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# On Approximation Hardness of Dense TSP and Other Path Problems

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**Abstract**

TSP(1,2), the Traveling Salesman Problem with distances 1 and 2, is the problem of finding a tour of minimum length in a complete weighted graph where each edge has length 1 or 2. Let $d_o$ satisfy $0 < d_o < 1/2$. We show that TSP(1,2) has no PTAS on the set of instances where the density of the subgraph spanned by the edges with length 1 is bounded below by $d_o$. We also show that LONGEST PATH has no PTAS on the set of instances with density bounded below by $d_o$ for all $0 < d_o < 1/2$.

**Key words:** Approximation Schemes, Traveling Salesman Problem, Longest Path, Hamiltonian Cycle, Approximation Hardness, Dense Instances

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# 1 Introduction

In a recent breakthrough Arora [A96], [A97] developed PTASs (polynomial time approximation schemes) for the Euclidean TSP and the Steiner Tree Problems. There have been also several lower bound proven recently on the special cases of the TSP and the LONGEST PATH problems. Trevisan [T97] succeeded in proving that geometric TSP is Max-SNP-hard in $\mathbb{R}^{\log n}$ for every $\ell_p$ metric. Papadimitriou and Yannakakis [PY93] proved also that TSP(1,2), the Traveling Salesman Problem with distances one and two, is Max-SNP-hard. Using this result, Karger, Motwani and Ramkumar [KMR93] proved that LONGEST PATH is not constant factor approximable, unless P=NP, even for graphs with maximum degree 4. Moreover, they proved that LONGEST PATH has no PTAS on Hamiltonian graphs. Both results were improved by Bazgan, Santha and Tuza [BST98] who showed that LONGEST PATH is not constant factor approximable for cubic Hamiltonian graphs, unless P=NP.

On the other hand there has been a success recently in designing new PTASs for the dense instances of many NP-hard optimization problems, cf., e.g., [AKK95], [FV96], [GGR96], [FK97], and [K97]. The problems left aside in [AKK95] were the dense instances of the LONGEST PATH, HAMILTONIAN CYCLE, and the TSP.

The purpose of this note is to clarify the status of the above *dense* problems in proving that LONGEST PATH and TSP(1,2) are both Max-SNP-hard for "dense" instances. We define the density $d$ of a graph $G$ as the ratio $\delta(G)/|V(G)|$ where $\delta(G)$ is the minimum valency of $G$. We call TSP(1,2) *dense* if th subgraph spanned by the edges of length 1 is *dense*. We prove the following theorems

**Theorem 1** *Let $H$ be the graph spanned by the edges of length 1 in an instance $G$ of* TSP*(1,2) and let $d_o$ satisfy* $0 < d_o < 1/2$. *Then,* $\mathrm{TSP}(1,2)$ *is Max-SNP-hard when restricted to the instances in which the density of $H$ is* *at least* $d_o$

**Theorem 2** *Let* $d_o$ *satisfy* $0 < d_o < 1/2$*. Then,* LONGEST PATH *is Max-SNP-hard when restricted to instances with density at least* $d_o$

The next theorem is immediate from Theorem 2 and the fact, observed in [KMR93], that, for any set of instances, a PTAS for LONGEST PATH implies a PTAS for TSP(1,2) on the corresponding subset of Hamiltonian instances

**Theorem 3** *Let* $d_o$ *satisfy* $0 < d_o < 1/2$*. Then,* LONGEST PATH *has no PTAS when restricted to Hamiltonian instances with density at least* $d_o$

Before turning to the proofs of Theorems 1 and 2, let us remind the reader of the following theorem of Dirac.

**Dirac's Theorem** *A graph* $G$ *on* $n$ *vertices with minimum degree* $\delta(G) \geq \frac{n}{2}$ *is Hamiltonian.*

The proof of Dirac is completely constructive: it allows one to compute quickly an Hamiltonian cycle (see also [DHK93]) in any graph which satisfies to the condition of the theorem. In view of Dirac's theorem our theorems are the best possible in the sense that in none of them can we replace the upper bound for $d_o$ by any number greater than or equal to $1/2$.

## 2 The Proofs

We consider simple undirected graphs. The vertex set and the edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. For any $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ spanned by $X$. By a covering of a graph we mean a covering of the vertices of this graph by pairwise vertex-disjoint paths.

**Proof of Theorem 1** Let $G$ be an instance of TSP(1,2), i.e. $G$ is a complete graph where each edge has length 1 or 2. Let $H$ denote the subgraph of $G$ with $V(H) = V(G)$ and which contains only the edges of $G$ of length 1. Let

$\mathcal{C}$ denote a covering of $V(H)$ by disjoint paths. (The paths in $\mathcal{C}$ may contain just one vertex.). Let $e(\mathcal{C})$ denote the number of edges in $\mathcal{C}$. Clearly, we can always extend $\mathcal{C}$ to a tour with length

$$e(\mathcal{C}) + 2(n - e(\mathcal{C})) = 2n - e(\mathcal{C}).$$

Therefore, we can reformulate TSP$(1,2)$ as the problem of finding a covering of $V(H)$ containing the maximum number of edges of $H$. Fix $\epsilon > 0$ and split the vertex set of $H$ into three parts $X, Y$ and $Z$ with $|X| = \epsilon n$, $|Y| = |Z| = (1-\epsilon)n/2$. Assume that $Y$ is an independent set, that all the edges linking $X$ to $Y$ and $Y$ to $Z$ are present and that there are no edges between $X$ and $Z$. Otherwise, $H$ is arbitrary. Clearly, $H$ has density $(1-\epsilon)/2$.

Let $l^*(H)$ denote the maximum number of edges in a covering of $V(H)$. Similarly, let $\ell^*(H[X])$ denote the maximum number of edges in a covering of $X$ by paths in the subgraph $H[X]$ of $H$ spanned by $X$. We claim that we have

$$\ell^*(H[X]) + (1-\epsilon)n - 1 \leq \ell^*(H) \leq \ell^*(H[X]) + (1-\epsilon)n.$$

The left-side of this inequality is clear: Any covering of $X$ using $m$ edges, say, can be augmented into a covering of V(G) with $m + (1-\epsilon)n - 1$ edges since the subgraph spanned by the set of vertices $Y \cup Z$ is Hamiltonian.

For the other direction, let $Q$ be an optimal covering of $V(G)$. Then the set $Q \cap E(H[X])$ of the edges of $H[X]$ belonging to $Q$ is a partial covering of X and thus it contains at most $\ell^*(H[X])$ edges. Now, for any covering of $H$ there are at most 2 edges adjacent to any vertex. Since every edge not in $X$ is incident to a vertex in $Y$, it follows immediately that $Q$ contains at most $\ell^*(H[X]) + (1-\epsilon)n$ edges. The claim implies that in order to approximate $\ell^*(H)$ with an arbitrary small relative error, we must approximate $\ell^*(H[X])$ with a relative error which will also be arbitrary small. But this is not possible since unrestricted TSP$(1,2)$ is Max-SNP hard.

□

**Proof of Theorem 2** Let us show that a PTAS for LONGEST PATH (in any given class $\mathcal{G}$ of simple graphs) implies a PTAS for TSP(1,2) in the corresponding class of instances. Thus, for each fixed $\delta > 0$, assume that we can obtain in polynomial time for each graph $H$ in our class, a path $P$ of length at least $(1-\delta)n^*$ where $n^*$ is the length of the longest path of $H$. Let us write

$$w^* = n^* + \alpha$$

where $w^*$ denotes the optimum value of TSP$(1,2)$ for the instance $G$ obtained from $H$ by adding all the edges of $K_n \backslash E(H)$ with lengths equal to 2.. Now, by adding edges of length 2 to the path $P$, we can clearly obtain a tour with length $w \leq (1-\delta)n^* + 2\delta n^* + \alpha = (1+\delta)n^* + \alpha$. We have thus

$$\frac{w}{w^*} \leq \frac{(1+\delta)n^* + \alpha}{n^* + \alpha} \leq 1 + \delta.$$

Since $\delta$ is arbitrarily small, this implies a PTAS for TSP(1,2)in $\mathcal{G}$. This contradicts theorem 1 when $\mathcal{G}$ is the class of all graphs with density at least $d$.

□

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