Evaluation of an Approximate Algorithm for the Everywhere Dense Vertex Cover Problem *

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Abstract

We generalize the DVC algorithm (see [4]) for the weighted case of vertex cover problem (VCP) and study the performance of this algorithm. An extension of result from [4] for the weighted case is proposed in terms of a new density parameter. Given a graph $G(V,E)$ let there be $\rho > 0$ such that $w(O(v)) \geq \rho w(V)$ for any vertex $v \in V$. Here $w(O(v))$ is the total weight of the neighbours of $v$, and $w(V)$ is the total weight of $V$. Then $\frac{2}{1+\rho}$ performance guarantee holds for an algorithm similar to DVC. The value $\rho$ can be easily estimated for everywhere $\varepsilon$-dense VCP, if there is such $d$ that $w_u \leq d w_v$ for any pair of vertices $u$ and $v$. The generalization of DVC allows us to propose another polynomial-time algorithm for the weighted VCP with the performance ratio as a function on $|V|$ which is less than $\frac{2}{1+\rho}$.

1 Definitions and Algorithms

Let $G = (V,E)$ be a graph with vertices $V$ and edges $E$. A set $C \subseteq V$ is called a vertex cover of $G$ if every edge has at least one endpoint in $C$. The vertex cover problem (VCP) is given a graph $G = (V,E)$ and weight $w_v \geq 0$ for each $v \in V$, find a vertex cover $OPT$ with minimum total weight $w(OPT)$. Here we denote $w(U) = \sum_{u \in U} w_u$ for any $U \subseteq V$. The unweighted VCP is a case when $w_v = 1$ for all $v \in V$.

In [1] Bar-Yehuda and Even proposed the following 2-approximation algorithm for the VCP formulated above.

Algorithm 2-Approximation

Set $C = \emptyset$.
While $E \neq \emptyset$ do

Let $e = (v,v') \in E$, and let $w(e) \in \{v,v'\}$ be such that $w_{u(e)} = \min\{w_v,w_{v'}\}$.

Set $C \leftarrow C \cup \{u(e)\}$, $w_v := w_v - w_{u(e)}$,

$w_{v'} := w_{v'} - w_{u(e)}$, $E \leftarrow E \setminus E_{u(e)}$.

Return the cover $C$.

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Denote the set of neighbours of vertex \( v \) as \( O(v) \). Let \( G(V') \) denote a subgraph induced by a vertex set \( V' \subseteq V \). Then the DVC algorithm suggested by Karpinski and Zelikovsky in [4] may be generalized in the following way.

**Algorithm DVC**

For all \( v \in V \) do

\[
V' \leftarrow V \setminus (O(v) \cup \{v\}).
\]

Find a vertex cover \( VC(v) \) for \( G(V') \) using some \( r \)-approximation scheme.

\[
VC(v) \leftarrow O(v) \cup VC(v).
\]

Return \( APPR \) where \( APPR = \arg \min_{v \in V} w(VC(v)) \).

Thus, the DVC algorithm is DVC\(_2\), where the 2-Approximation algorithm is used as \( r \)-approximation scheme. It is a \( \frac{2}{1+\varepsilon} \) -approximate algorithm for everywhere \( \varepsilon \)-dense unweighted VCP (see [4]). However, the following construction shows that for weighted everywhere \( \varepsilon \)-dense problems the ratio \( \frac{w(\text{APPR})}{w(OPT)} \) can become arbitrary close to 2 even if \( \varepsilon \in (0,1) \) remains constant.

Note that 2-Approximation algorithm returns a cover which consists of two vertices for an unweighted graph with \( |V| = 3 \) and \( |E| = 2 \), provided the vertices and edges are properly ordered. Let \( L_1, L_2, \ldots, L_k \) be copies of the 3-vertex graph mentioned above, thus the 2-Approximation algorithm returns a cover of twice the optimal weight, if applied to the graph \( \bigcup_{i=1}^{k} L_i \).

Suppose, \( n \) is big enough and let’s construct an \( n \)-vertex graph \( G \), which consists of \( k = \left\lfloor \frac{n-1-\varepsilon n}{3} \right\rfloor \) subgraphs \( L_1, L_2, \ldots, L_k \) and a clique \( K \) with \( n - 3k \geq \varepsilon n \) vertices. Let the weights of the clique vertices be all 0. Finally, connect each vertex of the subgraphs \( L_1, L_2, \ldots, L_k \) to all vertices of the clique \( K \). Graph \( G \) is everywhere \( \varepsilon \)-dense. The optimal cover for \( G \) includes only \( k \) vertices of weight 1 (one per each \( L_i, i = 1 \ldots k \)), while DVC\(_2\) covers optimally only one of \( L_i \)-s. Hence, \( w(\text{APPR}) = 2k - 1 \) and \( \lim_{n \to \infty} \frac{w(\text{APPR})}{w(OPT)} = 2 \).

## 2 Performance Guarantees

The following theorem is an extension of the theorem proved by Karpinski and Zelikovsky in [4].

**Theorem.** The weight of vertex cover returned by DVC\(_r\) algorithm is at most \( \frac{1+r(1-\varepsilon)}{1+r(1-\varepsilon)} \) times the weight of an optimal cover, if \( w(O(v)) \geq \rho w(V) \) for any vertex \( v \in V \).

**Proof.** Let \( v \in V \setminus OPT \). Then \( O(v) \subseteq OPT \) since all edges incident to \( v \) should be covered by \( OPT \). The vertices of \( O(v) \) cover all edges between \( O(v) \) and \( V' = V \setminus (O(v) \cup \{v\}) \) as well. So the rest of the vertices of \( OPT \) give an optimal cover for \( G(V') \).

Denote \( OPT' = OPT \setminus O(v) \) and let \( C \) be the cover of \( G(V') \) returned by the \( r\)-
approximation scheme. Then \( w(C) \leq r w(OPT') \) and therefore,

\[
\frac{w(\text{APPR})}{w(OPT)} \leq \frac{w(O(v)) + r w(OPT')}{w(O(v)) + w(OPT')} \leq \frac{\rho w(V) + r w(OPT')}{\rho w(V) + w(OPT')} = r - \frac{r - 1}{1 + \frac{w(OPT')}{\rho w(V)}}.
\]

If \( r w(OPT') \leq (1 - \rho) w(V) \) then

\[
\frac{w(\text{APPR})}{w(OPT)} \leq r - \frac{r - 1}{1 + \frac{(1-\rho) w(V)}{r \rho w(V)}} = \frac{r}{1 + \rho (r - 1)}.
\]

If \( r w(OPT') > (1 - \rho) w(V) \) then inequality \( w(C) \leq w(V) \) also yields the desired bound as follows.

\[
\frac{w(\text{APPR})}{w(OPT)} \leq \frac{w(V)}{w(O(v)) + w(OPT')} \leq \frac{w(V)}{\rho w(V) + \frac{(1-\rho) w(V)}{r}} = \frac{r}{1 + \rho (r - 1)}.
\]

\( \square \)

Note that if \( w_v > 0 \) for all \( v \in V \) then the proved inequality is strict. Indeed, the equality would imply that on one hand, \( r w(OPT') = (1 - \rho) w(V) \). On the other, \( w(O(v)) = \rho w(V) \), \( r w(OPT') = w(C) \) and thus, \( r w(OPT') \leq w(V) = w(V) - \rho w(V) - w_v = r w(OPT') - w_v \), which contradicts the assumption that \( w_v > 0 \).

The theorem implies that DVC_2 is a \( \frac{4}{1 + \rho} \)-approximation algorithm for the weighted VCP.

In the special case of application of DVC_2 to the unweighted everywhere \( \varepsilon \)-dense VCP, the obtained bound yields the approximation guarantee \( \frac{|\text{APPR}|}{|OPT|} < \frac{2}{1 + \rho} \), like the one proved in [4].

**Corollary** If \( d \) is such that \( w_u \leq dw_v \) holds for any vertices \( u \) and \( v \) of graph \( G \), then the algorithm DVC_2 has an approximation ratio not more than \( \frac{2^{1+\frac{\varepsilon}{2(\rho+1)}}}{1+\varepsilon} \) for everywhere \( \varepsilon \)-dense weighted VCP.

**Proof.** Note that

\[
\frac{w(V)}{w(O(v))} = 1 + \frac{w(V) - w(O(v))}{w(O(v))} \leq 1 + \frac{|V\setminus O(v)|}{|O(v)|} \max_{u \in V} w_u \leq 1 + d(\varepsilon^{-1} - 1).
\]

Application of the theorem gives the required approximation bound. \( \square \)

A \( \left( 2 - \frac{\log k}{2 \log n} \right) \)-approximation algorithm, suggested by Bar-Yehuda and Even [2] may be used as \( r \)-approximation scheme in DVC_r as well. This gives an algorithm for the weighted VCP with the approximation ratio not more than \( (4 \log n - \log \log n)/(2(\rho + 1) \log n - \rho \log \log n) \). It’s not difficult to see that this bound is stronger than \( \frac{2}{1 + \rho} \) if \( n \geq 2 \), since the parameter \( r < 2 \) gives a better approximation bound for DVC_r.
References


