Eq. 3, p. 4 of
Parallel Complexity of Integer Coprimality

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Eq. 3, p. 4 is correct as it stands (up to a constant factor.) Eq. 3, p. 4 is based on multiplying the two partial fraction expansions of

\[ \frac{1}{z^a - 1/\rho^a} \]

and

\[ \frac{1}{z^b - 1/\rho^b} \]

In general if \( Q \) is a polynomial in \( w \) with simple zeros, then

\[ \frac{1}{Q} = \sum_{\beta} \frac{1}{Q'(\beta) \cdot (w - \beta)} \]

where the sum is over the zeros \( \beta \) of \( Q \), and \( Q' \) is the derivative of \( Q \) w.r.t. \( w \).

In the case of \( \frac{1}{z^a - 1/\rho^a} \), we have \( Q = z^a - 1/\rho^a \), i.e., \( z \) plays the role of \( w \), and in case of \( \frac{1}{z^b - 1/\rho^b} \), \( Q = (1/z)^b - 1/\rho^b \), and \( 1/z \) plays the role of \( w \).

In the first case \( Q \) can be written as

\[ \prod_{j=0}^{a-1} (z - \omega^j / \rho) \]

where \( \omega \) is a primitive \( a \)-th root of unity. In the second case \( Q \) can be written as

\[ \prod_{k=0}^{b-1} (1/z - \nu^k / \rho) \]

where \( \nu \) is a primitive \( b \)-th root of unity.

In the first case \( Q' = a z^{a-1} \), so \( Q'(\omega^j / \rho) = a \omega^{-j} / \rho^{a-1} \). In the second case \( Q' = b (1/z)^{b-1} \) (remember that the variable is \( 1/z \)), so \( Q'(\nu^k / \rho) = b \nu^{-k} / \rho^{b-1} \). This yields

\[ \frac{1}{z^a - 1/\rho^a} = \sum_{j=0}^{a-1} \frac{\omega^j}{z - \omega^j / \rho} \]

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and
\[
\frac{1}{z^{-b} - 1/p^b} = \sum_{k=0}^{b-1} \frac{\nu^k}{1/z - \nu^k/p}
\]

In turn, Eq. 3, p.4 arises from
\[
\frac{1}{z^{-a} - 1/p^a} = \sum_{j=0}^{a-1} \frac{\omega^j}{\omega^j \cdot (z \omega^{-j} - 1/p)}
\]

and
\[
\frac{1}{z^{-b} - 1/p^b} = \sum_{k=0}^{b-1} \frac{\nu^k}{\nu^k \cdot (\nu^{-k} / z - 1/p)}
\]

Note that the root of unity factors cancel in the numerator and denominator of each term.