} = a_{[1,i]}) \geq m - l_i - s = m_{i+1} \geq \bar{m}$. Therefore, for any non-tiny prefix $a_{[1,i]}$, $(A_{i+1} \mid A_{[1,i]} = a_{[1,i]})$ is ϵ quasi-random. Therefore by lemma 3.3, $\overline{A_{[1,i+1]}}$ is $2^{-s} + \epsilon$ close to the distribution $\overline{A_{[1,i]}} \times U$, and by induction $\overline{A_{[1,i+1]}}$ is $(i + 1)(2^{-s} + \epsilon)$ quasi-random. □

How big do we need k to be? Let us denote $q_i = m - l_i$, i.e., q_i are the number of bits still missing. Notice that $q_i = m - l_i = m - (l_{i-1} + \frac{m_i}{f(n)}) = q_{i-1} - \frac{m_i}{f(n)} = q_{i-1} - \frac{q_{i-1} - t(n)}{f(n)}$. Therefore, if $\frac{q_{i-1}}{2} \geq t(n)$, then $q_i \leq (1 - \frac{1}{2f(n)})q_{i-1}$. Thus, it must happen that after $O(f(n)log(n))$ steps, either $q_{i-1} \leq 2t(n)$, or else $m_i \leq \bar{m}$. In the first case, $q_{i-1} \leq 2t(n)$, and we can stop and fill all the $2t(n)$ missing bits with a truly random string. In the second case, $m_i \leq \bar{m}$, i.e., $q_{i-1} \leq \bar{m} + s$, so if we add $s = t(n)$ truly random bits, there are only $\bar{m}$ missing bits as required.

Therefore it is sufficient to take $k = O(f(n)log(n))$, and let the final extractor be $E(x, r) \circ y$, where y is of length $2t(n)$ and is truly random.

□

B Main Theorem

B.1 Proof of lemma 4.1

Proof: [Of lemma 4.1]

- By lemma 3.2 there is an explicit $(m, \frac{m}{f(m)}, O(log^2 m \cdot log(\frac{1}{\epsilon})), \frac{m}{f^2(m)}, \epsilon)$– extractor.
- By lemma 3.4 there is an explicit $(2m, m, O(f^2(m) \cdot log^2 m \cdot log(\frac{1}{\epsilon})), m - \frac{m}{f(m)}, poly(m) \cdot \epsilon)$– extractor.
- By Theorem 3 there is an explicit $(n, m, O(log(n) \cdot polylog(m) \cdot f^2(m) \cdot log(\frac{1}{\epsilon}), m - log(n) \cdot \frac{m}{f(m)}, log(n) \cdot poly(m) \cdot \epsilon)$– somewhere random merger. For any $m \geq m_0$ $\frac{m}{f(m)} \leq \frac{m}{b \cdot log(n)}$, hence $log(n) \cdot \frac{m}{f(m)} \leq \frac{m}{b}$.

□

B.2 Proof of lemma 4.3

Proof: The lemma follows easily from lemma B.1 using lemma 3.5. □

Lemma B.1 *Suppose for any $\bar{m} \leq m \leq \bar{\bar{m}}$ there is an explicit $(n, m, t, m - \frac{m}{\bar{c}}, \epsilon)$ somewhere random merger, where $\bar{c}$ is the constant in lemma 4.3. Then, for any $\bar{m} \leq m \leq \bar{\bar{m}}$ there is an explicit $(n, m, O(\bar{m} \cdot log(\frac{1}{\epsilon}) + log(n) \cdot t), \Omega(m), poly(n) \cdot \epsilon)$, extractor E.*

Proof: Let $b = c_{sz} \cdot (1 - \frac{1}{\bar{c}})$. Clearly b is a constant, and $1 < b < c_{sz}$. Define $m_i = b^i \cdot \bar{m} \cdot log(\frac{1}{\epsilon})$, and let l be the first integer s.t. $\Sigma_{i=1}^{l} 2m_i \leq \frac{m}{2}$.

Define $E = E_l \overset{M_{l-1}}{\odot} E_{l-1} \ldots \overset{M_1}{\odot} E_1$, where:

- E_i is the $(n, 2m_i, m_i, c_{sz} \cdot m_i, 2^{-m_i/2})$–extractor A_{m_i} from lemma 3.1
- M_i is the $(n, c_{sz} \cdot m_{i+1}, t, b \cdot m_{i+1}, \epsilon)$–somewhere random merger promised in the hypothesis of the lemma.

Now we use theorem 5 with $t_i = m_i$ and $s_i = (c_{sz} - b)m_{i-1}$, and we also take $s = \frac{m}{2l}$. By Theorem 5, E is an $(n, \Sigma_{i=1}^{l} 2m_i + (l-1)s, t_1 + \Sigma_{i=1}^{l-1} t, t_{l+1}, \Sigma_{i=1}^{l} 2^{-m_0/2} + \Sigma_{i=1}^{l-1} \epsilon + (l-1)n2^{-s/3+3})$ –extractor. Since $l = O(log(n))$ and $\epsilon \geq 2^{-s/3}$ (otherwise the result is trivial), E is an extractor as required. Since A_i, M_i are explicit, so is E. □

B.3 Proof of lemma 4.5

Proof: Choose $\delta = \frac{1-\gamma}{2}$ and $\beta = 1 - \frac{\delta}{2}$. Let the extractor E be $E = B_m \overset{M}{\odot} E_{sz}$ where

- E_{sz} is the $(n, n^\beta, O(log^2 n \cdot log(\frac{1}{\epsilon})), n^{2\beta-1}, \epsilon)$ –extractor of lemma 3.2.
- B_m is the extractor from corollary 4.4.
- M is the merger from corollary 4.2.

Since $n^{2\beta-1} = n^\delta \cdot n^\gamma = \Omega(2^{\sqrt{log(n)}} \cdot log(\frac{1}{\epsilon}))$, $E = B_m \overset{M}{\odot} E_{sz}$ is well-defined. By theorem 4, for every m, E is an explicit $(n, m + n^\beta + n^\gamma, polylog(n) \cdot log(\frac{1}{\epsilon}), \Omega(m), poly(n) \cdot \epsilon)$–extractor. In particular if $H_\infty(X) = \Omega(n^\beta)$ we extract $\Omega(H_\infty(X))$ as required. □

B.4 Proof of theorem 4.5

Proof:

- By lemma 3.5, lemma 4.5 implies an explicit $(2n, n, polylog(n) \cdot log(\frac{1}{\epsilon}), n - n^{\beta}, poly(n) \cdot \epsilon)$ extractor.
- There is some constant d (that depends only on γ) s.t. for every $log^d n \leq m \leq n$, $log(n) \cdot m^{\beta} \leq \frac{m}{c}$. Therefore by theorem 3, for every m there is an explicit $(n, m, polylog(n) \cdot log(\frac{1}{\epsilon}), m - \frac{m}{c}, poly(n) \cdot \epsilon)$ somewhere random merger.
- By lemma 4.3, this implies an explicit $(n, m, polylog(n) \cdot log(\frac{1}{\epsilon}), m, poly(n) \cdot \epsilon)$–extractor, for any m. Plugging $\epsilon' = \frac{\epsilon}{poly(n)}$, gives the theorem.

□

C Somewhere Random Mergers

C.1 A Somewhere Random source has large min-entropy

Lemma C.1 *If $X = X_1 \circ \ldots \circ X_d$ is an (m, ϵ, η) somewhere random source, then X is η–close to an $(m, \epsilon, 0)$ somewhere random source X'.*

Proof: [of lemma C.1]

Let Y be an (m, ϵ, η) selector for X. Denote $p = Prob(Y = 0) \leq \eta$. Define the distribution $\overline{D}$ by:

$$\overline{D}(i,x) = \begin{cases} 0 & \text{If } i = 0 \\ \frac{Prob(\ (Y,X)=(i,x)\ )}{1-p} & \text{otherwise} \end{cases}$$

It is easy to see that $\overline{D}$ is a distribution. Define the random variable $Y' \circ X'$ as the result of the picking $(i,x) \in \overline{D}$, i.e. $\overline{Y' \circ X'} = \overline{D}$. It is clear that $d(\overline{X}, \overline{X'}) \leq d(\overline{Y \circ X}, \overline{Y' \circ X'}) = p \leq \eta$.

Now we want to show that Y' is an $(m, \epsilon, 0)$ selector for X'. It is clear that $Prob(Y' = 0) = 0$. For $i > 0$ we have:

$$Prob(Y' = i) = \Sigma_x\ Prob((Y', X') = (i,x)) = \Sigma_x \frac{Prob((Y,X) = (i,x))}{1-p} = \frac{Prob(Y = i)}{1-p}$$

$$Prob(X' = x \mid Y' = i) = \frac{Prob((Y', X') = (i,x))}{Prob(Y' = i)} = \frac{Prob((Y,X) = (i,x))}{1-p} \cdot \frac{1-p}{Prob(Y = i)} =$$

$$\frac{Prob((Y,X) = (i,x))}{Prob(Y = i)} = Prob(X = x \mid Y = i)$$

Therefore, since we know that $(X_i | Y = i)$ is ϵ–close to U_m, we also know that $(X'_i | Y' = i)$ is ϵ close to U_m, thus completing the proof.

□

Lemma C.2 *Let $X = X_1 \circ \ldots \circ X_d$ be an $(m, \epsilon, 0)$–somewhere random source, then X is ϵ close to an $(m, 0, 0)$–somewhere random source Z.*

Proof: [of lemma C.2]

Let Y be an $(m, \epsilon, 0)$ selector for X. Fix some $i \in [1..d]$. We know that $d((X_i \mid Y = i), U_m) \leq \epsilon$. Define a distribution $Z^{(i)}$ by:

$$Z^{(i)}(x) = \begin{cases} \frac{1}{2^m} \cdot Prob(X = x \mid X_i = x_i \text{ and } Y = i) & \text{if } Prob(X_i = x_i \text{ and } Y = i) > 0 \\ \frac{1}{2^m} \cdot 1 & \text{if } Prob(X_i = x_i \text{ and } Y = i) = 0 \\ & \text{and for every } j \neq i:\ x_j = 0^m \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that $Z^{(i)}$ is indeed a distribution, and that $Z_i^{(i)} = U_m$. Define $Y \circ Z$ to be the random variable obtained by choosing i according to Y, then choosing z according to $Z^{(i)}$, i.e., for all $i > 0$, $(Z \mid Y = i) = Z^{(i)}$. Also, denote $X^{(i)} = (X \mid Y = i)$. Then:

$$Prob(Z_i = z_i \mid Y = i) = Z_i^{(i)}(z_i) = 2^{-m}$$

We soon prove that:

Claim C.1 $d(X^{(i)}, Z^{(i)}) \leq \epsilon$.

Thus:

$$d(\overline{X}, \overline{Z}) \leq d(\overline{Y \circ X}, \overline{Y \circ Z}) = \Sigma_{i>0}\ Pr(Y = i) \cdot d((X \mid Y = i), (Z \mid Y = i)) =$$
$$\Sigma_{i>0}\ Pr(Y = i) \cdot d(X^{(i)}, Z^{(i)}) \leq \epsilon$$

Hence Z satisfies the requirements of the lemma.

□

Proof: [of claim C.1]

We need to show that for any $A \subseteq \Lambda_X$, $|X^{(i)}(A) - Z^{(i)}(A)| \leq \epsilon$. It is sufficient to show this for the set A containing all $x \in \Lambda_X$ s.t. $X^{(i)}(x) > Z^{(i)}(x)$.

$$X^{(i)}(A) - Z^{(i)}(A) =$$
$$\Sigma_{x \in A} Prob(X = x \mid Y = i) - Prob(Z = x \mid Y = i) =$$
$$\Sigma_{a_i \in A_i} Prob(X_i = a_i \mid Y = i) \cdot \Sigma_{x \in A} Prob(X = x \mid X_i = a_i \ and\ Y = i) -$$
$$Prob(Z_i = a_i \mid Y = i) \cdot \Sigma_{x \in A} Prob(Z = x \mid Z_i = a_i \ and\ Y = i) =$$
$$\Sigma_{a_i \in A_i} Prob(X_i = a_i \mid Y = i) \cdot \Sigma_{x \in A} Prob(X = x \mid X_i = a_i \ and\ Y = i) -$$
$$Prob(Z_i = a_i \mid Y = i) \cdot \Sigma_{x \in A} Prob(X = x \mid X_i = a_i \ and\ Y = i) =$$
$$\Sigma_{a_i \in A_i} (Prob(X_i = a_i \mid Y = i) - Prob(Z_i = a_i \mid Y = i)) \cdot \Sigma_{x \in A} Prob(X = x \mid X_i = a_i \ and\ Y = i) \leq$$
$$\Sigma_{a_i \in A_i} (Prob(X_i = a_i \mid Y = i) - Prob(Z_i = a_i \mid Y = i)) \cdot 1 \leq d(X_i^{(i)}, Z_i^{(i)}) = d(X_i^{(i)}, U_m) \leq \epsilon$$

□

Lemma C.3 *Let $X = X_1 \circ \ldots \circ X_d$ be an $(m, 0, 0)$ somewhere random source, then $H_\infty(\overline{X}) \geq m$.*

Proof: Suppose Y is an $(m, 0, 0)$ selector for X.

$$Prob(X = x) = \Sigma_{i \in [1..d]}\ Prob(Y = i) \cdot Prob(X = x \mid Y = i) \leq$$
$$\Sigma_{i \in [1..d]}\ Prob(Y = i) \cdot Prob(X_i = x_i \mid Y = i) \leq$$
$$\Sigma_{i \in [1..d]}\ Prob(Y = i) \cdot 2^{-m} = 2^{-m}$$

□

Combining lemmas C.1, C.2 and lemma C.3 we get lemma 5.1.

C.2 Proof of claim 5.1

Proof:

The proof is by downward induction on j. The basis $j = l$ simply says that for any i , $d((X_i \mid Y = i)\ ,\ U_m\ ) = 0$, which is exactly the hypothesis. Suppose it is true for j, we prove it for $j - 1$. By the induction hypothesis:

- $d(\ (Z^j_{2i-1} \mid Y \in [2^{l-j}(2i-2)+1, 2^{l-j}(2i-1)])\ ,\ U_{m_j}\ ) \leq (l-j)\epsilon(m)$
- $d(\ (Z^j_{2i} \mid Y \in [2^{l-j}(2i-1)+1, 2^{l-j}2i])\ ,\ U_{m_j}\ ) \leq (l-j)\cdot\epsilon(m)$

Soon we prove the following lemma:

Lemma C.4 *Let A, B and Y be any random variables. Suppose that $d((A \mid Y \in S_1), U_m) \leq \epsilon$ and $d((B \mid Y \in S_2), U_m) \leq \epsilon$ for some disjoint sets S_1 and S_2. Then $(A \circ B \mid Y \in S_1 \cup S_2)$ is ϵ–close to some $\overline{X}$ with $H_\infty(\overline{X}) \geq m$.*

Therefore:

- $(Z^j_{2i-1} \circ Z^j_{2i} \mid Y \in [2^{l-j+1}(i-1)+1, 2^{l-j+1}i])$ is $(l-j)\cdot\epsilon(m)$ close to some $\overline{W}$ with $H_\infty(\overline{W}) \geq m_j$.
- Since $Z^{j-1}_i = E(Z^j_{2i-1} \circ Z^j_{2i}\ ,\ t_j)$, it follows that $(Z^{j-1}_i \mid Y \in [2^{l-j+1}(i-1)+1, 2^{l-j+1}i])$ is $(l-j)\cdot\epsilon(m)$ close to $E_{m_j}(x, t_k)$ where $x \in \overline{X}$ and $H_\infty(\overline{X}) \geq m_j$. Therefore, it is $(l-j)\cdot\epsilon(m) + \epsilon(m)$ close to random, as required.

□

C.2.1 Proof of lemma C.4

Proof: We define random variables $Y', A' \circ B'$ as follows:

- Pick $Y' = i \in S_1 \cup S_2$ with: $Pr(Y' = i) = Pr(Y = i \mid Y \in S_1 \cup S_2)$.
- Choose $a' \circ b' \in (A \circ B \mid Y = i)$.

Claim: $Pr(A' = a' \mid Y' = i) = Pr(A = a' \mid Y = i)$ and $Pr(B' = b' \mid Y' = i) = Pr(B = b' \mid Y = i)$.

Proof:

$$Pr(A' = a' \mid Y' = i) = \Sigma_{b'} Pr(A' \circ B' = a' \circ b' \mid Y' = i) =$$
$$\Sigma_{b'} Pr(A \circ B = a' \circ b' \mid Y = i) = Pr(A = a' \mid Y = i)$$

And similarly for B. □

Define

$$Z' = \begin{cases} 1 & \text{If } Y' \in S_1 \\ 2 & \text{Otherwise, i.e. } Y' \in S_2 \end{cases}$$

Claim C.2 $(A' \mid Z' = 1) = (A \mid Y \in S_1)$ *and* $(B' \mid Z' = 2) = (B \mid Y \in S_2)$.

Proof: First we observe that for $i \in S_1$:

$$Pr(Y' = i \mid Y' \in S_1) = \frac{Pr(Y' = i)}{Pr(Y' \in S_1)} = \frac{Pr(Y = i \mid Y \in S_1 \cup S_2)}{Pr(Y \in S_1 \mid Y \in S_1 \cup S_2)} = Pr(Y = i \mid Y \in S_1)$$

Therefore,

$$Pr(A' = a' \mid Z' = 1) \;=\; \Sigma_{i \in S_1} Pr(Y' = i \mid Z' = 1) \cdot Pr(A' = a' \mid Y' = i) \;=$$
$$\Sigma_{i \in S_1} Pr(Y = i \mid Y \in S_1) \cdot Pr(A = a' \mid Y = i) \;=\; Pr(A = a' \mid Y \in S_1)$$

□

Hence, Z' is an $(m, \epsilon, 0)$ selector for $A' \circ B'$.

Therefore by lemma 5.1, $\overline{A' \circ B'}$ is ϵ–close to some $\overline{X}$ with $H_\infty(\overline{X}) \geq m$. However,

Claim: $\overline{A' \circ B'} \;=\; (A \circ B \mid Y \in S_1 \cup S_2)$.

Proof:

$$Pr(A' \circ B' = a' \circ b') \;=$$
$$\Sigma_{i \in S_1 \cup S_2} Pr(A' \circ B' = a' \circ b' \mid Y' = i) \cdot Pr(Y' = i) \;=$$
$$\Sigma_{i \in S_1 \cup S_2} Pr(A \circ B = a' \circ b' \mid Y = i) \cdot Pr(Y = i \mid Y \in S_1 \cup S_2) \;=$$
$$Pr(A \circ B = a' \circ b' \mid Y \in S_1 \cup S_2)$$

□

Thus, $(A \circ B \mid Y \in S_1 \cup S_2) \;=\; \overline{A' \circ B'}$ is ϵ–close to some $\overline{X}$ with $H_\infty(\overline{X}) \geq m$, completing the proof.

□

D Composing Extractors

D.1 Composing Two Extractors

Proof: [of claim 6.1]

For any w s.t. $Y(w) = i$:

$$Prob(X_{[i,n]} = w_{[i,n]} \mid X_{[1,i-1]} = w_{[1,i-1]},\ Y(x) = i) \quad \leq \quad \text{(Since } Prob(A \mid B) \leq \tfrac{Prob(A)}{Prob(B)} \text{)}$$

$$\frac{Prob(\ X_{[i,n]}=w_{[i,n]} \mid X_{[1,i-1]}=w_{[1,i-1]}\ )}{Prob(Y(x)=i \mid X_{[1,i-1]}=w_{[1,i-1]}\ )} \quad \leq \quad \text{(Since } f(w) = i\text{)}$$

$$\frac{(\epsilon_2-\epsilon_3)\cdot 2^{-m_1}}{Prob(Y(x)=i \mid X_{[1,i-1]}=w_{[1,i-1]}\ )} \quad \leq \quad \text{(By Claim D.1)}$$

$$\frac{(\epsilon_2-\epsilon_3)\cdot 2^{-m_1}}{\epsilon_2-\epsilon_3} = 2^{-m_1}$$

□

Proof: [of claim 6.2]

Take any $w_{[1,i-1]}$ that can be extended to some w with $Y(w) = i$.

$$Prob(X_{[1,i-1]} = w_{[1,i-1]}) \ = \ \frac{Prob(X_{[1,n]} = w_{[1,n]})}{Prob(X_{[i,n]} = w_{[i,n]} \mid X_{[1,i-1]} = w_{[1,i-1]})} \ = \ \frac{Prob(X_{[1,n]} = w_{[1,n]})}{Prob(X_i = w_i \mid X_{[1,i-1]} = w_{[1,i-1]}) \ \cdot Prob(X_{[i+1,n]} = w_{[i+1,n]} \mid X_{[1,i]} = w_{[1,i]})}$$

However,

- $Prob(X_{[i+1,n]} = w_{[i+1,n]} \mid X_{[1,i]} = w_{[1,i]}) \geq (\epsilon_2 - \epsilon_3)2^{-m_1}$ (Since $f(w) = i$)
- $Prob(X_i = w_i \mid X_{[1,i-1]} = w_{[1,i-1]}) \geq \epsilon_3$ (Since $w \notin B_3$)
- $Prob(X_{[1,n]} = w_{[1,n]}) \leq 2^{-(m_1+m_2+s)}$ (Since $H_\infty(X) \geq m_1 + m_2 + s$)

Thus,

$$Prob(X_{[1,i-1]} = w_{[1,i-1]}) \leq \frac{2^{-m_1-m_2-s}}{\epsilon_3 \ \cdot(\epsilon_2 - \epsilon_3)2^{-m_1}} \ = \ \frac{2^{-m_2-s}}{\epsilon_3 \ \cdot(\epsilon_2 - \epsilon_3)} \qquad (1)$$

Therefore,

$$Prob(X_{[1,i-1]} = w_{[1,i-1]} \mid Y(x) = i) \quad \leq \quad \text{(Since } Prob(A \mid B) \leq \tfrac{Prob(A)}{Prob(B)} \text{)}$$

$$\frac{Prob(\ X_{[1,i-1]}=w_{[1,i-1]})}{Prob(Y(x)=i)} \quad \leq \quad \text{(By Eq. (1)}$$

$$\frac{2^{-m_2-s}}{\epsilon_3\cdot(\epsilon_2-\epsilon_3)\cdot Prob(Y(x)=i)} \quad \leq \quad \text{(By Claim D.2)}$$

$$\frac{2^{-m_2-s}}{\epsilon_3 \ \cdot(\epsilon_2-\epsilon_3)\cdot(\epsilon_1-\epsilon_2-\epsilon_3)} \ = \ \frac{2^{-m_2-s}}{\epsilon_3^3} \ = \ 2^s \cdot 2^{-m_2-s} \ = \ 2^{-m_2}$$

□

D.2 Technical Lemmas

In this subsection we prove some technical lemmas used in subsection 6.1.

Claim D.1 *For any i and $w_{[1,i-1]}$, if $Prob_{x\in X}(Y(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) > 0$, Then $Prob_{x\in X}(Y(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \geq \epsilon_2 - \epsilon_3$.*

Proof:

Since $w_{[1,i-1]}$ can be extended to some w with $Y(w) = i \neq 0$, by definition 6.2:

$$Prob(f(x) = i) \geq \epsilon_1 \ , and$$
$$Prob(f(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \geq \epsilon_2$$

However, this implies that for *any* extension w' of $w_{[1,i-1]}$, with $f(w') = i$, it holds that $w' \notin B_1 \cup B_2$. Hence,

$$Prob(Y(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \ =$$
$$Prob(f(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \ - \ Prob(f(x) = i \ and \ x \in B \mid x_{[1,i-1]} = w_{[1,i-1]}) \ =$$
$$Prob(f(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \ - \ Prob(f(x) = i \ and \ x \in B_3 \mid x_{[1,i-1]} = w_{[1,i-1]}) \ \geq$$
$$\epsilon_2 - \epsilon_3$$

The last inequality uses claim D.3.

□

Claim D.2 *For any i, if $Prob_{x\in X}(Y(x) = i) > 0$, then $Prob_{x\in X}(Y(x) = i) \geq \epsilon_1 - \epsilon_2 - \epsilon_3$.*

Proof:

Since there is some w' s.t. $Y(w') = i \neq 0$, by definition 6.2:

$$Prob(f(x) = i) \geq \epsilon_1$$

This implies that for *any* w' with $f(w') = i$, we know that $w' \notin B_1$. Hence,

$$Prob(Y(x) = i) \ =$$
$$Prob(f(x) = i) \ - Prob(f(x) = i \ and \ x \in B) \ \geq$$
$$Prob(f(x) = i) \ - Prob(f(x) = i \ and \ x \in B_2) - Prob(f(x) = i \ and \ x \in B_3) \ \geq$$
$$\epsilon_1 - \epsilon_2 - \epsilon_3$$

The last inequality uses claim D.3.

□

Claim D.3

1. $Prob(x \in B_1) \leq n\epsilon_1$
2. *For any i:* $Prob(f(x) = i \text{ and } x \in B_2) \leq \epsilon_2$
3. $Prob(x \in B_2) \leq n\epsilon_2$
4. *For any i and* $w_{[1,i-1]}$*:* $Prob(f(x) = i \text{ and } x \in B_3 \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_3$
5. *For any i:* $Prob(f(x) = i \text{ and } x \in B_3) \leq \epsilon_3$
6. $Prob(x \in B_3) \leq n\epsilon_3$

Proof:

1) If there is an x with $f(x) = i \text{ and } x \in B_1$, then $Prob(f(x) = i) \leq \epsilon_1$. Thus, $Prob(x \in B_1 \text{ and } f(x) = i) \leq \epsilon_1$, and $Prob(x \in B_1) \leq \Sigma_{i=1}^{n} Prob(x \in B_1 \text{ and } f(x) = i) \leq n\epsilon_1$.

2) If for some $w_{[1,i-1]}$ $Prob(f(x) = i \text{ and } x \in B_2 \mid x_{[1,i-1]} = w_{[1,i-1]}) > 0$ then there is an extension w of $w_{[1,i-1]}$ s.t.: $f(w) = i \text{ and } w \in B_2$, and therefore, $Prob(f(x) = i \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_2$. Thus, for all $w_{[1,i-1]}$, $Prob(f(x) = i \text{ and } x \in B_2 \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_2$. Therefore, $Prob(f(x) = i \text{ and } x \in B_2) = \Sigma_{w_{[1,i-1]}} Prob(x_{[1,i-1]} = w_{[1,i-1]}) \cdot Prob(f(x) = i \text{ and } x \in B_2 \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \Sigma_{w_{[1,i-1]}} Prob(x_{[1,i-1]} = w_{[1,i-1]}) \cdot \epsilon_2 \leq \epsilon_2$.

3) $Prob(x \in B_2) \leq \Sigma_{i=1}^{n} Prob(x \in B_2 \text{ and } f(x) = i) \leq n\epsilon_2$.

4) If for some $w_{[1,i-1]}$ $Prob(f(x) = i \text{ and } x \in B_3 \mid x_{[1,i-1]} = w_{[1,i-1]}) > 0$ then there is an extension w of $w_{[1,i-1]}$ s.t.: $f(w) = i \text{ and } w \in B_3$, and therefore, $Prob(x_i = w_i \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_3$. In particular, $Prob(x \in B_3 \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq Prob(x_i = w_i \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_3$. Thus, for all $w_{[1,i-1]}$, $Prob(f(x) = i \text{ and } x \in B_3 \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \epsilon_3$.

5) $Prob(f(x) = i \text{ and } x \in B_3) \leq \Sigma_{w_{[1,i-1]}} Prob(x_{[1,i-1]} = w_{[1,i-1]}) \cdot Prob(f(x) = i \text{ and } x \in B_3 \mid x_{[1,i-1]} = w_{[1,i-1]}) \leq \Sigma_{w_{[1,i-1]}} Prob(x_{[1,i-1]} = w_{[1,i-1]}) \cdot \epsilon_3 \leq \epsilon_3$.

6) $Prob(x \in B_3) \leq \Sigma_{i=1}^{n} Prob(x \in B_3 \text{ and } f(x) = i) \leq n\epsilon_3$.

□

D.3 Composing Many Extractors

Given the extractors E_i and mergers M_i, we want to define the composition $E_k \overset{M_{k-1}}{\odot} E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2 \overset{M_1}{\odot} E_1$ as a series of compositions of two extractors. Clearly, when defining this composition we can choose between left or right associativity, which turns out to be very different.

We compare these two alternatives by the number of truly random bits needed to compose the extractors, and the running time of the composed extractor.

Number of truly random bits :

- left associativity- $E = (E_k \overset{M_{k-1}}{\odot} E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2) \overset{M_1}{\odot} E_1$

  This way the extractor E_1 extracts enough randomness for the extractor $(E_k \overset{M_{k-1}}{\odot} E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_2)$, including its inner merges. Thus, this composition is very efficient, and we pay only for one merge.

- right associativity- $E = E_k \overset{M_{k-1}}{\odot} (E_{k-1} \overset{M_{k-2}}{\odot} \ldots E_1)$

  This way we have to invest truly random bits for all the merges.

Thus, left associativity is more efficient.

Running Time :

For both alternatives the trivial running time is n^k. However, as can be seen from the proof of Theorem 5 , when using right associativity, we know how to compose extractors in polynomial time using a dynamic programming algorithm. This does not seem to work in the left associativity case.

Thus, we choose to use right associativity. However, we still feel unsure about the right way extractors should be composed.

E Applications

E.1 Review of Results

E.1.1 a-expanding Graphs

DEFINITION E.1 *[Pip87] An undirected graph is a–expanding if any two disjoint sets of vertices of size at least a are joined by an edge.*

Using our extractor we get:

Corollary E.1 *(following [WZ93]) For every N and $1 \leq a \leq N$, there is an efficiently constructible a–expanding graph with N vertices, and maximum degree $O(\frac{N}{a} 2^{polyloglog(N)})$.*

Corollary E.2 *(following [Pip87], see [WZ93] lemma 5) There are explicit algorithms for sorting in k rounds using $O(n^{1+\frac{1}{k}} \cdot 2^{polyloglog(n)})$ comparisons, and for selecting in k rounds using $O(n^{1+\frac{1}{2^k-1}} \cdot 2^{polyloglog(n)})$ comparisons.*

Corollary E.3 *(following [AKSS89], see [WZ93] lemma 6) There are explicit algorithms to find all relations except $O(a \cdot nlog(n))$ among n elements, in one round and using $O(\frac{n^2}{a} \cdot 2^{polyloglog(n)})$ comparisons.*

E.1.2 Superconcentrators of small depth

DEFINITION E.2 *$G = ((A, C, B), E)$ is a superconcentrator if G is a layered graph with input vertices A, output vertices B, and for any sets $X \subseteq A$,$Y \subseteq B$ of size k, there are at least k vertex-disjoint paths from X to Y.*

Much research was done on finding small explicit superconcentrators of small depth (see [WZ93] for references). We achieve:

Corollary E.4 *(following [WZ93]) For every N there is an efficiently constructible depth 2 superconcentrator over N vertices with size $O(N \cdot 2^{polyloglog(N)})$.*

Corollary E.5 *(Following [WZ93],lemma 10) For every N there is an explicitly constructible superconcentrator over N vertices, with linear size and $polyloglog(N)$ depth .*

E.1.3 Hardness of Approximating The Iterated Log of Max Clique.

DEFINITION E.3 *We denote $\underbrace{loglog \ldots log}_{k} n$ by $log^{(k)}(n)$. We define $P_{e,k}(n)$ by:*

$$P_{e,k}(n) = 2^{2^{\cdot^{\cdot^{elog^{(k)}n}}}} \Big\} \; k \; 2's$$

We denote the size of the biggest clique in G by $\omega = \omega(G)$.

Corollary E.6 *(following [Zuc93] [6]) If for any constant b, approximating $log^{(k)}\omega$ to within a factor of b is in $\bigcup_e DTime(P_{e,k}(n))$, then $\bigcup_e NTime(P_{e,k}(n)) = \bigcup_e DTime(P_{e,k}(n))$.*

[6] This lemma is a deterministic version of the lemma appearing in [Zuc93]. The only difference is that we are able to replace the randomly chosen extractor in [Zuc93] with our new constructible extractor. A proof of this corollary appears in subsection E.4.

E.1.4 Simulating BPP Using a Weak Random Source

A direct corollary of Theorem 2:

Corollary E.7 *For any $\delta > 0$ and $k > 0$, BPP can be simulated in time $n^{O(log^{(k)}n)}$ using a weak random source X with min entropy at least n^{δ} .*

E.2 a–expanders

Here we prove corollary E.1 which easily follows from the following lemma:

Lemma E.8 *If for every m there is an $(n, m, t = t(n), m, \epsilon = \frac{1}{4})$–extractor, then for every $1 \leq a \leq N = 2^n$ there is an efficiently constructible a–expanding graph with N vertices, and maximum degree $O(\frac{N}{a}2^{2t})$.*

Proof: (following [WZ93] [7])

Let V be a set with $N = 2^n$ vertices, and W a set with $a = 2^m$ vertices.

- Define a bipartite graph $G = (V, W, E)$, s.t. $(x, y) \in E \iff \exists r \in \{0,1\}^t \; s.t. \; y = E(x, r)$.
- Denote $BAD = \{w \in W \mid deg(w) > 2d_{avg}\}$ where $d_{avg} = \frac{2^n \cdot 2^t}{2^m}$ is the average degree of vertices in W.
- Let H be the induced graph of G over $V \cup (W \setminus BAD)$, and let the output be H^2.

Claim: $|BAD| \leq \epsilon \cdot 2^m$.

Proof: Consider the uniform distribution $\overline{D}$ over $V = \{0,1\}^n$. Then:

$\cdot$ $Prob_{x \in \overline{D}, r \in U_t}(y \in BAD) \leq \frac{|BAD|}{2^m} + \epsilon$ (Since E is an extractor)

$\cdot$ $Prob_{x \in \overline{D}, r \in U_t}(y \in BAD) \geq \Sigma_{b \in BAD} \frac{2d_{avg}}{2^{n+t}} = \frac{|BAD| \cdot 2 \cdot d_{avg}}{2^{n+t}} = \frac{2 \cdot |BAD|}{2^m}$ (Since $Pr(y = b) \geq \frac{2d_{avg}}{2^{n+t}}$)

Hence, $\frac{|BAD|}{2^m} \leq \epsilon$. □

Claim: For any $X \subseteq V$ s.t. $|X| \geq 2^m$, $|\Gamma(X)| \geq (1 - \epsilon)2^m$

Proof: Suppose not. Take the uniform distribution $\overline{D}$ over X. Clearly, $H_{\infty}(\overline{D}) \geq m$. However, since $|\Gamma(X)| \leq (1 - \epsilon)2^m$, it follows that $d(E(\overline{D}, U_t), U_m) \geq \epsilon$, which is a contradiction. □

Claim: For every $X_1, X_2 \subseteq V$ s.t. $|X_1|, |X_2| \geq 2^m$, $|(\Gamma(X_1) \cap \Gamma(X_2)) \setminus BAD| > 0$.

Proof:

$$|(\Gamma(X_1) \cap \Gamma(X_2)) \setminus BAD| \geq |\Gamma(X_1) \cap \Gamma(X_2)| - |BAD| \geq$$

$$(1 - 2\epsilon)2^m - \epsilon 2^m = (1 - 3\epsilon)2^m \geq \frac{2^m}{4} > 0$$

□

Hence in H^2:

[7] The proof follows a simple idea from [WZ93]. [WZ93] used also a complicated recursive construction that we can avoid completely because we use our stronger extractor.

- For every $X_1, X_2 \subseteq V$, s.t. $|X_1|, |X_2| \geq 2^m = a$, there is an edge going from X_1 to X_2.
- For every vertex in V the degree is bounded by $2^t \cdot 2d_{avg} = 2^t \cdot 2\frac{2^n \cdot 2^t}{2^m} = 2 \cdot 2^t \cdot \frac{N}{a} \cdot 2^t$.

□

Corollaries E.2 and E.3 follow corollary E.1 by [WZ93] lemmas 5 and 6 respectively.

E.3 Superconcentrators

First we prove corollary E.4:

Lemma E.9 *[Mes84] $G = ((A, C, B), E)$ is a superconcentrator of depth 2, iff for any $1 \leq k \leq n$ and any sets $X \subseteq A$,$Y \subseteq B$ of size k, $|\Gamma(X) \cap \Gamma(Y)| \geq k$.*

Following [WZ93], good extractors yield small superconcentrators of depth 2.

Lemma E.10 *(following [WZ93]) If for every m there is an $(n, m, t, m+3, \frac{1}{3})$ extractor E_m, then we can efficiently build a superconcentrator over N vertices of depth 2 and size $N \cdot log(N) \cdot 2^t$*

Proof:

We describe the superconcentrator we build:

Input and output layers. The input and output layers A and C are of sizes $N = 2^n$. Thus we can identify each input/output vertex with a string in $\{0,1\}^n$.

The middle layer. We let the middle layer B be the union of n disjoint sets $B_1, \ldots, B_n$, $|B_m| = 4 \cdot 2^{m+1}$. Again, we describe each vertex in B_m as a string in $\{0,1\}^{m+3}$.

Edges going from A to B. For every $x \in A = \{0,1\}^n$, $1 \leq m \leq n$ and $r \in \{0,1\}^t$ we add an edge going from x to $E_m(x, r) \in \{0,1\}^{m+3} = B_m$.

Edges going from C to B. are the mirror image of the edges going from A to B.

Claim E.1 *For any $X \subseteq A$ of size $2^m \leq k \leq 2^{m+1}$, $|\Gamma(X) \cap B_m| \geq \frac{2}{3}|B_m|$.*

Proof: Consider the uniform distribution $\overline{D}$ over X. Clearly, $H_\infty(\overline{D}) \geq m$. Hence, $d(E(\overline{D}, U_t), U_{m+3}) \leq \frac{1}{3}$. However, $|\Gamma(X) \cap B_m| < \frac{2}{3}|B_m|$ implies that $d(E(\overline{D}, U_t), U_{m+3}) > \frac{1}{3}$ - a contradiction. □

Therefore, for any $X \subseteq A$ and $Y \subseteq C$ of size $2^m \leq k \leq 2^{m+1}$, $|\Gamma(X) \cap \Gamma(Y) \cap B_m| \geq \frac{1}{3}|B_m| \geq k$. Hence by lemma E.9, our graph is a superconcentrator. □

Now we turn to the proof of corollary E.5:

Proof: [Of corollary E.5]

[WZ93] show that an explicit depth 2 superconcentrator of size $O(N \cdot f(N))$ implies an explicit linear-size superconcentrator of depth $O(log(f(N)))$. Hence, the corollary follows corollary E.4. □

E.4 Approximating the iterated log of clique size

Fact E.1 *[ALM+92, AS92]* $NP \subseteq PCP(O(log(n), O(1), O(log(n)), O(1), \frac{1}{2})$.

Lemma E.11 *(following [Zuc91]) If there is an* $(r+k, r, t, r, \frac{1}{2})$*-extractor, then* $PCP(r, m, q, a, \frac{1}{2}) \subseteq PCP(r+k, 2^t m, q, a, 2^{-k})$.

Proof: Define the bipartite graph $G = (A, B, E)$ where $A = \{0,1\}^{r+k}$,$B = \{0,1\}^r$, and $(a, b) \in E$ iff $\exists r \in \{0,1\}^t$ s.t. $E(a, r) = b$. Now define the following protocol: the verifier V chooses $a \in A$ randomly, and runs the original protocol 2^t times, with the strings $E(a, r)$ for all possible values of $r \in \{0,1\}^t$. Using the extractor's properties, it is easy to see that the error rate (those $a \in A$ that do not have "bad" neighbors) is less than $\frac{2^r}{2^{r+k}} = 2^{-k}$. □

Corollary E.12 *For any* $k \geq log(n)$, $NP \subseteq PCP(O(log(n))+k, 2^{polylog(k)}, O(log(n)), O(1), 2^{-k})$ [8].

We now turn to proving corollary E.6. Instead of proving this corollary in its generality, we prefer to prove a special case that demonstrates the idea and has somewhat "nicer" parameters. The proof of the general case is almost the same.

Lemma E.13 *(following [FGL+91, Zuc91, SZ94]) If for any constant* b*, approximating* $log^{(3)}(\omega(G))$ *to within* b *is in* P*, then* $NP \subseteq \bigcup_e DTime(P_{e,3}(n))$.

Proof: [FGL+91]

Let $L \in NP$, $x \in \{0,1\}^n$. By corollary E.12, for any $k \geq log(n)$: $L \in PCP(r = O(log(n)) + k, m = O(2^{polylog(k)}), q = O(log(n)), a = O(1), \epsilon = 2^{-k})$.

Build the transcript graph G [9] of L. G has 2^{r+ma} vertices. If $x \in L$ then $\omega(G) = 2^r$, otherwise $\omega(G) \leq 2^r \epsilon = 2^{O(log(n))}$. Thus, if we take $k = 2^{(loglog(n))^2}$, $log^{(3)}(\omega(G))$ is either bigger than $2log^{(3)}(n)$ or less than $log^{(3)}(n) + O(1)$.

Since we assume we know how to approximate $log^{(3)}(\omega(G))$ to within any constant factor, we can check whether $x \in L$ or not. Thus $L \in DTime(poly(2^m))$. Now, $2^m = 2^{2^{polylog(k)}} = 2^{2^{polyloglog(n)}} = 2^{2^{2^{O(loglogloglog(n))}}}$. Thus $L \in \bigcup_e DTime(P_{e,3}(n))$. □

[8] It is well known that using expanders it is possible to achieve a similar result, however the randomness then is $O(log(n) + k)$

[9] See [FGL+91]

F Sketch of proof of Theorem 2

The results in this subsection are based on the work done in [NZ93, SZ94]. In subsection F.1 we present a short sketch of the [SZ94] extractor, which was given in lemma 3.2, and works for sources having $n^{1/2+\gamma}$ min-entropy using $O(log^2 n)$ truly random bits. In subsection F.2 we show how our new extractor can reduce the number of truly random bits to $O(log(n)loglog(n))$. In subsection F.3 we show how to build such extractors for sources having n^δ min-entropy for any constant $\delta > 0$. In subsection F.4 we put everything together to get the extractor of theorem 2. We only give short sketches of the proofs here, the full proofs will appear in the full version of the paper.

F.1 The Srinivasan and Zuckerman Extractor

Proof: [An informal sketch of the [SZ94] proof of lemma 3.2]

First we need the following lemma from [NZ93]:

Lemma F.1 *[NZ93, SZ94] Let X be a random variable over $\{0,1\}^n$, and suppose $H_\infty(\overline{X}) \geq \delta n$. Choose l indices $i_1, \ldots, i_l$ pairwise independently from $[1,n]$* [10] *and let $b = x_{i_1} \circ \ldots \circ x_{i_l}$. Let B be the random variable (depending on x and the choice of indices) whose value is b. Then the distribution $\overline{B}$ is ϵ close to some distribution $\overline{W}$ with $H_\infty(\overline{W}) \geq \delta' n$, where $\delta' = c' \cdot \delta / log(\delta^{-1})$ for some constant c' and $\epsilon = O(1/\sqrt{\delta' l})$.*

Now we build the extractor:

Algorithm :

1. Get $x \in \overline{X}$, where $H_\infty(\overline{X}) \geq n^{1/2+\gamma}$.
2. Choose $k = O(log(n))$ blocks $b_1, \ldots, b_k$ of length $l = n^{1/2}$. The bits of each block are chosen pairwise independently, and the indices of each block are independent of the indices of the other blocks.
3. Take $t = O(log(n))$ truly random bits, and extract ct bits from B_k using the extractor of lemma 3.1. Then use these ct bits to extract $c^2 t$ bits from B_{k-1}, and continue this "doubling" procedure until we have some n^δ bits.

Correctness (sketch) : Denote by B_i the distribution of the i'th block. By lemma F.1 B_1 has at least $l \cdot \frac{n^{1/2+\gamma}}{n} = n^\gamma$ min-entropy, with high probability. Since $|B_1| = l = n^{1/2}$, for most prefixes b_1, $H_\infty(X \mid B_1 = b_1)$ is at least $n^{1/2+\gamma} - n^{1/2}$, which is almost $n^{1/2+\gamma}$. Repeating this process we see that for most prefixes each block B_i conditioned on the prefix $b_{[1,i-1]}$ has at least $l \cdot \frac{n^{1/2+\gamma}}{n} = n^\gamma$ min-entropy.

Thus, by lemma 3.3 the doubling procedure in step (3) gives us a quasi-random string.

Number of truly random bits : To choose a block pairwise independently we need $O(log(n))$ random bits. Thus, to choose the k blocks we need $O(k \cdot log(n))$ truly random bits.

□

[10] This is almost accurate. For the precise way of choosing the indices see [SZ94].

F.2 $O(log(n)loglog(n))$ truly random bits suffice

Using our new extractor, we can easily reduce the number of truly random bits needed in the above algorithm. Notice that it is sufficient to take only $k = O(loglog(n)) + 1$ blocks, use the doubling technique to get $polylog(n)$ quasi-random bits, and then use them in our new extractor to extract all the n^γ min-entropy present in B_1. Thus we see that:

Lemma F.2 *For every constant* $1/2 > \gamma > 0$ *there is an* $(n, n^{1/2+\gamma}, O(log(n)loglog(n)), \Omega(n^\gamma), \frac{1}{n})$ *–extractor.*

In fact this actually shows that:

Lemma F.3 *If for every* $\delta > 0$ *there is* $\delta_1 > 0$ *and an* $(n, n^\delta, O(log(n)log^{(k)}n), n^{\delta_1}, \frac{1}{n})$*– extractor, then for every* $1/2 > \gamma > 0$ *there is* $\delta_2 > 0$ *and an* $(n, n^{1/2+\gamma}, O(log(n)log^{(k+1)}n), \Omega(n^{\delta_2}), \frac{1}{n})$ *–extractor.*

F.3 Extracting Randomness from Weaker Sources

Now we consider what happens when the source X has less min-entropy. First let us see why we required that $H_\infty(\overline{X}) \geq n^{1/2+\gamma}$, in the above algorithms. Suppose $H_\infty(\overline{X}) = n^{1/2}$. Then, even if we randomly choose the l indices of B_1 from $[1, n]$, we still need l to be at least $n^{1/2}$ to ensure that $H_\infty(\overline{B_1}) > 1$. Thus, it might happen that for most prefixes b_1, $H_\infty(X \mid B_1 = b_1)$ is very small, and we have no chance of getting a good second block.

However, if this happens and $H_\infty(X \mid B_1)$ is frequently small, then intuitively this means that with high probability B_1 takes most of the randomness present in X. Thus $H_\infty(B_1)$ should be large, and therefore B_1 is much more "condensed" than X. This suggests the following algorithm, to extract randomness from a source X with $H_\infty(\overline{X}) = n^{1/3+\gamma}$, using $O(log(n)loglog(n))$ truly random bits.

Algorithm :

- Choose $x \in X$, and truly random (short) strings t, t_i.
- Extract k ($k = O(loglog(n))$) blocks $b_1, \ldots, b_k$ of length $l = n^{2/3}$, pairwise independently as before.
- Compute $z_i = E(b_i, t)$, where E is the extractor of lemma 3.2.
- Let z_{k+1} be the result of running the extractor of lemma F.3 over these k blocks.
- Define $z = z_1 \circ \ldots \circ z_{k+1}$, and let the output be $E(z, t)$.

Correctness, a very informal sketch : Roughly speaking, if i is the first index s.t. $H_\infty(X \mid b_{[1,i]}) < \frac{n^{1/3+\gamma}}{2}$, then B_i (conditioned on $b_{[1,i-1]}$) has at least $n^{1/3+\gamma/2}$ min entropy, while B_i has only $l = n^{2/3}$ bits. Therefore B_i is condensed enough for the [SZ94] extractor, and z_i (with the appropriate preconditioning) is quasi-random. On the other hand, if for no i $H_\infty(X \mid b_{[1,i]}) < \frac{n^{1/3+\gamma}}{2}$, then the doubling procedure works, and z_{k+1} (under the appropriate preconditioning) is quasi-random.

Thus, for any $b = b_{[1,k]}$ we can assign $Y = Y(b) = i \in [1, k+1]$ according to the first i (if at all) s.t. $H_\infty(X \mid b_{[1,i]}) < \frac{n^{1/3+\gamma}}{2}$, and we can show that Y is an $(n^\delta, \frac{1}{n}, \frac{1}{n})$ selector for $Z = Z_1 \circ \ldots Z_{k+1}$.

By lemma 5.1, $H_\infty(\overline{Z}) \geq n^\delta$, while Z contains only $kn^\delta = O(log(n))n^\delta$ bits. Therefore, Z is very condensed, and in particular after applying the [SZ94] extractor of lemma 3.2 we get a quasi-random string.

This shows that:

Lemma F.4 *For every $\gamma > 0$ there is some constant $\delta > 0$ and an $(n, n^{1/3+\gamma}, O(log(n)loglog(n)), \Omega(n^\delta), \frac{1}{n})$– extractor.*

Using the same algorithm but with the extractor of lemma F.4, we can get an extractor for $n^{1/4+\gamma}$. Repeating this any constant number of times we get:

Lemma F.5 *For every $\gamma > 0$ there is some constant $\delta > 0$ and an $(n, n^\gamma, O(log(n)loglog(n)), \Omega(n^\delta), \frac{1}{n})$ extractor.*

F.4 Doing it with $O(log(n)log^{(k)}n)$ random bits

Lemma F.3 together with lemma F.5 show that there is an $(n, n^{1/2+\gamma}, O(log(n)log^{(3)}n), n^\delta, \frac{1}{n})$– extractor E_2.

However, now we can repeat algorithm F.3, for this smaller $k = O(log^{(3)}n)$, and with the new $n^{1/2+\gamma}$ extractor E_2. This gives us an $(n, n^{\delta_1}, O(log(n)log^{(3)}n), n^{\delta_2}, \frac{1}{n})$– extractor.

Repeating this process any constant number of times we get Theorem 2.