Electronic Colloquium on Computational Complexity - Reports Series 1997 - available via:

ECCC
TR97-024

FTP: ftp.eccc.uni-trier.de:/pub/eccc/
WWW: http://www.eccc.uni-trier.de/eccc/
Email: ftpmail@ftp.eccc.uni-trier.de with subject 'help eccc'

Polynomial Time Approximation Schemes for Some Dense Instances of NP-Hard Optimization Problems

Marek Karpinski *

Abstract

We survey recent results on the existence of polynomial time approximation schemes for some dense instances of NP-hard optimization problems. We indicate further some inherent limits for existence of such schemes for some other dense instances of the optimization problems.

1 Introduction

The computational efficiency of approximating different *NP-hard optimization* problems varies a great deal. We know by now, that unless P=NP, some problems, such as CLIQUE cannot be approximated in polynomial time even within a factor $n^{1-\epsilon}$ for any $\epsilon > 0$ (cf. Håstad [H96]). Some other problems like MAX-CUT (cf. Goemans and Williamson [GW94]) or STEINER TREE (cf. Karpinski and Zelikovsky [KZ97a]), can be approximated to within some small fixed constant factor. Till recently only a very few optimization problems were known to have polynomial time approximation schemes (PTAS), approximating to within arbitrary small constant factors.

*Dept. of Computer Science, University of Bonn, 53117 Bonn, Email: marek@cs.uni-bonn.de. Research partially supported by the International Computer Science Institute, Berkeley, California, by the DFG Grant KA 673 4-1, and by the ESPRIT BR Grants 7097 and EC-US 030, by DIMACS, and by the Max-Planck Research Prize.

Some of the approximation algorithms with small approximation ratios achieve also good practical performances, like some cases of STEINER TREE problems (cf. [KZ97a]), some other algorithms do not yield yet efficient practical methods of dealing with optimization problems.

In this paper we are concerned with the problem of efficient approximability of the *dense instances* of *NP-hard* optimization problems.

Recently, the first polynomial time approximation schemes have been designed *for these problems* in Arora, Karger and Karpinski [AKK95], Fernandez de la Vega [FV96], Arora, Frieze and Kaplan [AFK96], and Karpinski and Zelikovsky [KZ97b]. Later on, Goldreich, Goldwasser and Ron [GGR96], and Frieze and Kannan [FK97] gave a constant sample size approximation schemes for some dense optimization problems. Fernandez de la Vega and Karpinski [FK97] gave also the first polynomial time approximability characterization for dense *weighted* instances of NP-hard problems.

This development was in contrast to the fact that the existence of such schemes for general instances would imply that P=NP by results of Arora, Lund, Motwani, Sudan, and Szegedy [ALMSS92].

The development above was followed by the study of the dense covering problems, Karpinski and Zelikovsky [KZ97b], and the dense bandwidth minimization problems, Karpinski, Wirtgen and Zelikovsky [KWZ97].

It is also a very interesting artifact that the recent successes in design of the polynomial time approximation schemes for dense optimization problems parallel the successes of the past attacks on dense approximate counting problems, Broder [B86], Jerrum and Sinclair [JS89], Dyer, Frieze, Jerrum [DFJ94], and Alon, Frieze and Welsh [AFW95].

2 MAX-SNP and Dense MAX-SNP Classes, and BEYOND

We consider in this Section the *dense* instances of the MAX-SNP class of optimization problems introduced by Papadimitriou and Yannakakis [PY91]. MAX-SNP class contains constraint-satisfaction problems, where the *constraints* are definable by quantifier-free propositional formulas.

We recall:

Definition: A (*maximization*) *problem* A is in MAX-SNP if there exists a sequence of relation symbols $G_1, \ldots, G_m$, a relation symbol S, and a quantifier-free formula $\Phi(G_1, \ldots, G_m, S, x_1, \ldots, x_k)$, with x_i variables, such that the following is true:

1. there is a polynomial time algorithm that for any given instance I of the problem A produces a set $\mathcal{V}$ and a sequence of relations $G_1^{\mathcal{V}}, \ldots, G_m^{\mathcal{V}}$ over $\mathcal{V}$ ($G_i^{\mathcal{V}}$ preserve the *arity* of G_i);
2. The value of the optimum solution $OPT(I)$ of A on instance I, satisfies

$$OPT(I) = \\ \underset{S^{\mathcal{V}}}{\mathrm{MAX}}\left\{\left|\{(x_1, \ldots, x_k) \in \mathcal{V}^k \middle| \Phi(G_1^{\mathcal{V}}, \ldots, G_m^{\mathcal{V}}, S^{\mathcal{V}}, x_1, \ldots, x_k) = TRUE\}\right|\right\}$$

   for $S^{\mathcal{V}}$ the relation over $\mathcal{V}$ of the same arity as S.

Example: MAX-CUT (cf. [GJ79], [P94]) is in MAX-SNP, since its optimum solution can be written as

$$\underset{S\subseteq\mathcal{V}}{\mathrm{MAX}} \left\{ \mid \{(x,y) \mid (G(x,y) \vee G(y,x) \wedge S(x) \wedge \neg S(y))\} \mid \right\}$$

for V the set of vertices of the graph, $G(x,y)$ its adjacency relation, and S a unary relation describing the *one side* of the cut.

For the notions of MAX-SNP-*completeness*, and MAX-SNP-*hardness* see [P94], and [AL97].

We define next the problem MAX-k-FUNCTION-SAT for some fixed integer k. MAX-k-FUNCTION-SAT has as an input m boolean functions $f_1, f_2, \ldots, f_m$ in n variables, and each f_i depends only on k variables. The problem is to find an assignment to the variables as to satisfy as many f_i's as possible.

It is known that every problem A from MAX-SNP can be viewed as a MAX-k-FUNCTION-SAT problem for a fixed k (cf. [P94]). Following [AKK95] we call an instance of a MAX-SNP problem *dense* if the corresponding instance of MAX-k-FUNCTION-SAT has $\Omega(n^k)$ functions.

Given an *optimization problem* A, a (meta) algorithm $\mathcal{A}$ is called a *polynomial time approximation scheme* (PTAS) if for every fixed $\epsilon > 0$, $\mathcal{A}$ is a *polynomial* time algorithm with *approximation ratio* $1+\epsilon$ (meaning $\mathcal{A}$ outputs a solution S to every instance I of A such that MAX$\left\{\frac{S}{OPT(I)}, \frac{OPT(I)}{S}\right\} \leq 1+\epsilon$, for OPT(I) the optimal solution, and the running time of $\mathcal{A}$ is polynomial in the size of I).

Only a very few problems, such as KNAPSACK [IK75], and BIN PACKING [FL81], [KK82], were till recently known to have PTASs.

In Arora, Karger and Karpinski [AKK95] the following general result on the existence of PTASs was proven.

Proposition 1. ([AKK95]) Dense MAX-SNP problems have PTASs.

The proof method involves the representation of MAX-k-FUNCTION-SAT by *smooth degree–k integer* programs, and the general result on approximating such programs (cf. [AKK95]).

Below is the list of problems were the smooth integer programs can be applied directly to obtain the PTASs. (We call a graph *dense* if it has $\Theta(n^2)$ edges, a hypergraph of dimension d is dense if it does has $\Theta(n^k)$ edges.)

- MAX-CUT: For a given graph partition its vertices into two sets so as to maximize the number of edges between them.
- MAX-DCUT: The directed version of the MAX-CUT.
- MAX-HYPERCUT(d): A generalization of MAX-CUT to hypergraphs of dimension d (an edge is considered in a cut if it has at least one vertex on each side).
- DENSE-k-SUBGRAPH: Given a graph, find a subset of k vertices that induces a graph with the most edges (cf. [KP93]).

Following [AKK95], we have

Proposition 2. ([AKK95]) Dense instances of the following problems have PTASs:
MAX-CUT, MAX-DCUT, MAX-HYPERCUT(d), and DENSE-k-SUBGRAPH for $k = \Omega(n)$.

In what follows we call a graph G *everywhere dense* if its minimum degree is $\Omega(n)$. We consider *everywhere dense* instances of three further problems.

- SEPARATOR: Given a graph, partition its vertices into two sets, each with at least $\frac{1}{3}$ of the vertices, so as to minimize the number of edges between them.
- BISECTION: Given a graph, partition its vertices into two equal halves so as to minimize the number of edges between them.
- MIN-k-CUT: Given a graph with n vertices, and k *source* vertices, partition its vertices into k groups such that (1) each group contains one source, and (2) the number of edges between different groups is *minimized.*

Consider a graph with a *minimum* degree δn, and let c denote the capacity of its minimum bisection. The PTAS for BISECTION of [AKK95] consists of two algorithms, one of which is a PTAS when $c \geq \alpha n^2$, and the other when $c < \alpha n^2$ for α a small constant. The algorithm for $c \geq \alpha n^2$ uses the above mentioned method for approximating smooth integer programs. For the case $c < \alpha n^2$ we use the fact that in a *minimum bisection*, there must be one side whose every vertex has at most half of its neighbors on the other side, and construct a randomized exhaustive *correction* sample algorithm. The algorithm can be also easily derandomized (cf. [AKK95]). Similar PTASs work for the SEPARATOR and BISECTION problems.

Proposition 3. ([AKK95]) Everywhere dense instances of the following problems have PTASs: BISECTION, SEPARATOR and MIN-k-SAT.

Fernandez de la Vega [FV96] has independently developed a PTAS for everywhere dense instances of MAX-CUT problem. His algorithm does not appear to generalize though to the other problems listed above.

Arora, Frieze, and Kaplan [AFK96] constructed a new rounding procedure for the quadratic assignment problem and used it to obtain PTASs on the dense instances of the NP-hard problems like QUADRATIC-ASSIGNMENT, MIN-LINEAR-ARRANGEMENT, d-DIMENSIONAL-ARRANGEMENT, BETWEENESS, and MIN-CUT-LINEAR-ARRANGEMENT (cf. [AFK96]).

In the other development the Regularity Lemma of Szemerédi was used to obtain more efficient PTAS for the above problems (cf. Frieze and Kannan [FK96]). Using also independent methods Goldreich, Goldwasser and Ron [GGR96], and Frieze and Kannan [FK97] gave constant time approximation schemes for some dense problems in the oracle model of computation.

3 Dense Covering Problems

We turn now to the three dense covering problems: SET COVER, STEINER TREE, and VERTEX COVER (cf. [H97], [AL97]). They do not fall into the dense MAX-SNP class definition of section 2 (VERTEX COVER is in MAX-SNP only if the degree of the graph is bounded.)

- SET COVER: Given a finite set X and a family of its subsets $\mathcal{P}$, find a minimum size subfamily $\mathcal{M}$ of $\mathcal{P}$ such that $X \subseteq \bigcup \mathcal{M}$.

  We call an instance of SET COVER ($X = \{x_1, \dots x_n\}, \mathcal{P} = \{p_1, \dots, p_m\}$) *$\epsilon$-dense* (for $\epsilon > 0$) if every element of X belongs to at least ϵm sets from $\mathcal{P}$. (The instances of SET COVER are called *dense* if they are ϵ-dense for some $\epsilon > 0$. We call SET COVER restricted to dense instances a dense SET COVER accordingly.)

- STEINER TREE: Given a connected graph G and a set of its distinguished (terminal) vertices S. Find a minimum size tree within G that spans all distinguished vertices from S.

  We call an instance $G = (V, E)$ of the STEINER TREE problem *ϵ-dense* if every distinguished terminal vertex is adjacent to at least $\epsilon \cdot \mid V \setminus S \mid$ nonterminals.

- VERTEX COVER: Given a graph G, find a minimum size vertex set X of G which covers all edges of G (i.e. at least one endpoint of any edge belongs to X).

We start with dense SET COVER problem. The general SET COVER was proven recently to have a threshold $(1 - o(1)) \ln n$ for the polynomial time approximation (cf. Feige [F96]) which in fact is matching asymptotically the approximation ratio by the well known greedy heuristic algorithm.

It is shown in Karpinski and Zelikovsky [KZ96] that the greedy heuristic algorithm can be applied more efficiently towards ther dense SET COVER.

Proposition 4. ([KZ96]) For any constant $c > 0$ and any $\epsilon > 0$, there is a polynomial time approximation algorithm for the ϵ-dense SET COVER with the approximation ratio $c \cdot \log n$.

Interestingly, we cannot expect on the lower bound side of the dense SET COVER, its NP-hardness, as the results of Papadimitriou and Yannakakis [PY96] imply.

Proposition 5. ([KZ97b]) Unless NP $\subseteq$ DTIME $[n^{\log n}]$, the dense SET COVER is not NP-hard.

We conjecture that SET COVER cannot be approximated to within a constant factor.

Conjecture 1. The dense SET COVER cannot be approximated in polynomial time to within a constant approximation ratio.

The second problem we discuss in this section is the ϵ-dense STEINER TREE problem. We note first in passing that for $\epsilon > \frac{1}{2}$, ϵ-dense STEINER TREE problem is a special case of the network STEINER TREE problem with edge lengths 1 and 2, the problem which is still MAX-SNP-hard, Bern and Plassmann [BP89]. The best known approximation ratio for the general problem is 1.644, Karpinski and Zelikovsky [KZ97a]. For the dense STEINER TREE problem the existence of a PTAS has been recently proven in Karpinski and Zelikovsky [KZ97b].

Proposition 6. ([KZ97b]) There exists a PTAS for the ϵ-dense STEINER TREE problem.

It is not difficult to see that there is a polynomial time reduction of the ϵ-dense SET COVER to the ϵ-dense STEINER TREE problem, and vice versa. Therefore, the similar result to Proposition 5 holds also for the dense STEINER TREE problem.

Furthermore we conjecture,

Conjecture 2. The dense STEINER TREE problem cannot be computed exactly in polynomial time .

The third problem, VERTEX COVER, is one of the first NP-hard optimization problems for which the approximation algorithms were proposed ([GJ79]). The problem is known to be MAX-SNP-hard, and the well-known 2-approximation algorithm is also believed to be the best possible (cf. [H97]). In Karpinski and Zelikovsky [KZ97b] the new approximation algorithm is designed for dense VERTEX COVER problems beating the approximation ratio 2.

We call a graph $G = (V, E)$ *everywhere* ϵ*-dense* if its *minimum* degree is at least $\epsilon \cdot \mid V \mid$. We call G ϵ*-dense* if $\mid E \mid \geq \epsilon \cdot \mid V \mid^2$.

Proposition 7. ([KZ97b]) There exists a polynomial time approximation algorithm for the VERTEX COVER problem on ϵ-dense graphs with approximation ratio $\frac{2}{2-\sqrt{1-\epsilon}}$.

For the everywhere dense instances we get

Proposition 8. ([KZ97b]) There exists a polynomial time approximation algorithm for the VERTEX COVER problem on everywhere ϵ-dense graphs with approximation ratio $\frac{2}{1+\epsilon}$.

Proposition 7 and 8 show that the *density* do help essentially in approximating the VERTEX COVER problem. Can we expect though existence of a PTAS for the dense VERTEX COVER problem?

The answer is no, as the *everywhere* ϵ*-dense* (and ϵ*-dense*) VERTEX COVER is MAX SNP-hard. (cf. [KZ96], [CT96]). This is due to the following *densification* construction. Start with a general instance (a graph G with n vertices) of the VERTEX COVER, and densify it by joining its all vertices with all vertices of a clique of size $\frac{\epsilon}{1-\epsilon}n$. The resulting graph is everywhere ϵ*-dense*. An existence of α-approximation algorithm for dense instances of VERTEX COVER entails now also $\alpha(1+\epsilon)$-approximation algorithm for the general VERTEX COVER problem which is MAX-SNP-hard.

Proposition 9. ([CT96], [KZ96]) The dense VERTEX COVER problem is MAX-SNP-hard.

4 Dense BANDWIDTH MINIMIZATION

We discuss now the problem of approximability of dense instances of the BANDWIDTH problem. The BANDWIDTH problem has a long and very interesting history, and a number of important technical applications (cf., e.g. [CCDG82]). It belongs also to the class of so called layout problems and is one of hardest in this class ([DSS94]). Its approximability status resembles the BISECTION problem discussed in Section 2 in what there is a general lack of approximation algorithms with essentially sublinear approximation ratios and on the other hand it lacks also any unapproximability results. The situation on dense instances of the BANDWIDTH was even more difficult than for the dense BISECTION, for which we have constructed a PTAS (see Section 2). For the dense BANDWIDTH however, even the existence of a constant ratio approximation algorithms was an open problem. The positive result on existence of a PTAS for the dense BISECTION illustrates also the difficulty of proving unapproximability result for the general BISECTION problem. It indicates that the standard method of reducing *balanced* (50/50) MAX-CUT to BISECTION on the complementary graph cannot work for a good reason. The balanced MAX-CUT is MAX-SNP-hard (cf. [PY91]), however by complementing a sparse graph we get a dense one on which the BISECTION is approximable.

The situation with the BANDWIDTH is, in fact, even more subtle in this respect. The standard graph operations or a slight densification seem to destroy the structure of the instance completely.

We give now an exact formulation of the problem.

- BANDWIDTH: Given a graph $G = (V, E)$, compute the numbering of its vertices such that the maximum difference between the numbers of adjacent vertices is *minimal*.

We define also the directed BANDWIDTH problem.

- DBANDWIDTH: Given a directed graph $G = (V, E)$, compute the numbering of its vertices as above such that for every vertex v its numbering is greater than any numbering of a vertex u such that $(u, v) \in E$.

The DBANDWIDTH problem corresponds to that of minimizing the *bandwidth* of an upper triangular matrix by simultaneous row and column permutations (cf. [GGJK78]).

The problem is known to be NP-hard even if restricted to binary trees (cf. [GGJK78]), or caterpillars with hairs of length at most 3 [M83]. This makes the BANDWIDTH one of the very rare combinatorial problems which are computationally 'hard' for trees. Interestingly, the problem is efficiently computable for complete trees [Sm95]. Only a very few special cases of this problem are known to have sublinear approximation ratio algorithms, among them $\log n$-approximation algorithm for the caterpillars ([HMM91]). There are no sublinear n^{ϵ}-approximation algorithms known for the BANDWIDTH problem even if restricted to trees.

We consider here the BANDWIDTH problem on the everywhere dense graphs. Using a randomized placing technique combined with the special perfect matching construction Karpinski, Wirtgen and Zelikovsky [KWZ97] proved.

Proposition 10. ([KWZ97]). There exists a randomized polynomial time approximation algorithm for the BANDWIDTH problem on everywhere dense graphs with approximation ratio 3.

Using a more constrained nature of DBANDWIDTH the similar techniques yield.

Proposition 11. ([KWZ97]). There exists a randomized polynomial time approximation algorithm for the DBANDWIDTH problem on everywhere dense graphs with approximation ratio 2.

It is still an open problem whether there are constant ratio approximation algorithms for '*dense*' instances of the BANDWIDTH, and the DBANDWIDTH. A challenging question remains whether there exist PTASs for the dense BANDWIDTH problems, or whether some of these problem are in fact MAX-SNP-hard.

5 Summary of Dense Approximation Results

We present here a table summarizing the results of Sections 2-4 with the best known approximation results, and the best up to date nonapproximability results on dense problems.

Problem	**Approx. Ratio**	**Approx. Hardness**	**Ref.**
DENSE MAX-SNP	PTAS	—	[AKK95]
DENSE MAX-CUT	PTAS	—	[AKK95],[FV96]
DENSE MAX-DCUT	PTAS	—	[AKK95]
DENSE MAX-HYPERCUT(d)	PTAS	—	[AKK95]
DENSE DENSE-K-SUBGRAPH	PTAS	—	[AKK95]
EVERYWHERE DENSE SEPERATOR	PTAS	—	[AKK95]
EVERYWHERE DENSE BISECTION	PTAS	—	[AKK95]
EVERYWHERE DENSE MIN-K-CUT	PTAS	—	[AKK95]
DENSE MIN-LINEAR-ARRANGEMENT	PTAS	—	[AFK96]
DENSE d-DIMENSIONAL-ARRANGEMENT	PTAS	—	[AFK96]
DENSE MIN-CUT-LINEAR-ARRANGEMENT	PTAS	—	[AFK96]
DENSE SET COVER	$\bigcap_c c \cdot \ln n$	OPEN	[KZ97b]
DENSE STEINER TREE	PTAS	—	[KZ97b]
DENSE VERTEX COVER	$\frac{2}{2-\sqrt{1-\varepsilon}}$	MAX-SNP-hard	[KZ97b]
EVERYWHERE DENSE VERTEX COVER	$\frac{2}{1+\varepsilon}$	MAX-SNP-hard	[KZ97b]
EVERYWHERE DENSE BANDWIDTH	3	OPEN	[KWZ97]
EVERYWHERE DENSE DBANDWIDTH	2	OPEN	[KWZ97]

Table 1: Table of known dense approximability results.

6 Polynomial Time Approximability of Dense Weighted Instances of NP-Hard Problems

The natural instances of optimization problems involve also weights (cf. [GJ79]) while the results studied before were concerned mainly with $0, 1$ cases. In Arora, Karger, Karpinski [AKK95], the dense MAX-CUT PTAS can be adjusted as to work also for the dense MAX WEIGHT CUT problem ([GJ79]) for the case of weights being bounded by B. In this case the algorithm produces a cut of weight at least maximum weight of a cut minus $\epsilon n^2 B$. This and also other bounded weight problems were considered briefly in Goldreich, Goldwasser and Ron [GGR96], and Frieze and Kannan [FK97]. Both papers evaluate the additional costs of handling bounded weights instead of 0,1 weights.

In a recent paper Fernandez de la Vega and Karpinski [FK97] gave the first polynomial time approximability characterization of dense (unbounded) weighted instances of MAX WEIGHT CUT, and MAX WEIGHT BISECTION, and some other dense weighted NP-hard optimization problems, in terms of their empirical weight distribution. The crucial point of this paper is a new unbounded weight Sampling Lemma. The reader is referred to [FK97] for details.

7 Further Research and Open Problems

It remains to be seen whether the techniques used with success in the dense instances of NP-hard optimization problems, like approximating smooth higher degree integer programs by linear programs, might be useful in approximating general problems. Perhaps some other, different from exhaustive sampling methods can be developed for the nondense instances as well. Another interesting issue is to develop new more efficient techniques for the dense unbounded weight instances of the optimization problems for which costs of allowing weights are not prohibitively high.

On the level of specific dense problems discussed before, it would be interesting to shed some light on the Conjectures 1 and 2. Is there even more dramatic improvement in approximation ratio for the dense SET COVER, like $o(\log n)$, still possible (cf. Proposition 4)?

One of the most challenging open *dense* problems today is the *dense* (*everywhere dense*) BANDWIDTH problem. Is there an approximation ratio below 3 (cf. Proposition 10), and more strongly, is there a PTAS possible for this problem, or on the lower bound side, is this problem 'approximation hard' in some sense?

Acknowledgment

My thanks to Sanjeev Arora, Dorit Hochbaum, Haim Kaplan, Seffi Naor, and Uri Zwick for helpful discussions, and to Jürgen Wirtgen for a careful reading of the manuscript.

References

[AFW95] N. Alon, A. Frieze and D. Welsh, *Polynomial Time Randomized Approximation Schemes for the Tutte Polynomial of Dense Graphs*, Random Structures and Algorithms 6 (1995), pp. 459-478.

[AFK96] S. Arora, A. Frieze and H. Kaplan, *A New Rounding Procedure for the Assignment Problem with Applications to Dense Graph Arrangements*, Proc. 37th IEEE FOCS (1996), pp. 21-30.

[AKK95] S. Arora, D. Karger, and M. Karpinski, *Polynomial Time Approximation Schemes for Dense Instances of NP-Hard Problems*, Proc. 27th ACM STOC (1995), pp. 284-293.

[AL97] S. Arora and C. Lund, *Hardness of Approximations*, in *Approximation Algorithms for NP-Hard Problems* (D. Hochbaum, ed.), PWS Publ. Co. (1997), pp. 399-446.

[ALMSS92] S. Arora, C. Lund, R. Motwani, M. Sudan and M. Szegedy, *Proof Verification and Hardness of Approximation Problems*, Proc. 33rd IEEE FOCS (1992), pp. 14-20.

[BP89] M. Bern and P. Plassmann, *The Steiner Problem with Edge Lengths 1 and 2*, Inform. Process. Lett. 32 (1989), pp. 171-176.

[B86] A.Z. Broder, *How Hard is it to Marry at Random (On the Approximation of the Permanent)*, Proc. 18th ACM STOC (1986), pp. 50-58, Erratum in Proc. 20th ACM STOC (1988), p. 551.

[CCDG82] P. Chinn, J. Chvatalova, A. Dewdney, N. Gibbs, *The Bandwidth Problem for Graphs and Matrices - A Survey*, Journal of Graph Theory 6 (1982), pp. 223-254.

[CT96] A. Clementi and L. Trevisan, *Improved Nonapproximability Result for Vertex Cover with Density Constraints*, Proc. 2nd Int. Conference, COCOON '96, Springer-Verlag (1996), pp. 333-342.

[DSS94] J. Diaz, M. Serna and P. Spirakis, *Some Remarks on the Approximability of Graph Layout Problems*, Technical Report LSI-94-16-R, Univ. Politec, Catalunya (1994).

[DFJ94] M.E. Dyer, A. Frieze and M.R. Jerrum, *Approximately Counting Hamilton Cycles in Dense Graphs*, Proc. 4th ACM-SIAM SODA (1994), pp. 336-343.

[F96] U. Feige, *A Threshold of* $\ln n$ *for Approximating Set Cover*, Proc. 28th ACM STOC (1996), pp. 314-318.

[FV96] W. Fernandez-de-la-Vega, *MAX-CUT has a Randomized Approximation Scheme in Dense Graphs*, Random Structures and Algorithms 8 (1996), pp. 187-999.

[FK97] W. Fernandez-de-la-Vega and M. Karpinski, *Polynomial Time Approximability of Dense Weighted Instances of MAX-CUT*, Research Report No. 85171-CS, University of Bonn (1997).

[FL81] W. Fernandez-de-la-Vega, G. S. Lueker, *Bin Packing Can be Solved Within* $1+\epsilon$ *in Linear Time*, Combinatorica 1 (1981), pp. 349-355.

[FK96] A. Frieze and R. Kannan, *The Regularity Lemma and Approximation Schemes for Dense Problems*, Proc. 37th IEEE FOCS (1996), pp. 12-20.

[FK97] A. Frieze and R. Kannan, *Quick Approximation to Matrices and Applications*, Manuscript (1997).

[GGJK78] M. Garey, R. Graham, D. Johnson, D. Knuth, *Complexity Results for Bandwidth Minimization*, SIAM J. Appl. Math. 34 (1978), pp. 477-495.

[GJ79] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman (1979).

[GW94] M. Goemans and D. Williamson, *.878-approximation Algorithms for MAX-CUT and MAX2SAT*, Proc. 26th ACM STOC (1994), pp. 422-431.

[GGR96] O. Goldreich, S. Goldwasser and D. Ron, *Property Testing and its Connection to Learning and Approximation*, Proc. 37th IEEE FOCS (1996), pp. 339-348.

[HMM91] J. Haralambides, F. Makedon, B. Monien, *Bandwidth Minimization: An Approximation Algorithm for Caterpillars*, Math. Systems Theory 24 (1991), pp. 169-177.

[H96] J. Håstad, *Clique is Hard to Approximate within* $n^{1-\epsilon}$, Proc. 37th IEEE FOCS (1986), pp. 627-636.

[H97] D. Hochbaum, *Approximating Covering and Packing Problems: Set Cover, Vertex Cover, Independent Set, and Related Problems*, in *Approximation Algorithms for NP-hard Problems* (D. Hochbaum, ed.), PWS Publ. Co. (1997), pp. 94-143.

[IK75] O. H. Ibarra, C. E. Kim, *Fast Approximation Algorithms for the Knapsack and Sum of Subsets Problems*, J. ACM 22 (1975), pp. 463-468.

[JS89] M. R. Jerrum and A. Sinclair, *Approximating the Permanent*, SIAM J. Comput. 18 (1989), pp. 1149-1178.

[J94] D. S. Johnson, *Approximation Algorithms for Combinatorial Problems*, J. Comput. System Sciences 9 (1974), pp. 256-278.

[KK82] N. Karmarkar and R. M. Karp, *An Efficient Approximation Scheme for the One-dimensional Bin-Packing Problem*, Proc. 23rd IEEE FOCS (1982), pp. 312-320.

[K72] R.M. Karp, *Reducibility among Combinatorial Problems*, in Complexity of Computer Computations (R. Miller and J. Thatcher, ed.), Plenum Press (1972), pp. 85-103.

[KWZ97] M. Karpinski, J. Wirtgen and A. Zelikovsky, *An Approximation Algorithm for the Bandwidth Problem on Dense Graphs*, ECCC Technical Report TR 97-017 (1997).

[KZ96] M. Karpinski and A. Zelikovsky, *Approximating Dense Cases of Covering Problems (Preliminary Version)*, Technical Report TR-96-059, International Computer Science Institute, Berkeley (1996).

[KZ97a] M. Karpinski and A. Zelikovsky, *New Approximation Algorithms for the Steiner Tree Problem*, J. of Combinatorial Optimization 1 (1997), pp. 47-65.

[KZ97b] M. Karpinski and A. Zelikovsky, *Approximating Dense Cases of Covering Problems*, ECCC Technical Report TR 97-004, 1997, to appear in Proc. DIMACS Workshop on Network Design: Connectivity and Facilities Location, Princeton (1997).

[KP93] G. Kortsarz and D. Peleg, *On Choosing a Dense Subgraph*, Proc. 34th IEEE FOCS (1993), pp. 692-701.

[M83] B. Monien, *The Bandwidth Minimization Problem for Caterpillars with Hair Length 3 is NP-Complete*, SIAM J. Alg. Disc. Math. 7 (1986), pp. 505-514.

[P94] C. Papadimitriou, *Computational Complexity*, Addison-Wesley, (1994).

[PY91] C. Papadimitriou and M. Yannakakis, *Optmization, Approximation and Complexity Classes*, J. Comput. System Sciences 43 (1991), pp. 425-440.

[PY96] C. Papadimitriou and M. Yannakakis, *On Limited Nondeterminism and the Complexity of the VC-dimension*, J. Comput. System Sciences 53 (1996), pp. 161-170.

[R88] P. Raghavan, *Probabilistic Construction of Deterministic Algorithms: Approximate Packing Integer Programs*, J. Comput. System Sciences 37 (1988), pp. 130-143.

[Sm95] L. Smithline, *Bandwidth of the Complete k-ary Tree*, Discrete Mathematics 142 (1995), pp. 203-212.

[Y92] M. Yannakakis, *On the Approximation of Maximum Satisfiability*, Proc. 3rd ACM-SIAM SODA (1992), pp. 1-9.