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In this remark I am sticking to the notation of TR98-042; the reader of [Lok95, KP98] should substitute in what follows $A \mapsto H$, $C \mapsto A$ and $c \mapsto \theta$.

[Lok95, Theorem 2.1(ii)] proved a lower bound on the restricted rigidity function $R_A(r,c)$ for Hadamard matrices A, and [KP98, Theorem 2.6(b)] gave a numerical improvement on this bound. The proof method is based upon bounding the Frobenius norm $||A-\alpha B||_F$ (where $B = A-C$) using Hoffman-Wielandt inequality. The parameter α was set to $(1/c)$ in [Lok95] and to (r/n) in [KP98]. This careful choice was needed to ensure the optimality of the bound. If we follow TR98-042 and restrict ourselves to the case $c = const$, this subtleties become unnecessary. We can simply set $\alpha = 1$ and bound $||C||_F$.

Now, [Lok95, KP98] dealt only with Hadamard matrices since for them calculations are substantially simpler and results are the best (cf. Corollary 1 in TR98-042). Should we bother about other matrices, we can simply expand the right-hand side of (2) in [Lok95] ((7) in [KP98], respectively) to its assumed value:

$$||C||_F^2 \geq \sigma_{r+1}^2(A) + \cdots + \sigma_n^2(A) \tag{1}$$

(the case of Hadamard matrices A is characterized by $\sigma_{r+1}(A) = \ldots = \sigma_n(A) = \sqrt{n}$). As we already noted before, calculations become totally obvious in case $c = const$ and lead to the bound

$$R_A(r,c) \geq (\sigma_{r+1}^2(A) + \cdots + \sigma_n^2(A)) \cdot c^{-O(1)}. \tag{2}$$

And now this bound opens plenty of opportunities for the research in elementary linear algebra on lower-bounding its right-hand side in terms of other matrix invariants. One can consider for example the absolute value of the determinant calculated in terms of singular values as follows:

$$|Det(A)| = \sigma_1(A) \cdot \ldots \cdot \sigma_r(A) \cdot \sigma_{r+1}(A) \cdot \ldots \cdot \sigma_n(A). \tag{3}$$

It is clear already from the visual comparison with (2) that the product (3) naturally splits into two terms, and the second term $\sigma_{r+1}(A) \cdot \ldots \cdot \sigma_n(A)$ presents no difficulties. We simply apply the inequality between geometric and quadratic means:

$$\sigma_{r+1}(A) \cdot \ldots \cdot \sigma_n(A) \leq \left(\frac{\sigma_{r+1}^2(A) + \cdots + \sigma_n^2(A)}{n-r} \right)^{(n-r)/2}. \tag{4}$$

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The remaining term $\sigma_1(A) \cdot \ldots \cdot \sigma_r(A)$ is much worse. Since we can not bound it in general, we apply another simplifying assumption from TR98-042 and require that the absolute values of entries in A are also bounded by c. Then $\sigma_1^2(A) + \cdots + \sigma_r^2(A) \leq ||A||_F^2 \leq n^2 \cdot c^{O(1)}$, and the same inequality gives us

$$\sigma_1(A) \cdot \sigma_2(A) \cdot \ldots \cdot \sigma_r(A) \leq \left(\frac{n^2 \cdot c^{O(1)}}{r}\right)^{r/2}. \tag{5}$$

Combining (2), (3), (4) and (5), we get

$$R_A(r,c) \geq (n-r) \cdot \left(|Det(A)| \cdot \left(\frac{r}{n^2 \cdot c^{O(1)}}\right)^{r/2}\right)^{2/(n-r)} \cdot c^{-O(1)}.$$

Finally, the term $(c^{O(1)})^{(r/2)\cdot(2/(n-r))}$ gets absorbed by $c^{-O(1)}$ since $r \leq n/2$. For the same reasons, if we care we can replace the remaining portion $\left(\frac{r}{n^2}\right)^{r/(n-r)}$ by $\left(\frac{1}{r}\right)^{r/(n-r)}$. This is because $(x^2)^{1/(x-1)} \leq O(1)$ in the range $x \geq 2$ $(x = n/r)$. And this gives us Theorem 1 from TR98-042.

References

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