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The Operators min and max on the Polynomial Hierarchy

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Abstract

Starting from Krentel's class OptP [Kre88] we define a general maximization operator max and a general minimization operator min for complexity classes and show that there are other interesting optimization classes beside OptP. We investigate the behavior of these operators on the polynomial hierarchy, in particular we study the inclusion structure of the classes $\max\cdot$P, $\max\cdot$NP, $\max\cdot$coNP, $\min\cdot$P, $\min\cdot$NP, and $\min\cdot$coNP. Furthermore we prove some very powerful relations regarding the interaction of the operators max, min, U, Sig, C, $\oplus$, $\exists$, and $\forall$. This gives us a tool to show that all the min and max classes are distinct under reasonable structural assumptions. Besides that we are able to characterize the polynomial hierarchy uniformly by three operators.

1 Introduction

The complexity of maximization problems is still a major field of research in structural complexity. Different lines of approach were chosen in the past. In 1988 Krentel [Kre88] defined the class OptP, which was subsequently studied in a series of papers [Kre88, Köb89, Kre92, GKR95]. In all these papers it was pointed out that OptP and its subclasses *max*-P and *min*-P (in Köbler's notation) are constructed from NP in a natural way. We ask, what happens if this construction is applied to P, coNP, and other complexity classes? In this paper we will take a strong complexity theoretic approach, extending the work started by Krentel.

Recall that OptP $=$ *max*-P $\cup$ *min*-P, where *max*-P is the set of all functions f such that there exists a nondeterministic polynomial-time Turing machine N such that every accepting path of $N(x)$ writes a binary number and the largest among the binary numbers over all accepting paths of $N(x)$ equals $f(x)$. Similarly one defines *min*-P.

In [Kre88] it is shown that the function MAXIMUM SATISFYING ASSIGNMENT, which given a boolean formula outputs the lexicographically largest satisfying assignment, is, though in *max*-P, metric complete for $\mathrm{F}\Delta_2^p$ which is a seemingly larger class then *max*-P. But note that computing MAXIMUM SATISFYING ASSIGNMENT does not need the full computational power of the class *max*-P, namely it can be computed by a nondeterministic polynomial-time Turing machine in the sense that the largest *accepting path* (not output) is the value of the function (we guess all assignments and accept if the guessed assignment satisfies the input formula). Does this observation lead to a smaller class of optimization functions?

There have been various attempts to relate function and complexity classes by defining operators in order to map one kind of class to the other. Two major examples of operators which map complexity classes to function classes are # [Tod91] as a generalization of Valiants class #P [Val79a, Val79b] and Mēd [VW93]. In these papers the authors define for a complexity class $\mathcal{K}$:

$$f \in \# \cdot \mathcal{K} \iff \bigvee_{A \in \mathcal{K}} \bigvee_{p \in Pol} \bigwedge_{x \in \Sigma^*} f(x) = ||\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y \rangle \in A\}||.$$

$$f \in \text{M\={e}d} \cdot \mathcal{K} \iff \bigvee_{A \in \mathcal{K}} \bigvee_{p \in Pol} \bigwedge_{x \in \Sigma^*} f(x) = median\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y \rangle \in A\}.$$

Here Pol denotes the set of all polynomials. Both definitions have turned out to be of considerable interest in structural complexity as can be seen by a long series of papers, for instance [HV95, OH93, VW93] to name only a few of them. Note that the # and the Mēd operator capture the essence of counting and finding the middle element, respectively.

So it is quite natural to similarly define "pure" optimization operators. Let $\mathcal{K}$ be a complexity class. We define the function classes max $\cdot\mathcal{K}$ and min $\cdot\mathcal{K}$:

$$f \in \max\cdot\mathcal{K} \iff \bigvee_{A \in \mathcal{K}} \bigvee_{p \in Pol} \bigwedge_{x \in \Sigma^*} f(x) = max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y \rangle \in A\}$$

and if this set is empty let $f(x) = 0$.

$$g \in \min\cdot\mathcal{K} \iff \bigvee_{A \in \mathcal{K}} \bigvee_{p \in Pol} \bigwedge_{x \in \Sigma^*} g(x) = min\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y \rangle \in A\}$$

and if this set is empty let $g(x) = 2^{p(|x|)}$.

Obviously we have max $\cdot$NP = max-P. But we do not know similar equivalences for our classes max $\cdot$P and max $\cdot$ coNP offhand. So we study the inclusion relations between the min and max classes and also their inclusion relations with the other well known function classes such as # $\cdot$ P, span-P, # $\cdot$ coNP and the classes FP and FΔ_2^p. It turns out that none of the interesting new classes such as max $\cdot$P and max $\cdot$ coNP is equal to some known class. And under reasonable structural assumptions all min and max classes are distinct. Our results reveal an interesting asymmetry between maximization and minimization with respect to counting.

Our operator max should be compared with the maximization operator defined in [VW95, Vol94]. Vollmer and Wagner define for a complexity class $\mathcal{K}$:

$f \in \text{F}\cdot\mathcal{K} \iff$ there exist a set $A \in \mathcal{K}$ and functions $g_1, g_2 \in$ FP such that for all x:

(1.) $\langle x, y+1 \rangle \in A \Longrightarrow \langle x, y \rangle \in A$,

(2.) $f(x) = max\{y : g_1(x) \leq y \leq g_2(x) \wedge \langle x, y \rangle \in A\}$ if this set is not empty, $f(x) = g_1(x)$ otherwise.

In contrast to our definition we have here the additional condition (1.) which essentially allows a binary search to find the value of a maximization function. We want to study a pure optimization operator and thus concentrate on maximization without any further constraints.

Regarding the relationship of the operators F [VW95] and max we note that clearly $\text{F}\cdot\mathcal{K} \subseteq \max\cdot\mathcal{K}$ for any complexity class $\mathcal{K}$. Furthermore we are able to show that $\text{F}\cdot\mathcal{K} = \max\cdot\mathcal{K} \iff \mathcal{K} = \exists \cdot \mathcal{K}$ and $\text{F}\cdot\mathcal{K} = \min \cdot \text{co}\mathcal{K} \iff \mathcal{K} = \forall \cdot \mathcal{K}$. Hence in the case that $\mathcal{K} =$ P or $\mathcal{K} =$ coNP we get truly new function classes.

A different model of maximization was investigated by Chen and Toda [CT95]. Chen and Toda investigate the complexity of finding *maximal* solutions with respect to a partial order. Note that we search for *the maximum* solution with respect to the natural ordering of natural numbers.

We prove a number of powerful relations regarding the interaction of the operators max, min, U, Sig, C, $\oplus$, $\exists$, and $\forall$. In particular we show for complexity classes $\mathcal{K}$ having some reasonable closure properties, such as closure under $\leq_m^p$:

$$
\begin{array}{ll}
\mathrm{U} \cdot \max\cdot\mathcal{K} = \mathcal{K}, & \mathrm{Sig} \cdot \min\cdot\mathcal{K} = \mathrm{co}\mathcal{K}, \\
\mathrm{U} \cdot \min\cdot\mathcal{K} = \mathrm{co}\mathcal{K}, & \mathrm{C} \cdot \max\cdot\mathcal{K} = \exists \cdot \mathcal{K}, \\
\mathrm{Sig} \cdot \max\cdot\mathcal{K} = \exists \cdot \mathcal{K}. & \mathrm{C} \cdot \min\cdot\mathcal{K} = \forall \cdot \mathrm{co}\mathcal{K},
\end{array}
$$

$$\oplus \cdot \max\cdot\mathcal{K} = \oplus \cdot \min\cdot\mathcal{K} = \mathrm{P}^{\exists\cdot\mathcal{K}}.$$

Note that we define the operators C and $\oplus$ on function classes and not on complexity classes as originally done in the literature [Wag86, PZ83]. Our operator C appeared already in [VW95].

The above relations turn out to be a very useful tool in order to show that various inclusions between function classes are unlikely. Our study yields a huge number of new characterizations of central complexity classes such as P, NP, coNP and PP and allows a characterization of the polynomial hierarchy based on only three operators.

2 Preliminaries

We adopt the notations commonly used in structural complexity. For details we refer the reader to any of the following standard books, e.g., [BDG88, Pap94]. Denote the characteristic function of a set A by c_A. We usually identify strings over the alphabet $\Sigma = \{0,1\}$ with computation paths of nondeterministic Turing machines in the following manner: The sequence of 0's and 1's is the sequence of nondeterministic choices our machine will make, where 0 stands for taking the first and 1 for taking the second of the two possible computation steps. A nondeterministic polynomial-time Turing machine (NPTM for short) with time bound p is said to be *normalized* if for all inputs x all of its computation paths have the same length $p(|x|)$. Denote the set of all polynomials by Pol.

For a nondeterministic polynomial-time Turing machine M with output device we define for every input x, the sets $Acc_M(x)$ and $Out_M(x)$ and the function $maxpath_M(x)$ as follows:

$Acc_M(x) = \{z : z \text{ is an accepting computation path of } M(x)\}$,

$Out_M(x) = \{y : y \text{ is output on some accepting path of } M(x)\}$,

$maxpath_M(x) = max\{z : z \in Acc_M(x)\}$,

Let $\langle .,. \rangle$ and $\langle\!\langle .,. \rangle\!\rangle$ be pairing functions mapping from $\Sigma^* \times \mathbb{N}$ and $\Sigma^* \times \Sigma^*$ to Σ^*, respectively, and having the standard properties such as being polynomial-time computable and polynomial-time invertible. Let for every natural number n, $bin(n)$ denote the binary representation of n and let for every nonempty string $z \in \Sigma^*$, $number(z)$ denote the natural number whose binary representation, possibly with some leading 0's, forms the string z.

Let $\# \cdot \mathcal{K}$ be defined as done in the previous section.

As already noted we want to study classes of "pure" optimization functions and thus define for a complexity class $\mathcal{K}$,

$$f \in \max\cdot\mathcal{K} \iff \bigvee_{A\in\mathcal{K}} \bigvee_{p\in Pol} \bigwedge_{x\in\Sigma^*} f(x) = max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \in A\},$$

$$g \in \min\cdot\mathcal{K} \iff \bigvee_{A\in\mathcal{K}} \bigvee_{p\in Pol} \bigwedge_{x\in\Sigma^*} g(x) = min\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \in A\}.$$

Note that one can define $\max\cdot\mathcal{K}$ in a more elegant way, namely

$$f \in \max\cdot\mathcal{K} \iff \bigvee_{A\in\mathcal{K}} \bigvee_{p\in Pol} \bigwedge_{x\in\Sigma^*} f(x) = sup\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in A\}.$$

Because of reasons of tradition we stick to the notation max and min instead of introducing sup and inf. But when writing max or min we actually mean sup or inf. In the following we omit any statement concerning the empty set, since all complexity classes $\mathcal{K}$ in the context of this paper are at least closed under $\leq_m^p$, and thus we can without loss of generality assume that for all $A \in \mathcal{K}$ and all x, in the case of maximization $\langle x, 0\rangle \in A$ and in the case of minimization $\langle x, 2^{p(|x|)}\rangle \in A$. Note that we can, if $\mathcal{K}$ is closed under intersection with P sets, equivalently define

$$f \in \max\cdot\mathcal{K} \iff \bigvee_{A\in\mathcal{K}} \bigvee_{f_1\in \mathrm{FP}} \bigvee_{f_2\in \mathrm{FP}} \bigwedge_{x\in\Sigma^*} f(x) = max\{y : f_1(x) \leq y \leq f_2(x) \wedge \langle x, y\rangle \in A\}.$$

Recall that the polynomial hierarchy is defined as follows: $\Sigma_0^p = \Pi_0^p = \Delta_0^p = \Delta_1^p = \mathrm{P}$ and, for each $i \geq 0$, $\Sigma_i^p = \mathrm{NP}^{\Sigma_{i-1}^p}$, $\Pi_i^p = \{L : \overline{L} \in \Sigma_i^p\}$, and $\Delta_i^p = \mathrm{P}^{\Sigma_{i-1}^p}$. In this paper we want to concentrate on classes of optimization functions where the underlying complexity class $\mathcal{K}$ is a class Δ_i^p, Σ_i^p or Π_i^p from the polynomial hierarchy. For short we will call these classes the min-max classes.

Let $\mathrm{F}\Delta_i^p$ denote the set of all single valued total functions (mapping from Σ^* to $\mathbb{N}$) computable in polynomial-time with the help of an oracle from Σ_{i-1}^p. Let $\mathrm{FP} = \mathrm{F}\Delta_1^p$.

We say a set is trivial if it is $\emptyset$ or Σ^* and otherwise we say it is nontrivial. We often need that a complexity class $\mathcal{K}$ is closed under intersection with P sets. Note that closure under $\leq_m^p$ reductions together with the property that $\mathcal{K}$ contains nontrivial sets ensures that. In order to be able to state theorems in a very compact form we let for reasons of simplicity complexity class denote a set of languages over Σ^*, containing at least one nontrivial language.

Let a function class be a set of single valued total functions mapping from Σ^* to $\mathbb{N}$. For a function class $\mathcal{F}$ define its subset of polynomially bounded functions by

$$\mathcal{F}_{pol} = \{f \in \mathcal{F} : \bigvee_{p\in Pol} \bigwedge_{x\in\Sigma^*} f(x) \leq p(|x|)\}.$$

Note that the same class was denoted by $\mathcal{F}[O(\text{logn})]$ in [KST89].

For every set of operators OP defined on a (complexity or function) class $\mathcal{C}$ we denote the algebraic closure of $\mathcal{C}$ under the operators of OP by $\Gamma_{\mathrm{OP}}(\mathcal{C})$.

3 Inclusion Relations

We will study the inclusion relations among the min-max classes. Recall $\# \cdot \mathrm{NP} \subseteq \# \cdot \mathrm{coNP}$ from [KST89]. Do the operators max and min display a similar behavior with respect to their effect on the complexity classes NP and coNP? We will see that this is not the case.

The operators max and min are obviously monotone and one can easily verify the following claims.

Proposition 3.1 *1.* $\max\cdot\mathrm{NP} = max\text{-}\mathrm{P}$,

2. $\min\cdot\mathrm{NP} = min\text{-}\mathrm{P}$,

3. $\mathrm{FP} \subseteq \max\cdot\mathrm{P} \subseteq \max\cdot\mathrm{NP} \subseteq \mathrm{F}\Delta_2^p$,

4. $\mathrm{FP} \subseteq \min \cdot \mathrm{P} \subseteq \min \cdot \mathrm{NP} \subseteq \mathrm{F}\Delta_2^p$,

5. $\max \cdot \mathrm{P} \subseteq \max \cdot \mathrm{coNP} \subseteq \max \cdot \Delta_2^p$,

6. $\min \cdot \mathrm{P} \subseteq \min \cdot \mathrm{coNP} \subseteq \min \cdot \Delta_2^p$.

Note that all the above claims relativize.

Besides that we are able to prove another inclusion which establishes a close connection between $\max \cdot \Sigma_i^p$ and $\min \cdot \Pi_i^p$, and $\min \cdot \Sigma_i^p$ and $\max \cdot \Pi_i^p$, for all $i \in \mathbb{N}$.

Lemma 3.1 *For every $i \in \mathbb{N}$,*

1. $\max \cdot \Sigma_i^p \subseteq \min \cdot \Pi_i^p$,

2. $\min \cdot \Sigma_i^p \subseteq \max \cdot \Pi_i^p$.

Proof: (1.) Let $f \in \max \cdot \Sigma_i^p$, $A \in \Sigma_i^p$ and $p \in Pol$ such that for all x, $f(x) = max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in A\}$.

Define B to be the set

$$B = \{y : \bigvee_{y'} (y < y' < 2^{p(|x|)} \wedge \langle x, y'\rangle \in A)\}.$$

Since the class Σ_i^p is closed under $\exists$ we have $B \in \Sigma_i^p$. And note that

$$\begin{aligned} f(x) &= max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in A\} \\ &= min\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \notin B\}. \end{aligned}$$

Hence $f \in \min \cdot \Pi_i^p$.

(2.) The proof is similar to the proof of (1). ∎

Note that we have $\max \cdot \mathrm{NP} \subseteq \mathrm{F}\Delta_2^p$ according to Proposition 3.1. One would expect that also $\max \cdot \mathrm{coNP} \subseteq \mathrm{F}\Delta_2^p$ holds. We will see later, that this is not the case under reasonable structural assumptions.

Our next theorem shows that we have an inclusion between $\# \cdot \mathcal{K}$ and $\max \cdot \mathcal{C}$ if and only if there is also an inclusion between $\# \cdot \mathrm{co}\mathcal{K}$ and $\min \cdot \mathcal{C}$. We show furthermore that min and max are dual with respect to their inclusion relations.

Theorem 3.1 *For every complexity classes $\mathcal{K}$ and $\mathcal{C}$ closed under $\leq_m^p$,*

1. $\# \cdot \mathcal{K} \subseteq \max \cdot \mathcal{C} \iff \# \cdot \mathrm{co}\mathcal{K} \subseteq \min \cdot \mathcal{C}$,

2. $\max \cdot \mathcal{K} \subseteq \min \cdot \mathcal{C} \iff \min \cdot \mathcal{K} \subseteq \max \cdot \mathcal{C}$.

Proof: Let $\mathcal{K}$ and $\mathcal{C}$ be complexity classes closed under $\leq_m^p$. Before we start with the actual proof let us make some observations.

(I) Consider a function $f \in \# \cdot \mathcal{K}$, thus there exist a set $B \in \mathcal{K}$ and a polynomial p such that $f(x) = ||\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in B\}||$ for all x. Obviously

$$2^{p(|x|)} - f(x) = ||\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \notin B\}||$$

and thus is in $\# \cdot \mathrm{co}\mathcal{K}$.

(II) Now consider a function f such that $f \in \max\cdot\mathcal{C}$ via the set $B \in \mathcal{C}$ and $g \in \mathrm{FP}$, that is $f(x) = max\{y : 0 \le y \le g(x) \wedge \langle x, y\rangle \in B\}$ for all x. The set

$$B' = \{\langle x, y\rangle : 0 \le y \le g(x) \wedge \langle x, g(x) - y\rangle \in B\}$$

is also a set from $\mathcal{C}$, since $\mathcal{C}$ is closed under $\le_m^p$. Note that

$$g(x) - f(x) = min\{y : 0 \le y \le g(x) \wedge \langle x, y\rangle \in B'\}$$

and thus is in $\min\cdot\mathcal{C}$.

(III) Similarly one can see: If $f \in \min\cdot\mathcal{C}$ via $B \in \mathcal{C}$ and $g \in \mathrm{FP}$ then $g(x) - f(x) \in \max\cdot\mathcal{C}$. Now lets turn to the actual proof of our theorem.

(1.) Suppose $\# \cdot \mathcal{K} \subseteq \max\cdot\mathcal{C}$. Let $f \in \# \cdot \mathrm{co}\mathcal{K}$ via $B \in \mathcal{K}$ and polynomial p. We conclude $2^{p(|x|)} - f(x) \in \# \cdot \mathcal{K}$ from observation (I) and thus $2^{p(|x|)} - f(x) \in \max\cdot\mathcal{C}$ by our assumption.

Let $2^{p(|x|)} - f(x) \in \max\cdot\mathcal{C}$ via the set $C \in \mathcal{C}$ and the polynomial q, thus

$$2^{p(|x|)} - f(x) = max\{y : 0 \le y < 2^{q(|x|)} \wedge \langle x, y\rangle \in C\}.$$

Without loss of generality we have $p < q$ and hence

$$2^{p(|x|)} - f(x) = max\{y : 0 \le y \le 2^{p(|x|)} \wedge \langle x, y\rangle \in C\}.$$

But $2^{p(|x|)} - (2^{p(|x|)} - f(x)) = f(x) \in \min\cdot\mathcal{C}$ by observation (II).

This proves $\# \cdot \mathcal{K} \subseteq \max\cdot\mathcal{C} \Longrightarrow \# \cdot \mathrm{co}\mathcal{K} \subseteq \min\cdot\mathcal{C}$. Similarly one can show the other implication of claim (1.).

(2.) The proof of claim (2.) is quite similar and thus omitted. ∎

Recall from [KST89, Köb89] that $\max\cdot\mathrm{NP} \subseteq \# \cdot \mathrm{NP}$, $\mathrm{F}\Delta_2^p \subseteq \# \cdot \mathrm{coNP}$ and $\# \cdot \mathrm{coNP} = \# \cdot \Delta_2^p$. The known inclusion relations between the considered function classes are presented in Figure 1.

In Section 5 and 6 we will show that we can not expect to have more inclusion relations than shown in Figure 1. Though we are neither able to prove that $\min\cdot\mathrm{P} \subseteq \# \cdot \mathrm{NP}$ nor can give structural consequences for this case, there is also relativized evidence that the above inclusion picture can not be improved. Recently, Glaser and Wechsung [GW97] showed among other relativization results that there exists a relativized world in which $\min\cdot\mathrm{P} \subseteq \# \cdot \mathrm{NP}$. They also describe a relativized world in which $\min\cdot\mathrm{P} \not\subseteq \# \cdot \mathrm{NP}$. In other words, non relativizable proof techniques are not powerful enough to solve the question whether "$\min\cdot\mathrm{P} \subseteq \# \cdot \mathrm{NP}$?"

The original definition of the class OptP in [Kre88] gives us a machine based characterization of $\max\cdot\mathrm{NP}$ and $\min\cdot\mathrm{NP}$. The following two lemmas express similar characterizations for $\max\cdot\mathrm{P}$ and $\max\cdot\mathrm{coNP}$.

Before we state the lemmas let us make the following observation. Let $\mathcal{K}$ be a complexity class closed under $\le_m^p$. Consider the function f where $f \in \max\cdot\mathcal{K}$ via the set $A \in \mathcal{K}$ and the polynomial p, in other words $f(x) = max\{y : 0 \le y < 2^{p(|x|)} \wedge \langle x, y\rangle \in A\}$ for all x. Now define the function f' which maps from Σ^* to Σ^* such that for all strings x, $|f'(x)| = p(|x|)$ and $f'(x) = 000\cdots0\, bin(f(x))$. Note that f' can also be described via $f'(x) = max_{\le_{lex}}\{z : z \in \Sigma^{=p(|x|)} \wedge \langle\!\langle x, z\rangle\!\rangle \in A'\}$ where $A' = \{\langle\!\langle x, z\rangle\!\rangle : \langle x, number(z)\rangle \in A\}$. Clearly $A' \in \mathcal{K}$ if $\mathcal{K}$ is closed under $\le_m^p$ and thus f' is close to be a real $\max\cdot\mathcal{K}$ function, but formally speaking it is not since $\max\cdot\mathcal{K}$ functions map from Σ^* to $\mathbb{N}$.

Similarly to the above we find a true $\max\cdot\mathcal{K}$ analog of each function g defined by $g(x) = max_{\le_{lex}}\{z : z \in \Sigma^{=q(|x|)} \wedge \langle\!\langle x, z\rangle\!\rangle \in B\}$ where q is some polynomial and $B \in \mathcal{K}$.

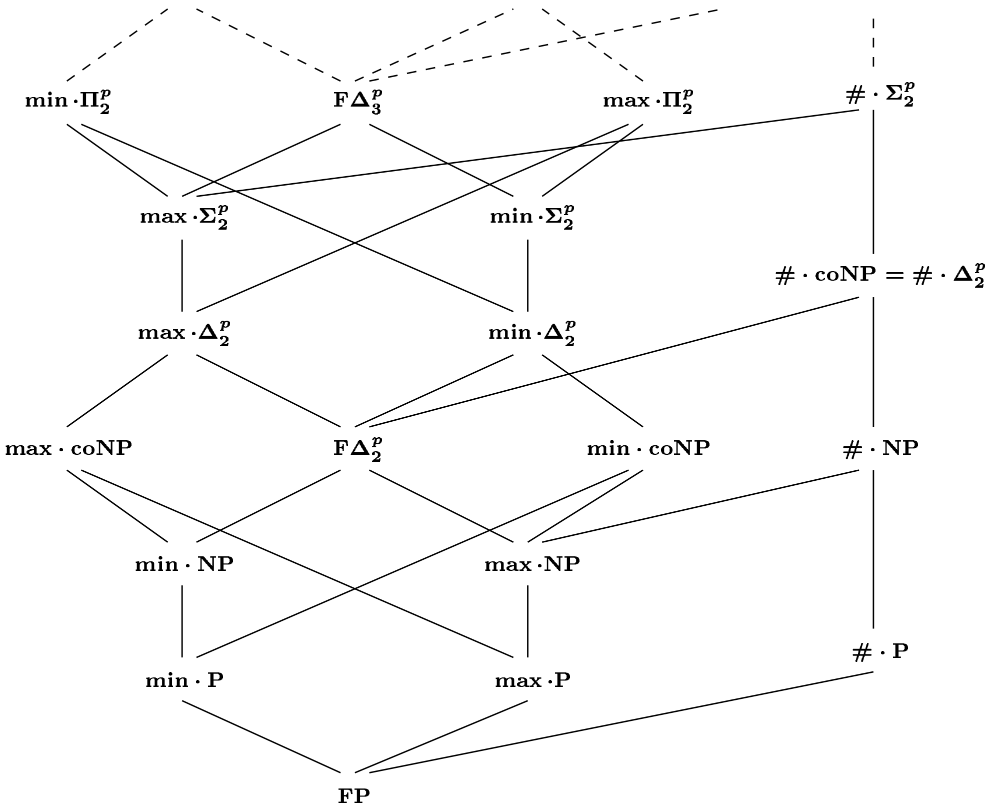


Figure 1: Inclusion structure of the min-max classes

The point of the above comment is that the function MAXIMUM SATISFYING ASSIGNMENT is, formally speaking, not a max ·P function.

Problem:	MAXIMUM SATISFYING ASSIGNMENT
Input:	boolean formula F.
Output:	lexicographically largest satisfying assignment of F.

Note that MAXIMUM SATISFYING ASSIGNMENT maps to Σ^* and not to $\mathbb{N}$ as all max ·P functions do, since a satisfying assignment is a string as leading zeros matter.

But clearly, MAXIMUM SATISFYING ASSIGNMENT NUMBER is a max ·P function, where

Problem:	MAXIMUM SATISFYING ASSIGNMENT NUMBER
Input:	boolean formula F.
Output:	*number*(lexicographically largest satisfying assignment of F).

and returns the same information as MAXIMUM SATISFYING ASSIGNMENT does.

Related to the above comment is the fact that we can understand every max ·P function as being computed by a NPTM as the largest accepting path—modulo the string integer correspondence, as the number of leading zeros is crucial for the outcome of a path. The lemma below states this fact in more detail.

Lemma 3.2 *For every function f,*

$$f \in \max\cdot\mathrm{P} \iff \bigvee_{\text{normalized NPTM } M} \bigwedge_{x\in\Sigma^*} f(x) = number(maxpath_M(x)).$$

Proof: Let $f \in \max\cdot\mathrm{P}$ via the set $A \in P$ and the polynomial p. Now let M be a nondeterministic Turing machine which on input x guesses a string $z \in \Sigma^{=p(|x|)}$ and accepts if and only if $\langle x, number(z)\rangle \in A$. M is obviously a normalized NPTM and we have for all inputs x, $f(x) = number(maxpath_M(x))$.

For the other implication let M be a normalized NPTM working in time p. The set $B = \{\langle x,y\rangle : M(x)$ accepts along path $0^{p(|x|)-|bin(y)|}y\}$ is clearly in P and hence

$$number(maxpath_M(x)) = max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \in B\}.$$

∎

Lemma 3.3 *For every function f,*

$$f \in \max\cdot\mathrm{coNP} \iff \bigvee_{\mathrm{NPTM} M} \bigwedge_{x\in\Sigma^*} f(x) = max\ \overline{Out_M(x)}.$$

Proof: Let $f \in \max\cdot\mathrm{coNP}$ via the set $A \in \mathrm{coNP}$ and the polynomial p. Let N be a NPTM accepting $\overline{A}$ in time q. Let M be a Turing machine working as follows: On input x M guesses y, $0 \leq y < 2^{p(|x|)}$, and z, $0 \leq z < 2^{q(|y|)}$, and simulates the work of $N(\langle x,y\rangle)$ along computation path z. $M(x)$ accepts and outputs y if and only if $N(\langle x,y\rangle)$ accepts. Note that

$$\begin{aligned} max\ \overline{Out_M(x)} &= max\{y : y \notin Out_M(x)\} \\ &= max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \in A\} \\ &= f(x). \end{aligned}$$

The other implication is true since we have for all NPTM M the set $A = \{\langle x,y\rangle : y \notin Out_M(x)\}$ is in coNP. ∎

Easily one can get similar characterizations for $\min\cdot\mathrm{P}$ and $\min\cdot\mathrm{coNP}$ and also in general for $\max\cdot\Delta_i^p, \min\cdot\Delta_i^p$ etc.

For the sake of self containment we will study the inclusion relations of the polynomially bounded counterparts of the min-max classes at the end of this section. The only known result is due to Köbler [Köb89], namely $\max_{\mathrm{pol}}\cdot\mathrm{NP} = \mathrm{span}_{\mathrm{pol}}\text{-}\mathrm{P}$. So lets take a look at the inclusion relations of the polynomial bounded subsets of the min-max classes.

Lemma 3.4 *1.* $\max_{\mathrm{pol}}\cdot\mathrm{P} = \min_{\mathrm{pol}}\cdot\mathrm{P} = \mathrm{FP}_{\mathrm{pol}} \subseteq \mathrm{FP}$*,*

2. $\max_{\mathrm{pol}}\cdot\mathrm{NP} = \min_{\mathrm{pol}}\cdot\mathrm{coNP}$*,*

3. $\min_{\mathrm{pol}}\cdot\mathrm{NP} = \max_{\mathrm{pol}}\cdot\mathrm{coNP}$*.*

Proof: (1.) can be seen as follows. Every $\max_{\mathrm{pol}}\cdot\mathrm{P}$ function can be computed in deterministic polynomial-time by brute force simulation. And thus $\max_{\mathrm{pol}}\cdot\mathrm{P} \subseteq \mathrm{FP}_{\mathrm{pol}}$. The inverse inclusion is trivial. The same argumentation works for $\min_{\mathrm{pol}}\cdot\mathrm{P}$.

(2.) The inclusion $\max_{\text{pol}} \cdot \text{NP} \subseteq \min_{\text{pol}} \cdot \text{coNP}$ can be shown by using the same ideas as in the proof of (1.) of Lemma 3.1. The other inclusion can be seen as follows: Let $f \in \min_{\text{pol}} \cdot \text{coNP}$ via the set $A \in \text{coNP}$ and the polynomial p. The set $B = \{\langle x, y\rangle : 0 \leq y \leq p(|x|) \wedge \bigwedge_z (z < y \Rightarrow \langle x, z\rangle \notin A)\}$ is in NP and we have $max\{y : 0 \leq y \leq p(|x|) \wedge \langle x, y\rangle \in B\} = f(x)$. Thus $f \in \max_{\text{pol}} \cdot \text{NP}$.

Similarly one can show (3.). ∎

4 Operators on Function Classes

In this section we define and study operators which map function classes to complexity classes. We investigate their images with respect to $\max \cdot \mathcal{K}$ and $\min \cdot \mathcal{K}$. Those investigations yield a huge number of new characterizations of central complexity classes and thus turn into a key tool for giving evidence that the picture of inclusion relations of Section 3 (see Figure 1) can not be improved, unless the polynomial hierarchy collapses.

One way of giving such consequences is applying a suitable monotone operator to the classes under investigation such that the non inclusion of the images is widely believed to be true. This idea was extensively exploited by Vollmer and Wagner in [VW93]. We will use the same method here to show that there are no more inclusions between the min-max classes, the $\text{F}\Delta_i^p$ classes and the # classes than shown in Figure 1 .

We define the following operators:

Definition 4.1 *For any function class $\mathcal{F}$ let*

$$\begin{aligned}
A \in \text{U} \cdot \mathcal{F} &\iff c_A \in \mathcal{F}, \\
A \in \text{Sig} \cdot \mathcal{F} &\iff \bigvee_{f \in \mathcal{F}} \bigwedge_{x \in \Sigma^*} (x \in A \iff f(x) > 0), \\
A \in \text{C} \cdot \mathcal{F} &\iff \bigvee_{f \in \mathcal{F}} \bigvee_{g \in \text{FP}} \bigwedge_{x \in \Sigma^*} (x \in A \iff f(x) \geq g(x)), \\
A \in \oplus \cdot \mathcal{F} &\iff \bigvee_{f \in \mathcal{F}} \bigwedge_{x \in \Sigma^*} (x \in A \iff f(x) \equiv 1 \bmod 2).
\end{aligned}$$

The above operators C and $\oplus$ are usually defined on complexity classes and not on function classes (see [Wag86, PZ83]). But note that we have for every operator $\text{OP} \in \{\text{C}, \oplus\}$ (denote its counterpart defined on complexity classes by op):

$$\text{op} \cdot \mathcal{K} = \text{OP} \cdot \# \cdot \mathcal{K}.$$

So it is quite natural to denote the above defined operators in that manner, since they capture the essential properties of their original definitions. Note that all of the above operators are indeed monotone.

Next we are going to study the effect of the above mentioned operators on the $\text{F}\Delta_i^p$ and the # classes.

Proposition 4.1 *For every positive $i \in \mathbb{N}$,*

$$\text{U} \cdot \text{F}\Delta_i^p = \text{Sig} \cdot \text{F}\Delta_i^p = \text{C} \cdot \text{F}\Delta_i^p = \oplus \cdot \text{F}\Delta_i^p = \Delta_i^p.$$

The proof is obvious and thus omitted. The results of the following proposition are also either well known or easy to prove. For the results 5 - 7 recall from [KST89] that $\# \cdot \mathrm{coNP} = \# \cdot \Delta_2^p$.

Proposition 4.2

1. $\mathrm{U} \cdot \# \cdot \mathrm{P} = \mathrm{UP}$,
2. $\mathrm{C} \cdot \# \cdot \mathrm{P} = \mathrm{PP}$,
3. $\oplus \cdot \# \cdot \mathrm{P} = \oplus\mathrm{P}$,
4. $\mathrm{U} \cdot \# \cdot \mathrm{NP} = \mathrm{NP}$,
5. $\mathrm{U} \cdot \# \cdot \mathrm{coNP} = \mathrm{UP}^{\mathrm{NP}}$,
6. $\mathrm{C} \cdot \# \cdot \mathrm{NP} = \mathrm{C} \cdot \# \cdot \mathrm{coNP} = \mathrm{PP}^{\mathrm{NP}}$,
7. $\oplus \cdot \# \cdot \mathrm{NP} = \oplus \cdot \# \cdot \mathrm{coNP} = \oplus\mathrm{P}^{\mathrm{NP}}$.

The following sequence of theorems investigates the images of classes of the form $\max\cdot\mathcal{K}$ or $\min\cdot\mathcal{K}$, where $\mathcal{K}$ is some complexity class, under the various operators.

Theorem 4.1 *For every complexity class $\mathcal{K}$ closed under $\leq_m^p$,*

1. $\mathrm{U} \cdot \max\cdot\mathcal{K} = \mathcal{K}$,
2. $\mathrm{U} \cdot \min\cdot\mathcal{K} = \mathrm{co}\mathcal{K}$.

Proof: (1.) Let $\mathcal{K}$ be a complexity class closed under $\leq_m^p$. Let $A \in \mathrm{U} \cdot \max\cdot\mathcal{K}$, hence there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all $x \in \Sigma^*$,

$$x \in A \iff max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in B\} = 1$$

$$x \notin A \iff max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in B\} = 0.$$

Hence

$$x \in A \iff \langle x, 1\rangle \in B.$$

Since $\mathcal{K}$ is closed under $\leq_m^p$ we get $A \in \mathcal{K}$.

Now let $A \in \mathcal{K}$. Define B to be the following set: $B = A \times \{1\}$. We conclude $B \in \mathcal{K}$ since $A \in \mathcal{K}$ and $\mathcal{K}$ is closed under $\leq_m^p$. Thus the following is true:

$$x \in A \iff c_A(x) = 1 = max\{y : 0 \leq y \leq 1 \wedge \langle x, y\rangle \in B\}$$

$$x \notin A \iff c_A(x) = 0 = max\{y : 0 \leq y \leq 1 \wedge \langle x, y\rangle \in B\}.$$

But this shows that $c_A \in \max\cdot\mathcal{K}$ and hence $A \in \mathrm{U} \cdot \max\cdot\mathcal{K}$.

Similarly one can show (2.). ∎

Note that the proof of Theorem 4.1 yields also $\mathrm{U} \cdot \max_{\mathrm{pol}}\cdot\mathcal{K} = \mathcal{K}$ and $\mathrm{U} \cdot \min_{\mathrm{pol}}\cdot\mathcal{K} = \mathrm{co}\mathcal{K}$.

Theorem 4.2 *For every complexity class $\mathcal{K}$ closed under $\leq_m^p$,*

1. $\mathrm{Sig} \cdot \max\cdot\mathcal{K} = \exists \cdot \mathcal{K}$,
2. $\mathrm{Sig} \cdot \min\cdot\mathcal{K} = \mathrm{co}\mathcal{K}$,
3. $\mathrm{Sig} \cdot \# \cdot \mathcal{K} = \exists \cdot \mathcal{K}$.

Proof: Let $\mathcal{K}$ be a complexity class closed under $\leq_m^p$.

(1.) Let $A \in \mathrm{Sig} \cdot \max\text{-}\mathcal{K}$. Hence there exists a function $f \in \max\text{-}\mathcal{K}$ such that for all $x \in \Sigma^*$,

$$x \in A \iff f(x) > 0.$$

By definition of the class $\max\text{-}\mathcal{K}$ there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in B\} > 0 \\ &\iff \bigvee_{0<y<2^{p(|x|)}} (\langle x, y\rangle \in B). \end{aligned}$$

We conclude $A \in \exists \cdot \mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$.

Now let $A \in \exists \cdot \mathcal{K}$. Hence there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$x \in A \iff \bigvee_{0\leq y<2^{p(|x|)}} (\langle x, y\rangle \in B).$$

Define C to be the set $C = \{\langle x, y+1\rangle : \langle x, y\rangle \in B\}$. $C \in \mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$.

Thus we have shown that there exist a set $C \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff \bigvee_{y} (1 \leq y \leq 2^{p(|x|)} \wedge \langle x, y\rangle \in C) \\ &\iff max\{y : 0 \leq y \leq 2^{p(|x|)} \wedge \langle x, y\rangle \in C\} > 0. \end{aligned}$$

This proves $A \in \mathrm{Sig} \cdot \max\text{-}\mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$ and thus completes the proof of (1.).

(2.) Let $A \in \mathrm{Sig} \cdot \min\text{-}\mathcal{K}$. Hence there exists a function $f \in \min\text{-}\mathcal{K}$ such that for all $x \in \Sigma^*$,

$$x \in A \iff f(x) > 0.$$

By definition of the class $\max\text{-}\mathcal{K}$ there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff min\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in B\} > 0 \\ &\iff \langle x, 0\rangle \notin B. \end{aligned}$$

We conclude $A \in \mathrm{co}\mathcal{K}$ since $\mathcal{K}$ and thus also $\mathrm{co}\mathcal{K}$ are closed under $\leq_m^p$.

Now let $A \in \mathrm{co}\mathcal{K}$ and $B = A \times \{0\}$. Thus $B \in \mathrm{co}\mathcal{K}$. Note that we have for all $x \in \Sigma^*$,

$$x \in A \iff min\{y : 0 \leq y \leq 1 \wedge \langle x, y\rangle \in \overline{B}\} > 0.$$

This proves $A \in \mathrm{Sig} \cdot \min\text{-}\mathcal{K}$ and completes the proof of (2.).

(3.) The proof is straightforward and thus omitted. ∎

Theorem 4.3 *For every complexity class $\mathcal{K}$ closed under $\leq_m^p$,*

$$\mathrm{C} \cdot \max\text{-}\mathcal{K} = \exists \cdot \mathcal{K}.$$

Proof: Let $\mathcal{K}$ be a complexity class closed under $\leq_m^p$.

Let $A \in \mathrm{C} \cdot \max\text{-}\mathcal{K}$. Hence there exist a function $f \in \max\text{-}\mathcal{K}$ and a function $g \in \mathrm{FP}$ such that for all $x \in \Sigma^*$,

$$x \in A \iff f(x) \geq g(x).$$

By definition of the class max·$\mathcal{K}$ there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff max\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \in B\} \geq g(x) \\ &\iff \bigvee_{g(x) \leq y < 2^{p(|x|)}} (\langle x,y\rangle \in B). \end{aligned}$$

We conclude $A \in \exists \cdot \mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$.

Now let $A \in \exists \cdot \mathcal{K}$. Hence there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$x \in A \iff \bigvee_{0 \leq y < 2^{p(|x|)}} (\langle x,y\rangle \in B.$$

Define C to be the set $C = \{\langle x, y+1\rangle : \langle x,y\rangle \in B\}$. $C \in \mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$.

Thus we have shown that there exist a set $C \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff \bigvee_y (1 \leq y \leq 2^{p(|x|)} \wedge \langle x,y\rangle \in C) \\ &\iff max\{y : 0 \leq y \leq 2^{p(|x|)} \wedge \langle x,y\rangle \in C\} \geq 1. \end{aligned}$$

This proves that $A \in \mathrm{C} \cdot \max\cdot\mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$. ∎

Theorem 4.4 *For every complexity class $\mathcal{K}$ closed under $\leq_m^p$,*

$$\mathrm{C} \cdot \min\cdot\mathcal{K} = \forall \cdot \mathrm{co}\mathcal{K}.$$

Proof: Let $\mathcal{K}$ be a complexity class closed under $\leq_m^p$.

Let $A \in \mathrm{C} \cdot \min\cdot\mathcal{K}$. Hence there exist a function $f \in \min\cdot\mathcal{K}$ and a function $g \in \mathrm{FP}$ such that for all $x \in \Sigma^*$,

$$x \in A \iff f(x) \geq g(x).$$

By definition of the class min·$\mathcal{K}$ there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff min\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \in B\} \geq g(x) \\ &\iff \bigwedge_{0 \leq y < g(x)} (\langle x,y\rangle \notin B). \end{aligned}$$

We conclude $A \in \forall \cdot \mathrm{co}\mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$.

Now let $A \in \forall \cdot \mathrm{co}\mathcal{K}$. Hence there exist a set $B \in \mathcal{K}$ and a polynomial p such that for all x,

$$\begin{aligned} x \in A &\iff \bigwedge_y (0 \leq y < 2^{p(|x|)} \wedge \langle x,y\rangle \notin B) \\ &\iff min\{y : 0 \leq y \leq 2^{p(|x|)} \wedge (\langle x,y\rangle \in B\} \geq 2^{p(|x|)}. \end{aligned}$$

This proves that $A \in \mathrm{C} \cdot \min\cdot\mathcal{K}$ since $\mathcal{K}$ is closed under $\leq_m^p$. ∎

A result similar in flavor to the one of Theorem 4.3 was proven in [VW95].

Theorem 4.5 *[VW95] If $\mathcal{K}$ is closed under $\leq_m^p$ and $\mathrm{P} \subseteq \mathcal{K}$, then $\mathrm{C} \cdot \mathrm{F} \cdot \mathcal{K} = \mathcal{K}$.*

Corollary 4.1 *For any complexity class $\mathcal{K}$ closed under $\leq_m^p$,*

1. $\mathrm{F}\cdot\mathcal{K} = \max\cdot\mathcal{K} \iff \mathcal{K} = \exists\cdot\mathcal{K}$,
2. $\mathrm{F}\cdot\mathcal{K} = \min\cdot\mathcal{K} \iff \mathcal{K} = \forall\cdot\mathrm{co}\mathcal{K}$.

The proof is immediate from the Theorems 4.3, 4.4 and 4.5.

Now lets take a look at the images of the min-max classes under the operator $\oplus$. One general result regarding the image of function classes under the operator $\oplus$ can be found in [VW95], namely if $\mathcal{K}$ has certain closure properties then $\oplus\mathrm{F}\cdot\mathcal{K} = \mathrm{P}^{\mathcal{K}}$. Note that this equality does not give us results related to $\max\cdot\mathrm{P}$ or $\max\cdot\mathrm{coNP}$ because of Corollary 4.1. Our result given below holds for a wider range of classes since we only require closure under $\leq_{ctt}^p$.

Theorem 4.6 *For every complexity class $\mathcal{K}$ closed under $\leq_{ctt}^p$,*

1. $\oplus\cdot\max\cdot\mathcal{K} = \mathrm{P}^{\exists\cdot\mathcal{K}}$,
2. $\oplus\cdot\min\cdot\mathcal{K} = \mathrm{P}^{\exists\cdot\mathcal{K}}$.

Proof: Let $\mathcal{K}$ be a complexity class closed under $\leq_{ctt}^p$.

(1.) Let $A \in \oplus\cdot\max\cdot\mathcal{K}$. Hence there exists a function $f \in \max\cdot\mathcal{K}$ such that for all $x \in \Sigma^*$,

$$x \in A \iff f(x) \equiv 1 \bmod 2.$$

Let B be the set $B = \{\langle x, z\rangle : f(x) \geq z\}$. Obviously $B \in \mathrm{C}\cdot\max\cdot\mathcal{K}$ and thus by Theorem 4.3 $B \in \exists\cdot\mathcal{K}$.

For a given $x \in \Sigma^*$, the value $f(x)$ can be determined deterministically in polynomial-time by binary search using queries to B. And given the value $f(x)$, checking wether $f(x)$ is odd or even can also be done deterministically in polynomial-time. So we have $A \in \mathrm{P}^B$ and thus $A \in \mathrm{P}^{\exists\cdot\mathcal{K}}$.

Now let $A \in \mathrm{P}^{\exists\cdot\mathcal{K}}$. So by definition there exist a deterministic oracle machine M, a set $B \in \exists\cdot\mathcal{K}$ and a nondecreasing polynomial p such that M^B accepts A in time p. Since $B \in \exists\cdot\mathcal{K}$ there exist a set $C \in \mathcal{K}$ and a polynomial q such that for all y,

$$y \in B \iff \bigvee_{|w|=2^{q(|y|)}} (\langle\!\langle y, w\rangle\!\rangle \in C).$$

Let us without loss of generality assume that the machine M on input x runs exactly $p(|x|)$ steps, making one query of length $p(|x|)$ to B in every step. Define a set D to be

$D = \{\langle x, z\rangle : x \in \Sigma^* \wedge z \in \mathbb{N}$ and

- **(1)** $bin(z) = 1a_1a_2\cdots a_{p(|x|)}y_1y_2\cdots y_{p(|x|)}w_1w_2\cdots w_{p(|x|)}c$ and
- **(2)** for every $1 \leq i \leq p(|x|)$: $a_i \in \{0,1\}$, $y_i \in \Sigma^* \wedge |y_i| = p(|x|)$, $w_i \in \Sigma^* \wedge |w_i| = q(p(|x|))$ and $c \in \{0,1\}$, and
- **(2)** $M^{(\cdot)}$ yields the result c given that on input x the queries $y_1, y_2, \cdots y_{p(|x|)}$ are asked in this order under the assumption that $a_1, a_2, \cdots a_{p(|x|)}$ are the answers, and
- **(3)** $\bigwedge_{1\leq i\leq p(|x|)} (a_i = 1 \Rightarrow \langle\!\langle y_i, w_i\rangle\!\rangle \in C)\}$.

Note that for every $\langle x, z\rangle \in D$, where $bin(z)$ is of the form

$$bin(z) = 1a_1a_2\cdots a_{p(|x|)}y_1y_2\cdots y_{p(|x|)}w_1w_2\cdots w_{p(|x|)}c,$$

we have, w_i is a witness for y_i (being an element of B) for all i with $a_i = 1$. With other words there is no element $\langle x, z\rangle$ in D such that for some i, $a_i = 1$ and $y_i \notin B$. Hence the largest z, call it $z_0(x)$, such that $\langle x, z\rangle \in D$ describes the computation process of $M^B(x)$ in the sense that $z_0(x)$ contains all queries asked by $M^B(x)$, all their answers and also the overall answer of $M^B(x)$. Thus the least significant bit of $bin(z_0(x))$ is equal to $c_A(x)$ and we have,

$$x \in A \iff max\{z : \langle x, z\rangle \in D\} \equiv 1 \bmod 2.$$

Note furthermore that condition (1) and (2) of D can be checked in deterministic polynomial-time and condition (3) is clearly a $\mathcal{K}$ predicate due to the closure of $\mathcal{K}$ under $\leq^p_{ctt}$. Hence $D \in \mathcal{K}$. This proves $A \in \oplus\cdot\max\cdot\mathcal{K}$.

(2.) The proof is similar. Note that for any complexity class $\mathcal{K}$ closed under $\leq^p_{ctt}$ we have, $\forall\cdot\mathrm{co}\mathcal{K} = \exists\cdot\mathcal{K}$. Furthermore one can easily modify the definition of the set D in such a way that the smallest z, such that $\langle x, z\rangle \in D$, describes the computation of $M^B(x)$. ∎

From the last five theorems, Theorem 4.1 to 4.6, we conclude a series of corollaries.

Corollary 4.2
1. $\mathrm{U}\cdot\max\cdot\mathrm{P} = \mathrm{P}$,
2. $\mathrm{Sig}\cdot\max\cdot\mathrm{P} = \mathrm{NP}$,
3. $\mathrm{C}\cdot\max\cdot\mathrm{P} = \mathrm{NP}$,
4. $\oplus\cdot\max\cdot\mathrm{P} = \Delta^p_2$,
5. $\mathrm{U}\cdot\min\cdot\mathrm{P} = \mathrm{P}$,
6. $\mathrm{Sig}\cdot\min\cdot\mathrm{P} = \mathrm{P}$,
7. $\mathrm{C}\cdot\min\cdot\mathrm{P} = \mathrm{coNP}$,
8. $\oplus\cdot\min\cdot\mathrm{P} = \Delta^p_2$.

Corollary 4.3
1. $\mathrm{U}\cdot\max\cdot\mathrm{NP} = \mathrm{NP}$,
2. $\mathrm{Sig}\cdot\max\cdot\mathrm{NP} = \mathrm{NP}$,
3. $\mathrm{C}\cdot\max\cdot\mathrm{NP} = \mathrm{NP}$,
4. $\oplus\cdot\max\cdot\mathrm{NP} = \Delta^p_2$,
5. $\mathrm{U}\cdot\min\cdot\mathrm{NP} = \mathrm{coNP}$,
6. $\mathrm{Sig}\cdot\min\cdot\mathrm{NP} = \mathrm{coNP}$,
7. $\mathrm{C}\cdot\min\cdot\mathrm{NP} = \mathrm{coNP}$,
8. $\oplus\cdot\min\cdot\mathrm{NP} = \Delta^p_2$.

The results (1.), (3.), (4.),(5.), (7.) and (8.) of Corollary 4.3 were previously known and mentioned in a series of papers [Kre88, Köb89, Wag87, GKR95, VW95].

Corollary 4.4
1. $\mathrm{U}\cdot\max\cdot\mathrm{coNP} = \mathrm{coNP}$,
2. $\mathrm{Sig}\cdot\max\cdot\mathrm{coNP} = \Sigma^p_2$,
3. $\mathrm{C}\cdot\max\cdot\mathrm{coNP} = \Sigma^p_2$,
4. $\oplus\cdot\max\cdot\mathrm{coNP} = \Delta^p_3$,
5. $\mathrm{U}\cdot\min\cdot\mathrm{coNP} = \mathrm{NP}$,
6. $\mathrm{Sig}\cdot\min\cdot\mathrm{coNP} = \mathrm{NP}$,
7. $\mathrm{C}\cdot\min\cdot\mathrm{coNP} = \Pi^p_2$,
8. $\oplus\cdot\min\cdot\mathrm{coNP} = \Delta^p_3$.

As all the proofs of this section relativize, the results of the corollaries also do.

Note that according to Theorem 4.3 and 4.6 we can verify the results of Ogiwara [Ogi91], namely,

1. $\mathrm{P}^{\mathrm{C}_=\mathrm{P}} = \oplus\cdot\min\cdot\mathrm{coC}_=\mathrm{P}$,
2. $\mathrm{NP}^{\mathrm{C}_=\mathrm{P}} = \mathrm{C}\cdot\max\cdot\mathrm{C}_=\mathrm{P}$,

3. $\mathrm{P}^{\mathrm{NP}^{\mathrm{C_=P}}} = \oplus \cdot \max \cdot \mathrm{C_=P}$.

Similarly we can now characterize all levels of the polynomial hierarchy relative to every class (having the closure properties mentioned in the theorems). Especially we are now able to characterize the polynomial hierarchy itself by three combined operators, namely $\mathrm{C} \cdot \max$, $\mathrm{C} \cdot \min$ and $\oplus \cdot \max$.

Corollary 4.5

$$\begin{aligned} \Sigma_0^p &= \Pi_0^p = \Delta_0^p = \mathrm{P}, \\ \Sigma_{i+1}^p &= \mathrm{C} \cdot \max \cdot \Delta_i^p, \\ \Pi_{i+1}^p &= \mathrm{C} \cdot \min \cdot \Delta_i^p, \\ \Delta_{i+1}^p &= \oplus \cdot \max \cdot \Delta_i^p, \\ \mathrm{PH} &= \Gamma_{\{\mathrm{C}\cdot\max, \mathrm{C}\cdot\min, \oplus\cdot\max\}}(\mathrm{P}). \end{aligned}$$

5 Structural Consequences

As already noted in [Vol94] the operator theoretical approach in order to provide evidence that two function (or complexity) classes are incomparable is very powerful and elegant. In the next two sections we will completely analyze the inclusion relations of the main function classes which are the min-max classes, the # classes and the $\mathrm{F}\Delta_i^p$ classes. The results of this section, that are the ones not including the # classes are presented in Figure 2.

To illustrate the method let us take a look at the following questions: Is $\max \cdot \mathrm{P} \subseteq \min \cdot \mathrm{P}$? Suppose $\max \cdot \mathrm{P} \subseteq \min \cdot \mathrm{P}$. Then by Corollary 4.2 we can immediately conclude $\mathrm{NP} \subseteq \mathrm{P}$, since the operator Sig is monotone. Hence we can claim

$$\max \cdot \mathrm{P} \subseteq \min \cdot \mathrm{P} \Longrightarrow \mathrm{P} = \mathrm{NP}.$$

Thus proving $\max \cdot \mathrm{P} \subseteq \min \cdot \mathrm{P}$ is at least as hard as proving $\mathrm{P} = \mathrm{NP}$. In the next theorem we will see that $\max \cdot \mathrm{P} \subseteq \min \cdot \mathrm{P}$ is even equivalent to $\mathrm{P} = \mathrm{NP}$.

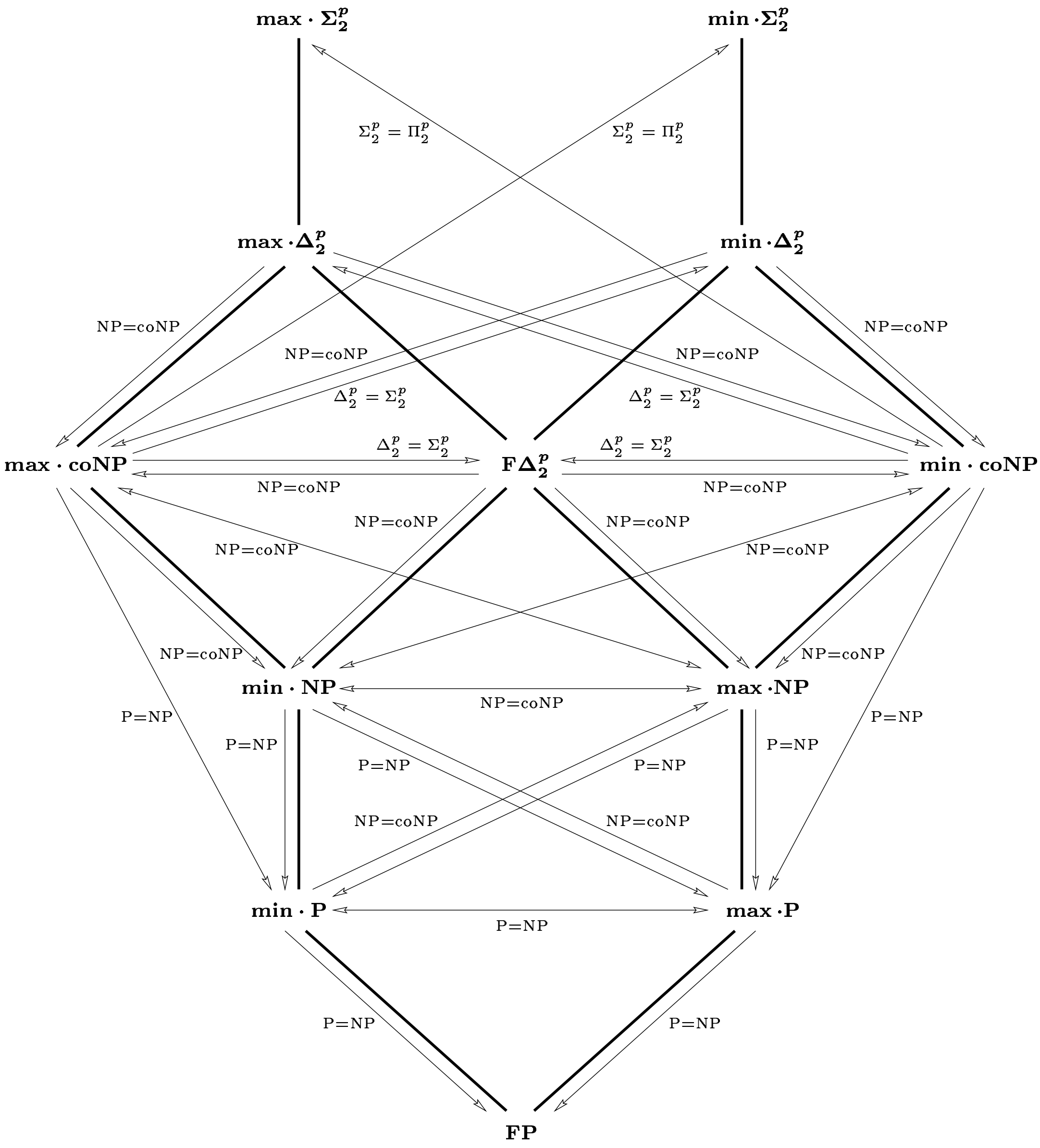
Before we state the main theorems of this section recall that for every $i, j \in \mathbb{N}$ we have,

$$\Delta_i^p \subseteq \Delta_j^p \iff \mathrm{F}\Delta_i^p \subseteq \mathrm{F}\Delta_j^p.$$

The following theorems show how closely the min-max classes and the polynomial hierarchy are related.

Theorem 5.1 *The following statements are equivalent:*

1. $\mathrm{FP} = \max \cdot \mathrm{P} = \min \cdot \mathrm{P} = \max \cdot \mathrm{NP} = \min \cdot \mathrm{NP} = \max \cdot \mathrm{coNP} = \min \cdot \mathrm{coNP} = \mathrm{F}\Delta_2^p$,
2. $\max \cdot \mathrm{P} \subseteq \mathrm{FP}$,
3. $\min \cdot \mathrm{P} \subseteq \mathrm{FP}$,
4. $\max \cdot \mathrm{P} = \min \cdot \mathrm{P}$,
5. $\max \cdot \mathrm{NP} \subseteq \max \cdot \mathrm{P}$,
6. $\min \cdot \mathrm{NP} \subseteq \min \cdot \mathrm{P}$,
7. $\max \cdot \mathrm{coNP} \subseteq \max \cdot \mathrm{P}$,
8. $\min \cdot \mathrm{coNP} \subseteq \min \cdot \mathrm{P}$,



Key: a bold line indicates an inclusion of the lower in the upper class

$\mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2$ means : $\mathcal{F}_1 \subseteq \mathcal{F}_2 \iff \alpha$

$\mathcal{F}_1 \xleftrightarrow{\alpha} \mathcal{F}_2$ means : $(\mathcal{F}_1 \subseteq \mathcal{F}_2 \iff \alpha) \wedge (\mathcal{F}_1 \supseteq \mathcal{F}_2 \iff \alpha)$

Figure 2: Structural consequences I

9. $\max \cdot \mathrm{NP} \subseteq \min \cdot \mathrm{P}$,

10. $\min \cdot \mathrm{NP} \subseteq \max \cdot \mathrm{P}$,

11. $\max \cdot \mathrm{coNP} \subseteq \min \cdot \mathrm{P}$,

12. $\min \cdot \mathrm{coNP} \subseteq \max \cdot \mathrm{P}$,

13. $\mathrm{P} = \mathrm{NP}$.

Proof: In the proof we want to show that (1.) implies (2.) $\cdots$ (12.), that any of (2.) $\cdots$ (12.) implies (13.) and that (13.) implies (1.) to complete the circular argument. Recall the results of Proposition 4.1 and the Corollaries 4.2 to 4.4.

Note that from (1.) the statements (2.) $\cdots$ (12.) follow directly.

By applying the operator C to (2.) or (3.) we conclude (13.). Similarly we can conclude (13.) from (4.) by applying the operator Sig and any of (5.), $\cdots$, (12.) by applying the operator U.

In order to complete the proof of our theorem it remains to show that (13.) implies (1.). Suppose $\mathrm{P} = \mathrm{NP}$. In this case the polynomial hierarchy collapses to P. Hence $\Delta_3^p = \mathrm{P}$ and equivalently $\mathrm{F}\Delta_3^p = \mathrm{FP}$. We conclude (1.) since all the min-max classes considered here are subsets of $\mathrm{F}\Delta_3^p$ (see Proposition 3.1 and Lemma 3.1). ∎

Theorem 5.2 *The following statements are equivalent:*

1. $\max \cdot \mathrm{NP} = \min \cdot \mathrm{NP} = \max \cdot \mathrm{coNP} = \min \cdot \mathrm{coNP} = \mathrm{F}\Delta_2^p$,

2. $\max \cdot \mathrm{P} \subseteq \min \cdot \mathrm{NP}$,

3. $\min \cdot \mathrm{P} \subseteq \max \cdot \mathrm{NP}$,

4. $\max \cdot \mathrm{NP} = \min \cdot \mathrm{NP}$,

5. $\max \cdot \mathrm{coNP} = \min \cdot \mathrm{coNP}$,

6. $\max \cdot \mathrm{NP} = \max \cdot \mathrm{coNP}$,

7. $\min \cdot \mathrm{NP} = \min \cdot \mathrm{coNP}$,

8. $\max \cdot \mathrm{coNP} \subseteq \min \cdot \mathrm{NP}$,

9. $\min \cdot \mathrm{coNP} \subseteq \max \cdot \mathrm{NP}$,

10. $\mathrm{F}\Delta_2^p \subseteq \max \cdot \mathrm{NP}$,

11. $\mathrm{F}\Delta_2^p \subseteq \min \cdot \mathrm{NP}$,

12. $\mathrm{F}\Delta_2^p \subseteq \max \cdot \mathrm{coNP}$,

13. $\mathrm{F}\Delta_2^p \subseteq \min \cdot \mathrm{coNP}$,

14. $\max \cdot \Delta_2^p \subseteq \max \cdot \mathrm{coNP}$,

15. $\min \cdot \Delta_2^p \subseteq \min \cdot \mathrm{coNP}$,

16. $\max \cdot \Delta_2^p \subseteq \min \cdot \mathrm{coNP}$,

17. $\min\cdot\Delta_2^p \subseteq \max\cdot \mathrm{coNP}$,

18. $\mathrm{NP} = \mathrm{coNP}$.

Proof: Recall the results of Proposition 4.1 and the Corollaries 4.2 to 4.4. We will show that (1.) implies any of (2.) to (17.), that any of (2.) to (17.) implies (18.) and that (18.) implies (1.).

(1.) obviously implies the statements from (2.) to (13.) by the inclusion relations of the considered classes (see Proposition 3.1, Lemma 3.1 or Figure 1).

Note that we can conclude $\mathrm{NP} = \Delta_2^p$ from (1.) by applying operator U and due to the monotonicity of the operators max and min we further conclude statements (14.) and (15.). Since (1.) implies also (5.) we have (16.) and (17.).

By applying the operator C to (2.), (3.), (8.) and (9.) we get statement (18.). Similarly we can conclude (18.) from any of (4.) to (7.) or any of (10.) to (17.) by applying operator U.

Now lets show that (18.) implies (1.). Assume NP = coNP and thus PH = NP. In this case we conclude $\max\cdot\mathrm{NP} = \max\cdot\mathrm{coNP} = \max\cdot\Delta_2^p$ and $\min\cdot\mathrm{NP} = \min\cdot\mathrm{coNP}$ due to the monotonicity of max·. We can further conclude $\max\cdot\mathrm{NP} = \min\cdot\mathrm{NP}$ from $\max\cdot\mathrm{NP} \subseteq \min\cdot\mathrm{coNP}$ and $\min\cdot\mathrm{NP} \subseteq \max\cdot\mathrm{coNP}$. Also we know $\mathrm{F}\Delta_2^p \subseteq \max\cdot\Delta_2^p$ and $\max\cdot\mathrm{NP} \subseteq \mathrm{F}\Delta_2^p$ by Proposition 3.1. Combining all these results we have (1.).

This completes the proof of our theorem. ∎

The equivalence of (4.) and (17.) can already be found in Köbler [Köb89]. Furthermore note that $\max_{\mathrm{pol}}\cdot\mathrm{NP} = \min_{\mathrm{pol}}\cdot\mathrm{NP}$ is also equivalent to NP = coNP due to the monotonicity of the operator U.

In Figure 3 we present the inclusion structure of the considered function classes for the case that NP = coNP. Despite the fact that we have not yet discussed the effect of NP = coNP on the inclusion relations with respect to the # classes we will see in Section 6 that the picture given now can not be improved.

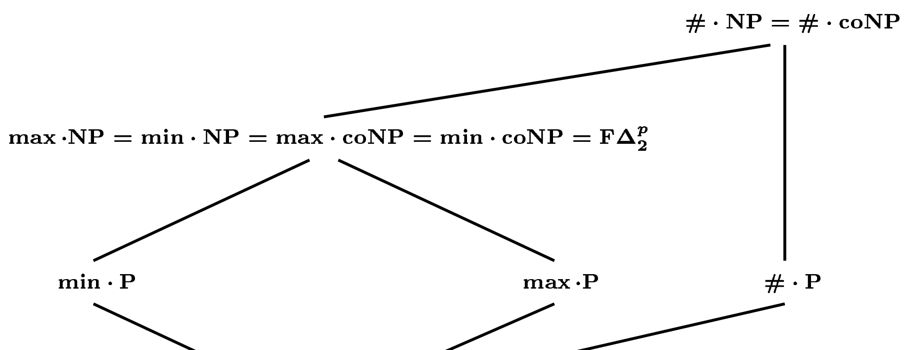


Figure 3: Inclusion structure if NP = coNP

Recall the known inclusion relations (see Figure 1). By the preceding theorem we know that $\mathrm{F}\Delta_2^p$ is contained in $\max\cdot\mathrm{coNP}$ if and only if PH collapses to its first level. What about the inverse inclusion? The answer is given in the next theorem.

Theorem 5.3 *The following statements are equivalent:*

1. $\max \cdot \text{coNP} \subseteq \text{F}\Delta_2^p$,
2. $\min \cdot \text{coNP} \subseteq \text{F}\Delta_2^p$,
3. $\max \cdot \text{coNP} \subseteq \min \cdot\Delta_2^p$,
4. $\min \cdot \text{coNP} \subseteq \max \cdot\Delta_2^p$,
5. $\Delta_2^p = \Sigma_2^p$.

Proof: We conclude (5.) from both (1.) and (2.) by applying the operator C. Similarly we conclude (5.) from (3.) by applying operator Sig. Furthermore, (3.) and (4.) are equivalent according to Theorem 3.1.

For the inverse implications suppose $\Delta_2^p = \Sigma_2^p$. This implies $\Delta_3^p = \Delta_2^p$ and hence $\text{F}\Delta_3^p = \text{F}\Delta_2^p$. But by the inclusion relations shown in Figure 1 we conclude $\min \cdot\Delta_2^p = \max \cdot\Delta_2^p = \text{F}\Delta_2^p$ and thus (1.), (2.), (3.) and (4.) hold. ∎

There are only a few still possible inclusions left for which we will give structural consequences in the last theorem of this section.

Theorem 5.4 *The following statements are equivalent:*

1. $\max \cdot \text{coNP} \subseteq \min \cdot\Sigma_2^p$,
2. $\min \cdot \text{coNP} \subseteq \max \cdot\Sigma_2^p$,
3. $\Sigma_2^p = \Pi_2^p$.

Proof: (1.) and (2.) are equivalent according to Theorem 3.1. We conclude (3.) from (1.) by applying the operator C. For the implication (3.)$\Longrightarrow$(1.) suppose $\Sigma_2^p = \Pi_2^p$. Recall that $\max \cdot \text{coNP} \subseteq \max \cdot\Sigma_2^p$ by the monotonicity of max and $\max \cdot\Sigma_2^p \subseteq \min \cdot\Pi_2^p$ by Lemma 3.1. Thus we get $\max \cdot \text{coNP} \subseteq \min \cdot\Sigma_2^p$. ∎

The theorems of this section give evidence that we either have already proven an inclusion between various min-max and $\text{F}\Delta_i^p$ classes in Section 3, or there is none.

6 Structural Consequences with respect to the # Classes

Recall from [KST89] that $\max \cdot\text{NP} \subseteq \# \cdot \text{NP}$. One might expect that similarly $\max \cdot\text{P} \subseteq \# \cdot \text{P}$ or, since $\min \cdot \text{coNP}$ is somehow closely related to $\max \cdot\text{NP}$ (see Lemma 3.4), also $\min \cdot \text{coNP} \subseteq \# \cdot \text{NP}$. We will prove that these expectations are false under widely accepted structural assumptions. The results of the following theorems are presented in Figure 5.

Recall from [KST89] $\# \cdot \text{coNP} = \# \cdot\Delta_2^p$. Regarding our task of showing that various min-max classes and # classes are incomparable only a few results were previously known. Statements (1.), (2.) and (5.) of the following theorem were proven in [KST89]. (5.) can be found together with (3.), (4.),(6.), (7.) and (8.) also in [Köb89]. A generalization of the fifth and sixth claim can be found in [Vol94].

Theorem 6.1 *1.* $\# \cdot \text{P} = \# \cdot \text{NP} \iff \text{UP} = \text{NP}$,

2. $\# \cdot \mathrm{NP} = \# \cdot \mathrm{coNP} \iff \mathrm{NP} = \mathrm{coNP}$,

3. $\max \cdot \mathrm{NP} \subseteq \# \cdot \mathrm{P} \iff \mathrm{NP} = \mathrm{UP}$,

4. $\min \cdot \mathrm{NP} \subseteq \# \cdot \mathrm{P} \iff \mathrm{NP} = \mathrm{coNP} = \mathrm{UP}$,

5. $\# \cdot \mathrm{P} \subseteq \max \cdot \mathrm{NP} \iff \mathrm{NP} = \mathrm{PP}$,

6. $\# \cdot \mathrm{P} \subseteq \min \cdot \mathrm{NP} \iff \mathrm{NP} = \mathrm{PP}$,

7. $\min \cdot \mathrm{NP} \subseteq \# \cdot \mathrm{NP} \iff \mathrm{NP} = \mathrm{coNP}$,

8. $\# \cdot \mathrm{NP} \subseteq \min \cdot \mathrm{NP} \vee \# \cdot \mathrm{NP} \subseteq \max \cdot \mathrm{NP} \Longrightarrow \mathrm{NP} = \mathrm{coNP}$.

It is interesting to note that NP = PP is equivalent to a seemingly stronger statement, namely using results of Toda [Tod91] one can prove:

Lemma 6.1 $\mathrm{NP} = \mathrm{PP} \iff \mathrm{NP} = \mathrm{PP}^{\mathrm{NP}}$.

Proof: The implication from right to left is obvious. For the other implication suppose NP = PP. According to [Tod91] we have $\mathrm{PP}^{\mathrm{PH}} \subseteq \mathrm{P}^{\mathrm{PP}}$. Furthermore we conclude PH = NP = coNP due to NP = PP. Thus $\mathrm{PP}^{\mathrm{PH}} \subseteq \mathrm{P}^{\mathrm{PP}} = \mathrm{P}^{\mathrm{NP}} = \mathrm{NP}$. ∎

It is clear that we have also $\mathrm{P} = \mathrm{PP} \iff \mathrm{P} = \mathrm{PP}^{\mathrm{NP}}$ and similar statements when replacing P and NP by Σ_i^p.

We will now turn to the actual task of this section, namely show that various min-max classes are incomparable with the # classes under reasonable structural assumptions. The first theorem shows that the containment of $\# \cdot \mathrm{coNP}$, $\# \cdot \mathrm{NP}$ or $\# \cdot \mathrm{P}$ in $\max \cdot \mathrm{P}$ or $\min \cdot \mathrm{P}$ is equivalent to the collapse of the counting hierarchy to P.

Theorem 6.2 *The following statements are equivalent:*

1. $\mathrm{FP} = \max \cdot \mathrm{P} = \min \cdot \mathrm{P} = \max \cdot \mathrm{NP} = \min \cdot \mathrm{NP} = \max \cdot \mathrm{coNP} = \min \cdot \mathrm{coNP} = \# \cdot \mathrm{P} = \# \cdot \mathrm{NP} = \# \cdot \mathrm{coNP} = \mathrm{F}\Delta_2^p$,

2. $\# \cdot \mathrm{P} \subseteq \mathrm{FP}$,

3. $\# \cdot \mathrm{P} \subseteq \max \cdot \mathrm{P}$,

4. $\# \cdot \mathrm{P} \subseteq \min \cdot \mathrm{P}$,

5. $\# \cdot \mathrm{NP} \subseteq \max \cdot \mathrm{P}$,

6. $\# \cdot \mathrm{NP} \subseteq \min \cdot \mathrm{P}$,

7. $\# \cdot \mathrm{coNP} \subseteq \max \cdot \mathrm{P}$,

8. $\# \cdot \mathrm{coNP} \subseteq \min \cdot \mathrm{P}$,

9. $\mathrm{P} = \mathrm{NP} = \mathrm{PP}$.

Proof: We want to prove our theorem showing that (1.) implies (2.)$\cdots$ (8.), that any of (2.)$\cdots$(8.) implies (9.) and that (9.) implies (1.).

Note that from (1.) the statements (2.)$\cdots$(8.) follow directly.

The implication (2.)$\Longrightarrow$(9.) follows from an application of C. Furthermore (3.) and (4.) are equivalent due to Theorem 3.1. We conclude $\mathrm{NP} \subseteq \mathrm{P}$ and $\mathrm{PP} \subseteq \mathrm{coNP}$ from (4.) by applying the operators Sig and C, respectively. Thus (3.)$\Longrightarrow$(9.) and (4.)$\Longrightarrow$(9.).

We can conclude (9.) from any of (5.)$\cdots$(8.) by combining the results when applying the operators C and U and using the fact that $\mathrm{P} = \mathrm{PP} \iff \mathrm{P} = \mathrm{PP}^{\mathrm{NP}}$.

For the implication (9.)$\Longrightarrow$(1.) suppose $\mathrm{P} = \mathrm{NP} = \mathrm{PP}$. Then we have $\mathrm{FP} = \max\cdot\mathrm{P} = \min\cdot\mathrm{P} = \max\cdot\mathrm{NP} = \min\cdot\mathrm{NP} = \max\cdot\mathrm{coNP} = \min\cdot\mathrm{coNP} = \mathrm{F}\Delta_2^p$ and $\#\cdot\mathrm{P} = \#\cdot\mathrm{NP} = \#\cdot\mathrm{coNP}$ according to Theorem 5.1 and 6.1, respectively. Furthermore we can make use of the fifth claim of Theorem 6.1 which yields $\#\cdot\mathrm{P} \subseteq \max\cdot\mathrm{NP} \iff \mathrm{NP} = \mathrm{coNP}$. Combining these results with the known inclusion $\max\cdot\mathrm{NP} \subseteq \#\cdot\mathrm{NP}$ we get (1.).

This completes the proof. ∎

In the next theorem we will strengthen the sixth claim of Theorem 6.1 and show that a # class is contained in $\max\cdot\mathrm{NP}$ or $\min\cdot\mathrm{NP}$ if and only if $\mathrm{NP} = \mathrm{PP}$ and thus the polynomial hierarchy collapses to its first level. We want to remind the reader that $\#\cdot\mathrm{P} \subseteq \max\cdot\mathrm{NP} \iff \mathrm{NP} = \mathrm{PP}$ and $\#\cdot\mathrm{P} \subseteq \min\cdot\mathrm{NP} \iff \mathrm{NP} = \mathrm{PP}$ were already shown in [Köb89].

Theorem 6.3 *The following statements are equivalent:*

1. $\max\cdot\mathrm{NP} = \min\cdot\mathrm{NP} = \max\cdot\mathrm{coNP} = \min\cdot\mathrm{coNP} = \#\cdot\mathrm{NP} = \#\cdot\mathrm{coNP} = \mathrm{F}\Delta_2^p$,
2. $\#\cdot\mathrm{NP} \subseteq \max\cdot\mathrm{NP}$,
3. $\#\cdot\mathrm{NP} \subseteq \min\cdot\mathrm{NP}$,
4. $\#\cdot\mathrm{coNP} \subseteq \max\cdot\mathrm{NP}$,
5. $\#\cdot\mathrm{coNP} \subseteq \min\cdot\mathrm{NP}$,
6. $\#\cdot\mathrm{NP} \subseteq \max\cdot\mathrm{coNP}$,
7. $\#\cdot\mathrm{NP} \subseteq \min\cdot\mathrm{coNP}$,
8. $\#\cdot\mathrm{coNP} \subseteq \max\cdot\mathrm{coNP}$,
9. $\#\cdot\mathrm{coNP} \subseteq \min\cdot\mathrm{coNP}$,
10. $\mathrm{NP} = \mathrm{PP}$.

Proof: In the proof we want to show that (1.) implies (2.) $\cdots$ (9.), that any of (2.) $\cdots$ (9.) implies (10.) and that (10.) implies (1.).

Note that from (1.) the statements (2.) $\cdots$ (9.) follow directly.

We get (10.) from (2.), (3.), (4.) and (5.) by applying operator C and using the equivalence given in Lemma 6.1. By applying the operators U and C to (6.) we conclude $\mathrm{NP} = \mathrm{coNP}$ and $\Sigma_2^p = \mathrm{PP}^{\mathrm{NP}}$, respectively. Thus (6.)$\Longrightarrow$(10.).

Note that (7.) and (8.) are equivalent according to Theorem 3.1. The implication (8.)$\Longrightarrow$(10.) can be seen as follows. Suppose $\#\cdot\mathrm{coNP} \subseteq \max\cdot\mathrm{coNP}$. We conclude $\mathrm{PP}^{\mathrm{NP}} \subseteq \Sigma_2^p$ and $\mathrm{UP}^{\mathrm{NP}} \subseteq \mathrm{coNP}$ by applying the operators C and U, respectively. Combining the two results with the equivalence given in Lemma 6.1 we get (10.).

Similarly one can show (9.)$\Longrightarrow$(10.).

It remains to show that (10.) implies (1.). Suppose NP = PP, thus PH = NP = coNP and also NP = $\mathrm{PP}^{\mathrm{NP}}$ according to Lemma 6.1. We conclude max$\cdot$NP = min $\cdot$ NP = max $\cdot$ coNP = min $\cdot$ coNP = $\mathrm{F}\Delta_2^p$ and # $\cdot$ NP = # $\cdot$ coNP according to Theorem 5.2 and 6.1, respectively. Since we have max$\cdot$NP $\subseteq$ # $\cdot$ NP it remains to show that # $\cdot$ NP $\subseteq$ max$\cdot$NP.

Let $f \in \# \cdot \mathrm{NP}$. Then there exists a $B \in \mathrm{NP}$ such that for some polynomial p it holds that $f(x) = ||\{y : 0 \leq y < 2^{p(|x|)} \wedge \langle x, y\rangle \in B\}||$.

$$C = \{\langle x, i\rangle : \text{ There exist at least } i \text{ distinct } y \text{ such that } \langle x, y\rangle \in B\}$$

and note that $C \in \mathrm{PP}^{\mathrm{NP}}$ and due to $\mathrm{NP} = \mathrm{PP}^{\mathrm{NP}}$ even $C \in \mathrm{NP}$.

Furthermore $f(x) = max\{i : 0 \leq i \leq 2^{p(|x|)} \wedge \langle x, i\rangle \in C\}$. This proves $f \in$ max$\cdot$NP and completes the proof of (10.)$\Longrightarrow$(1.). ∎

The last theorem shows that in the case of NP = PP only a few function classes remain, namely FP, max$\cdot$P, min $\cdot$ P, # $\cdot$ P, and max$\cdot$NP = min $\cdot$ NP = # $\cdot$ NP = $\mathrm{F}\Delta_2^p$. Their inclusion relations are shown in Figure 4.

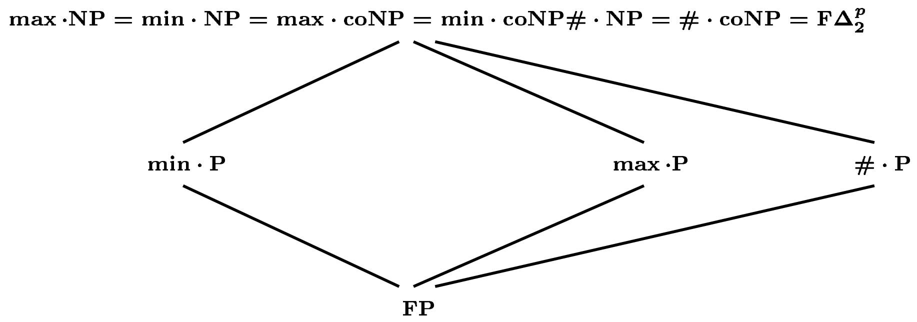


Figure 4: Inclusion structure if NP = PP

Note that we have # $\cdot$ P $\subseteq$ max $\cdot$ coNP $\iff$ # $\cdot$ P $\subseteq$ min $\cdot$ coNP due to Theorem 3.1. Until now we are unable to present structural equivalences for the case that # $\cdot$ P $\subseteq$ max $\cdot$ coNP, but we give a structural consequence which implies the collapse of the polynomial hierarchy to its second level.

Theorem 6.4 *1.* # $\cdot$ P $\subseteq$ max $\cdot$ coNP $\Longrightarrow$ PP $\subseteq \Sigma_2^p$,

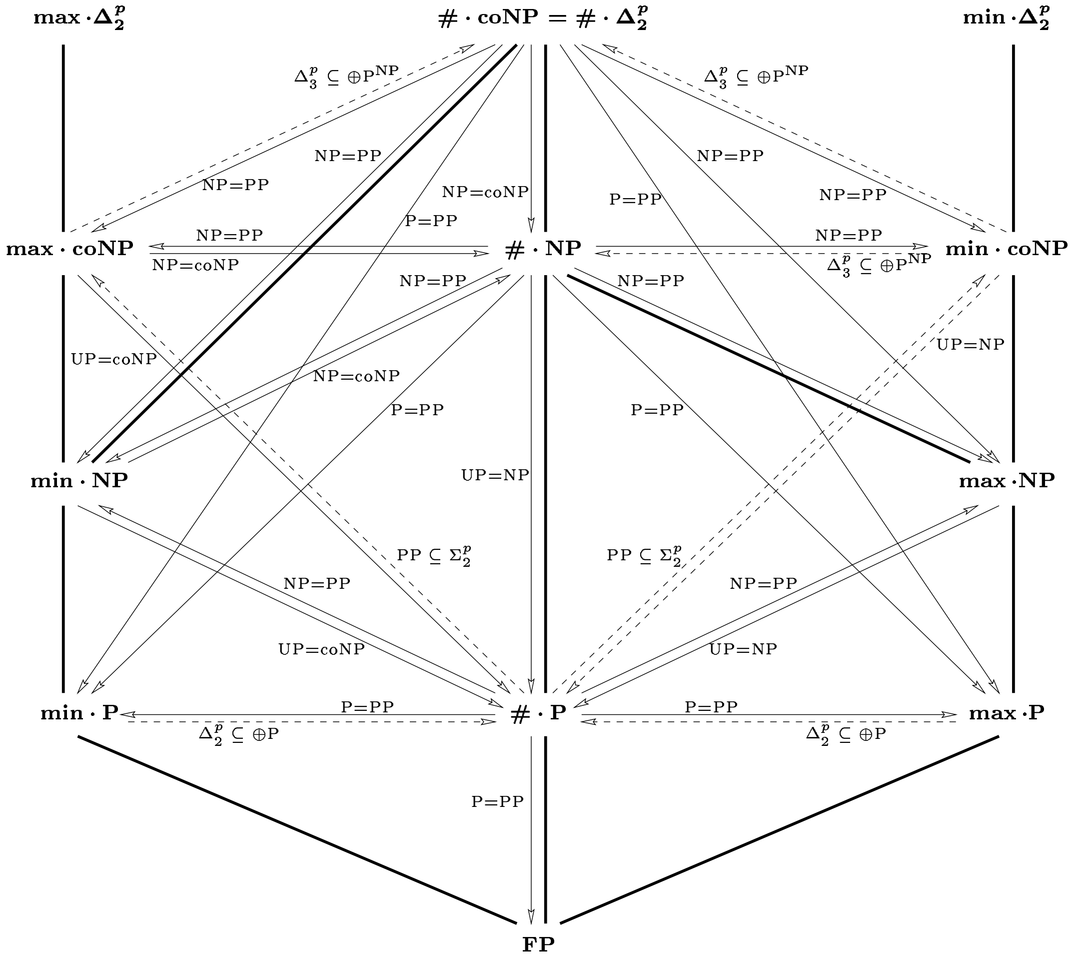
2. # $\cdot$ P $\subseteq$ min $\cdot$ coNP $\Longrightarrow$ PP $\subseteq \Sigma_2^p$.

The two claims follow directly from an application of the operator C and we can conclude $\Sigma_2^p = \Pi_2^p$ according to [Tod91]. Note that we can draw also the conclusions UP $\subseteq$ coNP and $\oplus$P $\subseteq \Delta_3^p$ from # $\cdot$ P $\subseteq$ max $\cdot$ coNP by applying the operators U and $\oplus$, respectively.

In the remainder of this section we will show that except the inclusions already shown, no min-max class is contained in any # class. Recall the third and fourth claim of Theorem 6.1 which show that max$\cdot$NP and min $\cdot$ NP can not be contained in # $\cdot$ P unless UP = coNP and UP = NP, respectively.

Theorem 6.5 *1.* max $\cdot$ coNP $\subseteq$ # $\cdot$ P $\iff$ NP = coNP = UP,

2. max $\cdot$ coNP $\subseteq$ # $\cdot$ NP $\iff$ NP = coNP.



Key: a bold line indicates an inclusion of the lower in the upper class

$\mathcal{F}_1 \overset{\alpha}{\dashrightarrow} \mathcal{F}_2$ means : $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Longrightarrow \alpha$

$\mathcal{F}_1 \overset{\alpha}{\longrightarrow} \mathcal{F}_2$ means : $\mathcal{F}_1 \subseteq \mathcal{F}_2 \iff \alpha$

Figure 5: Structural consequences II

Proof: (1.) The implication from left to right can be seen as follows. From $\max \cdot \text{coNP} \subseteq \# \cdot \text{P}$ we conclude $\min \cdot \text{NP} \subseteq \# \cdot \text{P}$, since $\min \cdot \text{NP} \subseteq \max \cdot \text{coNP}$. By the result of Theorem 6.1 we further conclude NP = coNP = UP.

For the inverse implication let NP = coNP = UP. We conclude $\max \cdot \text{coNP} = \min \cdot \text{coNP} = \min \cdot \text{NP}$ by Theorem 5.2. Furthermore we have $\min \cdot \text{NP} \subseteq \# \cdot \text{P}$ by Theorem 6.1. This completes the proof of Claim (1.).

(2.) The implication from left to right follows from the monotonicity of the operator U. For the inverse implication suppose NP = coNP. We conclude $\max \cdot \text{coNP} = \max \cdot \text{NP}$. But $\max \cdot \text{NP} \subseteq \# \cdot \text{NP}$ and thus $\max \cdot \text{coNP} \subseteq \# \cdot \text{NP}$ ∎

Theorem 6.6 *1.* $\max \cdot \text{P} \subseteq \# \cdot \text{P} \Longrightarrow \Delta_2^p \subseteq \oplus \text{P}$,

2. $\min \cdot \text{P} \subseteq \# \cdot \text{P} \Longrightarrow \Delta_2^p \subseteq \oplus \text{P}$,

3. $\min \cdot \text{coNP} \subseteq \# \cdot \text{P} \Longrightarrow \Delta_3^p \subseteq \oplus \text{P}$,

4. $\min \cdot \text{coNP} \subseteq \# \cdot \text{NP} \Longrightarrow \Delta_3^p \subseteq \oplus \text{P}^{\text{NP}}$,

5. $\max \cdot \text{coNP} \subseteq \# \cdot \text{coNP} \Longrightarrow \Delta_3^p = \oplus \text{P}^{\text{NP}}$,

6. $\min \cdot \text{coNP} \subseteq \# \cdot \text{coNP} \Longrightarrow \Delta_3^p = \oplus \text{P}^{\text{NP}}$.

Proof: All claims follow from an application of the operator $\oplus$. ∎

Note that from $\min \cdot \text{coNP} \subseteq \# \cdot \text{P}$ we can conclude also NP = UP by applying operator U.

7 Conclusions

We completely analyzed the inclusion relations among the min-max classes and the other central function classes. We showed that the inclusion structure of these function classes is closely related to the inclusion structure of central complexity classes, such as P, NP and PP. By defining and investigating the behavior of several operators, which map function classes to complexity classes, we were able to characterize known complexity classes. It turned out, that though $\max \cdot \text{NP}$ and $\min \cdot \text{NP}$ remain central classes of optimization functions, there are other interesting classes of optimization functions. In contrast to the the operator # where we have $\# \cdot \text{NP} \subseteq \# \cdot \text{coNP}$ (see [KST89]) the operator max displays a different behavior, namely $\max \cdot \text{NP}$ and $\max \cdot \text{coNP}$ are incomparable, unless NP = coNP.

As already noted in Section 3 we would like to have a structural consequence if $\min \cdot \text{P} \subseteq \# \cdot \text{NP}$.

Furthermore we would like to find structural equivalences for all inclusions listed in the Theorems 6.4 and 6.6.

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