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Comment 01 on ECCC TR97-043

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Some of the results of the paper have been based on the 'Triangle Conjecture'. Here we show that this conjecture, in its general form, is false, by describing the construction of an $n \times n$ matrix with ones on the main diagonal, with rank of the order of $n^{2/3}$, and such that the graph obtained by associating edges to nonzero entries of the matrix does not have 'transitive triangles'. As we will see, the construction still leaves open a number of possibilities to prove lower bounds on rigidity and/or on the size of small depth circuits. In fact we still hope to use variations of our techniques to base proofs of high rigidity, on the presence (in all small, but still linear, rank matrices) of other structures. As a byproduct of our construction, we find another constructive bound on the Ramsey number $R(3, n)$ which is far simpler than the construction obtained by Alon in the paper "Explicit Ramsey graphs and orthonormal labelings" (Electronic Journal of Combinatorics 1 (1994), R12).

In what follows we present our construction.

Let us consider a graph G with no cycles of length less than 6, and with a nonlinear number of edges, e.g., the bipartite graph given by a projective plane. Note that it is easy to construct such graphs with m vertices and $O(m^{3/2})$ edges.

Starting from G, we first describe an oriented graph H, and then a matrix which have nonzero entries corresponding to edges of H. For the sake of a simpler description, in the following we take G as the bipartite graph of a projective plane.

The vertices of the graph H will be pairs (P,L), where P corresponds to a point on a line L. The edges of H are given by pairs ((P,L),(P', L')), where P, P', L, L' are all different and $P \in L'$, i.e., the point P is incident to the line L'. It is straightforward to verify that there is no transitive triangle in H.

We now associate to H a matrix as follows. We index both rows and columns by pairs (P,L). To a row (P,L), we assign the vector whose coordinates are the vertices of G (that is, points and lines) and it has an entry equal to -1 on P and an entry equal to 1 on L, all the other entries being zero. To a column (P,L), we assign the characteristic vector of the set, i.e., all points of L except P and the vertex L. Thus the matrix obtained as the product of the vectors associated to its rows and those associated with its columns has 1's on the main diagonal, -1's on the entries corresponding to the edges of H, and 0 elsewhere. The rank of this matrix is m, the number of vertices of G, while its size is equal to the number of edges of G, which is of the order of $m^{3/2}$.

Notice that the above construction does not produce $\{0,1\}$ matrices, except for the field GF_2, thus leaving it possible that the Triangle Conjecture be true in this case.

It is an easy observation that the graph H might have oriented 3-cycles. This prevents from applying the construction to the symmetric case, where oriented 3-cycles 'become' triangles. However, to get a graph without oriented 3-cycles, one can start from a dense bipartite graph without cycles of length less than 8 (the density becomes smaller in this case, but still nonlinear, i.e., of the order of $n^{4/3}$). Then it is possible to proceed as above, and then symmetrize the construction by adding the matrix and its transpose, thus getting an $n \times n$ symmetric matrix still of rank $O(n^{3/4})$, and without triangles. However the above approach fails over GF_2, because symmetrization produces zeros on the main diagonal.

Using the fact that the rank (over any field) of a matrix is an upper bound on the size of the maximal independent set of a graph associated with the zero-nonzero pattern of the matrix, the above construction - after symmetrization - provides an explicit Ramsey graph. More precisely, it gives an n-vertex graph without triangles and with independent sets of size $O(n^{3/4})$. As it is, the bound is worse than Alon's bound. However, it is possible to work on the original construction, do another kind of symmetrization, and get the same asymptotic bound as Alon's, while significantly gaining in simplicity. The idea is to consider only the upper triangle of the matrix obtained from the graph H, and to copy it in the lower part of the matrix. In this way, we obtain a symmetric matrix without triangles, without having to start with a sparser graph G. On the other hand, we loose the 'low rank' property, whereas the construction clearly preserves the size of the maximal independent set.

As already mentioned at the beginning of this note, our approach has still the potential of offering lower bound techniques, either by looking for more general structures than triangles, such as transitive constant length odd cycles, or by looking at the symmetric case over GF_2 and at the non-symmetric case for $\{0,1\}$ matrices over fields different from GF_2.