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Evaluation of an Approximate Algorithm for the Everywhere Dense Vertex Cover Problem *

Anton Eremeev

e-mail: eremeev@iitam.omsk.net.ru

Abstract

We generalize the DVC algorithm (see [4]) for the weighted case of vertex cover problem (VCP) and study the performance of this algorithm. An extension of result from [4] for the weighted case is proposed in terms of a new density parameter. Given a graph $G(V, E)$ let there be $\rho > 0$ such that $w(O(v)) \geq \rho w(V)$ for any vertex $v \in V$. Here $w(O(v))$ is the total weight of the neighbours of v, and $w(V)$ is the total weight of V. Then $\frac{2}{1+\rho}$ performance guarantee holds for an algorithm similar to DVC. The value ρ can be easily estimated for everywhere ε-dense VCP, if there is such d that $w_u \leq d w_v$ for any pair of vertices u and v. The generalization of DVC allows us to propose another polinomial-time algorithm for the weighted VCP with the performance ratio as a function on $|V|$ which is less than $\frac{2}{1+\rho}$.

1 Definitions and Algorithms

Let $G = (V, E)$ be a graph with vertices V and edges E. A set $C \subseteq V$ is called a vertex cover of G if every edge has at least one endpoint in C. The vertex cover problem (VCP) is given a graph $G = (V, E)$ and weight $w_v \geq 0$ for each $v \in V$, find a vertex cover OPT with minimum total weight $w(OPT)$. Here we denote $w(U) = \sum_{u \in U} w_u$ for any $U \subseteq V$. The unweighted VCP is a case when $w_v = 1$ for all $v \in V$.

In [1] Bar-Yehuda and Even proposed the following 2-approximaton algorithm for the VCP formulated above.

Algorithm 2-Approximation

Set $C = \emptyset$.
While $E \neq \emptyset$ do
  Let $e = (v, v') \in E$, and let $u(e) \in \{v, v'\}$ be such that $w_{u(e)} = min\{w_v, w_{v'}\}$.
  Set $C \leftarrow C \cup \{u(e)\}$, $w_v := w_v - w_{u(e)}$,
  $w_{v'} := w_{v'} - w_{u(e)}$, $E \leftarrow E \backslash E_{u(e)}$.
Return the cover C.

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Denote the set of neighbours of vertex v as $O(v)$. Let $G(V')$ denote a subgraph induced by a vertex set $V' \subseteq V$. Then the DVC algorithm suggested by Karpinski and Zelikovsky in [4] may be generalized in the following way.

Algorithm DVC$_r$
For all $v \in V$ do
$V' \leftarrow V \backslash (O(v) \cup \{v\})$.
Find a vertex cover $VC(v)$ for $G(V')$ using some $r-$approximation scheme.
$VC(v) \leftarrow O(v) \cup VC(v)$.
Return $APPR$ where $APPR = argmin_{v \in V} w(VC(v))$.

Thus, the DVC algorithm is DVC$_2$, where the 2-Approximation algorithm is used as r-approximation scheme. It is a $\frac{2}{1+\varepsilon}$ - approximate algorithm for everywhere ε-dense unweighted VCP (see [4]). However, the following construction shows that for weighted everywhere ε-dense problems the ratio $\frac{w(APPR)}{w(OPT)}$ can become arbitrary close to 2 even if $\varepsilon \in (0,1)$ remains constant.

Note that 2-Approximation algorithm returns a cover which consists of two vertices for an unweighted graph with $|V| = 3$ and $|E| = 2$, provided the vertices and edges are properly ordered. Let $L_1, L_2, \ldots, L_k$ be copies of the 3-vertex graph mentioned above, thus the 2-Approximation algorithm returns a cover of twice the optimal weight, if applied to the graph $\bigcup_{i=1}^{k} L_i$.

Suppose, n is big enough and let's construct an n-vertex graph G, which consists of $k = \lfloor \frac{n(1-\varepsilon)}{3} \rfloor$ subgraphs $L_1, L_2, \ldots, L_k$ and a clique K with $n - 3k \geq \varepsilon n$ vertices. Let the weights of the clique vertices be all 0. Finally, connect each vertex of the subgraphs $L_1, L_2, \ldots, L_k$ to all vertices of the clique K. Graph G is everywhere ε-dense. The optimal cover for G includes only k vertices of weight 1 (one per each $L_i, i = 1 \ldots k$), while DVC$_2$ covers optimally only one of L_i-s. Hence, $w(APPR) = 2k - 1$ and $\lim\limits_{n \to \infty} \frac{w(APPR)}{w(OPT)} = 2$.

2 Performance Guarantees

The following theorem is an extension of the theorem proved by Karpinski and Zelikovsky in [4].

Theorem. *The weight of vertex cover returned by DVC$_r$ algorithm is at most* $\frac{r}{1+\rho(r-1)}$ *times the weight of an optimal cover, if* $w(O(v)) \geq \rho w(V)$ *for any vertex* $v \in V$.

Proof. Let $v \in V \backslash OPT$. Then $O(v) \subseteq OPT$ since all edges incident to v should be covered by OPT. The vertices of $O(v)$ cover all edges between $O(v)$ and $V' = V \backslash (O(v) \cup \{v\})$ as well. So the rest of the vertices of OPT give an optimal cover for $G(V')$.

Denote $OPT' = OPT \backslash O(v)$ and let C be the cover of $G(V')$ returned by the r-

approximation scheme. Then $w(C) \le rw(OPT')$ and therefore,

$$\frac{w(APPR)}{w(OPT)} \le \frac{w(O(v)) + rw(OPT')}{w(O(v)) + w(OPT')} \le \frac{\rho w(V) + rw(OPT')}{\rho w(V) + w(OPT')} = r - \frac{r-1}{1 + \frac{w(OPT')}{\rho w(V)}}.$$

If $rw(OPT') \le (1-\rho)w(V)$ then

$$\frac{w(APPR)}{w(OPT)} \le r - \frac{r-1}{1 + \frac{(1-\rho)w(V)}{r\rho w(V)}} = \frac{r}{1+\rho(r-1)}.$$

If $rw(OPT') > (1-\rho)w(V)$ then inequality $w(C) \le w(V)$ also yields the desired bound as follows.

$$\frac{w(APPR)}{w(OPT)} \le \frac{w(V)}{w(O(v)) + w(OPT')} < \frac{w(V)}{\rho w(V) + \frac{(1-\rho)w(V)}{r}} = \frac{r}{1+\rho(r-1)}.$$

□

Note that if $w_v > 0$ for all $v \in V$ then the proved inequality is strict. Indeed, the equality would imply that on one hand, $rw(OPT') = (1-\rho)w(V)$. On the other, $w(O(v)) = \rho w(V)$, $rw(OPT') = w(C)$ and thus, $rw(OPT') \le w(V') = w(V) - \rho w(V) - w_v = rw(OPT') - w_v$, which contradicts the assumption that $w_v > 0$.

The theorem implies that DVC_2 is a $\frac{2}{1+\rho}$-approximation algorithm for the weighted VCP.

In the special case of application of DVC_2 to the unweighted everywhere ε-dense VCP, the obtained bound yields the approximation guarantee $\frac{|APPR|}{|OPT|} < \frac{2}{1+\varepsilon}$, like the one proved in [4].

Corollary *If d is such that $w_u \le dw_v$ holds for any vertices u and v of graph G, then the algorithm DVC_2 has an approximation ratio not more than $2\frac{1+d(\varepsilon^{-1}-1)}{2+d(\varepsilon^{-1}-1)}$ for everywhere ε-dense weighted VCP.*

Proof. Note that

$$\frac{w(V)}{w(O(v))} = 1 + \frac{w(V) - w(O(v))}{w(O(v))} \le 1 + \frac{|V \setminus O(v)| \max_{u \in V} w_u}{|O(v)| \min_{u \in V} w_u} \le 1 + d(\varepsilon^{-1} - 1).$$

Application of the theorem gives the required approximation bound. □

A $\left(2 - \frac{\log \log n}{2 \log n}\right)$-approximation algorithm, suggested by Bar-Yehuda and Even [2] may be used as r-approximation scheme in DVC_r as well. This gives an algorithm for the weighted VCP with the approximation ratio not more than $(4 \log n - \log \log n)/(2(\rho+1)\log n - \rho \log \log n)$. It's not difficult to see that this bound is stronger than $\frac{2}{1+\rho}$ if $n > 2$, since the parameter $r < 2$ gives a better approximation bound for DVC_r.

References

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