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Comment 01 on ECCC TR96-023

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An Easy Extension

In ECCC TR96-023, I stated that, although I could prove that there *exists* a set in the counting hierarchy that cannot be computed by small uniform constant-depth threshold circuits, nonetheless I could not present any particular set that requires large size in this model. Indeed, for the particular bound stated there (the size must be larger than $T(n)$, if T is a function such that $T(T(n)) = 2^{o(n)}$), that is still the case.

However, if we weaken the bound only slightly, and consider functions such that, for *all* k $t^{(k)}(n) = 2^{o(n)}$ (where $t^{(k)}$ is t composed with itself k times), then we can show that any set hard for $\mathrm{C_=P}$ requires size greater than $t(n)$ to compute with uniform constant-depth threshold circuits. (Note that, for essentially every "natural" function T of interest, if $T^{(2)}(n) = 2^{o(n)}$, then for every k, $T^{(k)}(n) = 2^{o(n)}$. Thus this size bound, although much smaller in some sense, is not significantly weaker for functions that are likely to be of interest.)

In particular, as a consequence, the permanent cannot be computed by uniform constant-depth threshold circuits of size less than $t(n)$, for any such t.

Theorem 1. *Let t be a constructible function such that for all k, $t^{(k)}(n) = 2^{o(n)}$. Let A be any set that is hard for $C_=P$ under uniform TC^0 many-one reductions. Then A cannot be computed by uniform constant-depth threshold circuits of size $t(n)$.*

Proof: Assume otherwise. Then we will show that for every set B in the counting hierarchy, there is some k such that B has uniform constant-depth threshold circuits of size $T(n) = t^{(k)}(n)$. But since $T(T(n)) = 2^{o(n)}$, this contradicts Theorem 6 of the paper.

For the purposes of this proof, define CH_1 to be $\mathrm{C_=P}$, and for $i > 1$, define CH_i to be $\mathrm{C_=P}^{\mathrm{CH}_{i-1}}$.

First note that, under the assumption, $\mathrm{C_=P}$ has circuits of size $t(n^{O(1)}) < t(t(n))$. (The circuit consists of a poly-size TC^0 reduction from the $\mathrm{C_=P}$ set to B, followed by a circuit computing membership in B.) (Here, we are assuming without loss of generality that $t(n) \geq n^{\log n}$. Otherwise, we can take t' to be the maximum of $t(n)$ and $n^{\log n}$.)

Now assume that all sets in CH_i have uniform constant-depth circuits of size $t^{(3i)}(n)$, and consider a set $A \in \mathrm{CH}_{i+1}$. Thus there is some nondeterministic machine M and a set $D \in \mathrm{CH}_i$ such that M^D has exactly as many accepting paths as rejecting paths on input x if and only if $x \in A$. The set $\{(x, C) : M$ has exactly as many accepting paths as rejecting paths on input x, when all oracle queries are answered according to the circuit $C\}$ is in $\mathrm{C_=P}$, and by hypothesis has circuits of size $t(t(|(x, C)|)) \leq t^{(2)}(n + t^{(3i)}(n)) \leq t^{(3(i+1))}(n)$.

□