

The Descriptive Complexity Approach to LOGCFL

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Abstract

Building upon the known generalized-quantifier-based first-order characterization of LOGCFL, we lay the groundwork for a deeper investigation. Specifically, we examine subclasses of LOGCFL arising from varying the arity and nesting of *groupoidal* quantifiers. Our work extends the elaborate theory relating *monoidal* quantifiers to NC^1 and its subclasses. In the absence of the BIT predicate, we resolve the main issues: we show in particular that no single outermost unary groupoidal quantifier with FO can capture all the context-free languages, and we obtain the surprising result that a variant of Greibach’s “hardest context-free language” is LOGCFL-complete under quantifier-free BIT-free projections. We then prove that FO with unary groupoidal quantifiers is strictly more expressive with the BIT predicate than without. Considering a particular groupoidal quantifier, we prove that first-order logic with majority of pairs is strictly more expressive than first-order with majority of individuals. As a technical tool of independent interest, we define the notion of an aperiodic nondeterministic finite automaton and prove that FO translations are precisely the mappings computed by single-valued aperiodic nondeterministic finite transducers.

Keywords: finite model theory, descriptive complexity, computational complexity, automata and formal languages

1 Introduction

In *Finite Automata, Formal Logic, and Circuit Complexity* [Str94], Howard Straubing surveys an elegant theory relating finite semigroup theory, first-order logic, and computational complexity. The gist of this theory is that questions about the structure of the complexity class NC^1 , defined from logarithmic depth bounded fan-in Boolean circuits, can be translated back and forth into questions about the expressibility of first-order logic augmented with new predicates and quantifiers. Such a translation provides new insights, makes tools from one field available in the other,

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suggests tractable refinements to the hard open questions in the separate fields, and puts the obstacles to further progress in a clear perspective.

In this way, although, for example, the unresolved strict containment in NC^1 of the class ACC^0 , defined from bounded-depth polynomial-size unbounded fan-in circuits over $\{\text{AND}, \text{OR}, \text{MOD}\}$, remains a barrier since the work of Smolensky [Smo87], significant progress was made in (1) understanding the power of the BIT predicate and the related circuit uniformity issues [BIS90], (2) describing the regular languages within subclasses of NC^1 [BCST92, PMT91], and (3) identifying the all-important role of the interplay between arbitrary and regular numerical predicates in the status of the ACC^0 versus NC^1 question [Str94, p. 169, Conjecture IX.3.4].

Barrington, Immerman and Straubing [BIS90] introduced the notion of a *monoidal* quantifier and noted that, for any non-solvable group G , the class NC^1 can be described using first-order logic augmented with a monoidal quantifier for G . Loosely speaking, such a quantifier provides a constrained “oracle call” to the word problem for G (defined essentially as the problem of computing the product of a sequence of elements of G).

Bédard, Lemieux and McKenzie [BLM93] later noted that there is a fixed finite *groupoid* whose word problem is complete for the class LOGCFL of languages reducible in logarithmic space to a context-free language [Coo71, Sud78]. A groupoid G is a set with a binary operation satisfying no discernible property, and the word problem for G is that of computing the set of all legally bracketed products of a given sequence of elements of G . It is not hard to see that any context-free language is the word problem of some groupoid, and that any groupoid word problem is context-free (see [BLM93, Lemma 3.1]).

It followed that LOGCFL, a well-studied class which contains nondeterministic logarithmic space [Sud78] and is presumably much larger than NC^1 , can be described by first-order logic augmented with *groupoidal* quantifiers. These quantifiers can be defined formally as Lindström quantifiers [Lin66] for context-free languages.

In this paper, we take up the groupoidal first-order characterization of LOGCFL, and initiate an investigation of LOGCFL from the viewpoint of descriptive complexity. The rationale for this study, which encompasses the study of NC^1 , is that tools from logic might be of use in ultimately elucidating the structure of LOGCFL. We do not claim new separations of the major subclasses of LOGCFL here. But we make a first step, in effect settling necessary preliminary questions afforded by the first-order framework.

Our precise results concern the relative expressiveness of first-order formulas with ordering (written FO), interpreted over finite strings, and with: (1) nested versus unnested groupoidal quantifiers, (2) unary versus non-unary groupoidal quantifiers, (3) the presence versus the absence of the BIT predicate. Feature (3) was the focus of an important part of the work by Barrington, Immerman and Straubing [BIS90] on uniformity within NC^1 . Feature (2) was also considered, to a lesser extent, by the same authors, who left open the question of whether the “majority-of-pairs” quantifier could be simulated by a unary majority quantifier in the absence of the BIT predicate [BIS90, p. 297]. Feature (1) is akin to comparing many-one reducibility with Turing reducibility in traditional complexity theory.

Here we examine all combinations of features (1), (2) and (3). Our separation

results are summarized on Fig. 1 on p. 18. In the absence of the BIT predicate, we are able to determine the following relationships:

- FO to which a single unary groupoidal quantifier is applied, written $Q_{\text{Grp}}^{\text{un}}\text{FO}$, captures the CFLs, and is strictly less expressive than FO with nested unary quantifiers, written $\text{FO}(Q_{\text{Grp}}^{\text{un}})$, which in its turn is strictly weaker than LOGCFL. A consequence of this result, as we will see, is an answer to the above mentioned open question from [BIS90]: We show that first-order with the majority-of-pairs quantifier is strictly more expressive than first-order logic with majority of individuals.
- No single groupoid G captures all the CFLs as $Q_G^{\text{un}}\text{FO}$, i. e. as FO to which the single unary groupoidal quantifier Q_G^{un} is applied,
- FO to which a single *non-unary* groupoidal quantifier is applied, written $Q_{\text{Grp}}\text{FO}$, captures LOGCFL; our proof implies, remarkably, that adding a padding symbol to Greibach's hardest context-free language [Gre73], see also [ABB97], yields a language which is LOGCFL-complete under BIT-free quantifier-free projections.

When the BIT predicate is present, first-order with non-unary groupoidal quantifiers of course still describes LOGCFL. In the setting of monoidal quantifiers [BIS90], FO with BIT is known to capture uniform circuit classes, notably uniform ACC^0 , which have not yet been separated from NC^1 . We face a similar situation here: the BIT predicate allows capturing classes (for example $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$, verifying $\text{TC}^0 \subseteq \text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}}) \subseteq \text{LOGCFL}$), which only a major breakthrough would seem to allow separating from each other. We are able to attest to the strength of the BIT predicate in the setting of unary quantifiers, proving that:

- $Q_{\text{Grp}}^{\text{un}}\text{FO} \subsetneq Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}}$, i. e. (trivially) some non-context-free languages are expressible using BIT and a single unary groupoidal quantifier,
- $\text{FO}(Q_{\text{Grp}}^{\text{un}}) \subsetneq \text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$, i. e. (more interestingly) BIT adds expressivity even when unary groupoidal quantifiers can be nested.

We also develop a technical tool of independent interest, in the form of an aperiodic (a. k. a. group-free, a. k. a. counter-free) nondeterministic finite automaton. Aperiodicity has been studied intensively, most notably in connection with the star-free regular languages [Sch65], but, to the best of our knowledge, always in a deterministic context. Here we define a NFA A to be aperiodic if the DFA resulting from applying the subset construction to A is aperiodic. The usefulness of this notion lies in the fact, proved here, that first-order translations are precisely those mappings which are computable by single-valued aperiodic nondeterministic finite transducers.

Section 2 in this paper describes our first-order framework and exhibits a link between standard formal language operations and unary generalized quantifiers. Section 3 introduces nondeterministic finite transducers and proves that they characterize first-order translations. Section 4 forms the bulk of the paper and develops the relationships between our logic-based LOGCFL subclasses. Section 5 concludes with a number of suggestions how to extend the results obtained here.

2 Preliminaries

2.1 Complexity theory

REG and CFL refer to the regular and to the ϵ -free context-free languages respectively. The CFL results in this paper could be adapted to treat the empty string ϵ in standard ways. We will make scant reference to the inclusion chain

$$\text{AC}^0 \subsetneq \text{ACC}^0 \subseteq \text{TC}^0 \subseteq \text{NC}^1 \subseteq \text{NL} \subseteq \text{LOGCFL} = \text{SAC}^1 \subseteq \text{P},$$

where we assume familiarity with NC^1 , NL , and P , and recall that

- AC^0 (resp. ACC^0) (resp. TC^0) is the set of languages recognized by sufficiently uniform families of constant depth, polynomial size, unbounded fan-in circuits over the basis $\{\wedge, \vee, \neg\}$ (resp. over a basis consisting of $\{\wedge, \vee\}$ together with a single Boolean MOD_q gate, defined to output 0 iff q divides the sum of its input bits) (resp. over the basis consisting solely of \neg and the MAJORITY gate, defined to output 1 iff at least half of its input bits are set),
- LOGCFL is the set of languages logspace-reducible to a context-free language [Coo71, Sud78]; alternatively, this class is SAC^1 , namely the set of languages recognized by uniform families of log depth, polynomial size, Boolean circuits in which \wedge has bounded fan-in and the fan-in of \vee is unrestricted [Ven91].

2.2 The first-order framework

We consider first-order logic with linear order. We restrict our attention to *string signatures*, i. e. signatures of the form $\langle P_{a_1}, \dots, P_{a_s} \rangle$, where all the predicates P_{a_i} are unary, and in every structure \mathcal{A} , $\mathcal{A} \models P_{a_i}(j)$ iff the j th symbol in the input is the letter a_i . Such structures are thus words over the alphabet (a_1, \dots, a_s) , and first-order variables range over positions within such a word, i. e. from 1 to the word length n . For technical reasons that will become apparent shortly, we assume here, as in the rest of the paper, a linear order on each alphabet and we write alphabets as sequences of symbols to indicate that order.

Our basic formulas are built from variables in the usual way, using the Boolean connectives $\{\wedge, \vee, \neg\}$, the relevant predicates P_{a_i} together with $\{=, <\}$, the constants min and max , the quantifiers $\{\exists, \forall\}$, and parentheses. We will occasionally use the binary predicate $\text{BIT}(x, y)$, defined to be true iff the x th bit in the binary representation of y is 1. We write $\text{BC}(\mathcal{L})$ to denote the Boolean closure of the set \mathcal{L} of languages (i. e. closure under intersection, union, and complement) and $\text{BC}^+(\mathcal{L})$ to denote the closure under union and intersection only.

Definition 2.1. *Lindström quantifier.* Consider a language L over an alphabet $\Sigma = (a_1, a_2, \dots, a_s)$. Let \bar{x} be a k -tuple of variables (each of which ranges from 1 to the “input length” n , as we have seen). In the following, we assume the lexical ordering on $\{1, 2, \dots, n\}^k$, and we write X_1, X_2, \dots, X_{n^k} for the sequence of potential values taken on by \bar{x} . The groupoidal quantifier Q_L binding \bar{x} takes a meaning if $s - 1$ formulas, each having as free variables the variables in \bar{x} (and

possibly others), are available. Let $\phi_1(\bar{x}), \phi_2(\bar{x}), \dots, \phi_{s-1}(\bar{x})$ be these $s-1$ formulas. Then $Q_L \bar{x} [\phi_1(\bar{x}), \phi_2(\bar{x}), \dots, \phi_{s-1}(\bar{x})]$ holds on a string $w = w_1 \cdots w_n$, iff the word of length n^k whose i th letter, $1 \leq i \leq n^k$, is

$$\begin{cases} a_1 & \text{if } w \models \phi_1(X_i), \\ a_2 & \text{if } w \models \neg\phi_1(X_i) \wedge \phi_2(X_i), \\ \dots & \\ a_s & \text{if } w \models \neg\phi_1(X_i) \wedge \neg\phi_2(X_i) \wedge \dots \wedge \neg\phi_{s-1}(X_i), \end{cases}$$

belongs to L . Thus the formulas $[\phi_1(\bar{x}), \phi_2(\bar{x}), \dots, \phi_{s-1}(\bar{x})]$ fix a function mapping an input word/structure w of length n to a word of length n^k . This function is called the *reduction* or *transformation* defined by $[\phi_1(\bar{x}), \phi_2(\bar{x}), \dots, \phi_{s-1}(\bar{x})]$. In case we deal with the binary alphabet ($s = 2$) we omit the braces and write $Q_L \bar{x} \phi(\bar{x})$ for short.

Definition 2.2. A *groupoidal quantifier* is a Lindström quantifier Q_L where L is a context-free language.

The Lindström quantifiers of Definition 2.1 are more precisely what has been referred to as “Lindström quantifiers on string” [BV96]. The original more general definition [Lin66] uses transformations to arbitrary structures, not necessarily of string signature. However, in the context of this paper reductions to CFLs play a role of utmost importance, and hence the above definition seems to be the most natural.

The terminology “groupoidal quantifier” stems from the fact that any context-free language is a word problem over some groupoid [BLM93, Lemma 3.1], and vice-versa every word problem of a finite groupoid is context-free. Thus a Lindström quantifier on strings defined by a context-free language is nothing else than a Lindström quantifier (in the classical sense) defined by a structure that is a finite groupoid multiplication table.

Barrington, Immerman, and Straubing, defining *monoidal quantifiers* in [BIS90], in fact proceed along the same avenue: they first show how monoid word problems can be seen as languages, and then define generalized quantifiers given by such languages (see [BIS90, pp. 284f.]).

We refer the reader to standard texts for formal details on the semantics of our logical framework. For instance, Definition 2.1 skims over the semantics of a groupoidal quantifier in the case in which the underlying formulas contain free variables other than those in \bar{x} . We find Straubing’s handling of these issues [Str94] particularly convenient and we will occasionally refer to his treatment.

2.3 Unary quantifiers and homomorphisms

We will encounter unary groupoidal quantifiers repeatedly. Here we show how these relate to standard formal language operations. Recall that a length-preserving homomorphism $\Sigma^* \rightarrow \Delta^*$ is the unique free monoid morphism extending a map $h: \Sigma \rightarrow \Delta$ for finite alphabets Σ, Δ . In a different context, a result very similar to the next theorem is known as *Nivat’s Theorem* [MS97, Theorem 3.8, p. 207].

Theorem 2.3. *Let B be an arbitrary language, and let A be describable in $Q_B^{\text{un}}\text{FO}$, that is, by a first order sentence preceded by one unary Lindström quantifier (i. e. binding exactly one variable). Then there are length-preserving homomorphisms g, h and a regular language D such that $A = h(D \cap g^{-1}(B))$.*

Proof. Let A be defined by the formula $\psi \in Q_B^{\text{un}}\text{FO}$, $\psi = Q_B x \phi(x)$, $B \subseteq \Gamma^*$ (assuming $\Gamma = (0, 1)$ initially). Let Δ be the underlying alphabet determined by the string signature. ϕ thus defines a mapping from words over Δ to binary words. Define D to consist of all words $\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \cdots \begin{bmatrix} u_k \\ y_k \end{bmatrix}$ such that ϕ maps $u_1 \cdots u_k$ to $y_1 \cdots y_k$. Define the homomorphisms h and g by $h: \begin{bmatrix} a \\ b \end{bmatrix} \mapsto a$ and $g: \begin{bmatrix} a \\ b \end{bmatrix} \mapsto b$ for all $a \in \Delta$ and $b \in \Gamma$. Then $h(D \cap g^{-1}(B)) = A$. But why is D regular? Intuitively, D is regular because FO languages are regular. Arguing formally requires a bit of care because each y_i depends on the truth value of an FO formula in which the variable x is instantiated with i . A proof that a finite automaton is able to determine y_i can be found in Straubing [Str94, pp. 23–24]. To see that D itself is regular, note that an NFA N can guess an incorrect y_i (by guessing the position of the formal variable x in a \mathcal{V} -structure, borrowing notation from Straubing) and verify that y_i is incorrect. In this way N accepts the complement of D , so that D is regular¹.

The above strategy to show the regularity of D adapts to the case of a non-binary alphabet Γ , in which case N is a direct product of the NFAs accepting the languages defined by the relevant tuple of FO formulas. The homomorphisms g and h are unchanged. \square

Remark 2.4. FO precisely captures the variety of star-free regular languages [MP71], which allows us to even conclude that the D above is star-free.

2.4 Groupoid-based language classes

Here we define our first-order language classes precisely. Fix a finite groupoid G . Each $S \subseteq G$ defines a language $\mathcal{W}(S, G)$ composed of all words w , over the alphabet G , which “multiply out” to an element of S when an appropriate legal bracketing of w is chosen.

Definition 2.5. $Q_G\text{FO}$ is the set of languages describable by applying a single groupoidal quantifier Q_L to an appropriate tuple of FO formulas, where $L = \mathcal{W}(S, G)$ for some $S \subseteq G$.

$Q_{\text{Grp}}\text{FO}$ is the union, over each finite groupoid G , of $Q_G\text{FO}$.

$\text{FO}(Q_G)$ and $\text{FO}(Q_{\text{Grp}})$ are defined analogously, but allowing groupoidal quantifiers to be used as any other quantifier would (i. e. allowing arbitrary nesting).

$Q_G^{\text{un}}\text{FO}$ and $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$, etc, are defined analogously, but possibly allowing the BIT predicate (signaled by subscripting FO with bit) and/or restricting to *unary* groupoidal quantifiers (signaled by the exponent “un”).

¹An alternative proof that D is regular is immediate from Theorem 3.5.

3 An automaton characterization of FO-translations

As a technical tool, it will be convenient to have an automata-theoretic characterization of *first-order translations*, i. e. of reductions defined by FO-formulas with one free variable. Since FO precisely describes the (regular) languages accepted by aperiodic deterministic finite automata [MP71], one might expect aperiodic deterministic finite transducers to capture FO-translations. This is not the case however because, e.g. the FO-translation which maps every string $w_1 \cdots w_n$ to w_n^n cannot be computed by such a device.

We show in this section that the appropriate automaton model to use is that of a single-valued aperiodic nondeterministic finite transducer, which we define and associate with FO-translations in this section. But first, we discuss the notion of an aperiodic NFA.

Definition 3.1. A deterministic or nondeterministic FA M is *aperiodic* (or group-free) iff there is an $n \in \mathbb{N}$ such that for all states s and all words w ,

$$\delta(s, w^n) = \delta(s, w^{n+1}).$$

Here δ is the extension of M 's transition function from symbols to words. Observe that if M is nondeterministic then $\delta(t, v)$ is a set of states, i. e. locally here we abuse notation by not distinguishing between M 's extended transition function δ and the function δ^* as defined in the context of a nondeterministic transducer below.

Remark 3.2. This definition of aperiodicity for a DFA is the usual one (see [Ste85]). For a NFA, a statement obviously equivalent to Definition 3.1 would be that A is aperiodic iff applying the subset construction to A yields an aperiodic DFA. Hence [Sch65] a language L is star-free iff some aperiodic (deterministic or nondeterministic) finite automaton accepts L .

We now prepare the ground for the main result of this section, namely that single-valued aperiodic nondeterministic finite transducers characterize FO-translations.

Definition 3.3. A *finite transducer* is given by a set Q of *states*, an *input alphabet* Σ , an *output alphabet* Γ , an *initial state* q_0 , a *transition relation* $\delta \subseteq Q \times \Sigma \times \Gamma \times Q$ and a set $F \subseteq Q$ of *final states*. For a string $w = w_1 \cdots w_n \in \Sigma^*$ we define the *set* $O_M(w)$ of *outputs of M on input w* as follows. A string $v \in \Gamma^*$ of length n is in $O_M(w)$, if there is a sequence $s_0 = q_0, s_1, \dots, s_n$ of states, such that $s_n \in F$ and, for every i , $1 \leq i \leq n$, we have $(s_{i-1}, w_i, v_i, s_i) \in \delta$.

We say that M is *single-valued* if, for every $w \in \Sigma^*$, $|O_M(w)| = 1$. If M is single-valued it naturally defines a function $f_M : \Sigma^* \rightarrow \Gamma^*$.

For every string $u \in \Sigma^*$ and every state $s \in Q$ we write $\delta^*(s, u)$ for the set of states s' that are reachable from s on input u (i. e., there are $s_1, \dots, s_{|u|} = s'$ and $v_1 \cdots v_{|u|}$ such that, for every i , $1 \leq i \leq |u|$, we have $(s_{i-1}, u_i, v_i, s_i) \in \delta$).

As per Definition 3.1, M is *aperiodic* if there is an $n \in \mathbb{N}$ such that for all states q and all strings w , $\delta^*(q, w^n) = \delta^*(q, w^{n+1})$.

We will need some basic properties of FO-logic on strings.

Let k be a fixed natural number and Σ an alphabet. For every string u we write Φ_u^k for the set of FO-sentences of quantifier-depth k that hold in u . Let S^k denote the set $\{\Phi_u^k \mid u \in \Sigma^*\}$. It is well-known that S^k is finite, for every fixed k and Σ .

Lemma 3.4. *Let u, u', v, v' be strings such that $\Phi_u^k = \Phi_{u'}^k$, and $\Phi_v^k = \Phi_{v'}^k$. Then $\Phi_{uv}^k = \Phi_{u'v'}^k$.*

Proof. As $\Phi_u^k = \Phi_{u'}^k$ and $\Phi_v^k = \Phi_{v'}^k$, we know that the duplicator has a winning strategy in the k -round Ehrenfeucht game on u and u' and in the game on v and v' . These strategies can be easily combined to get a strategy on uv and $u'v'$. From the existence of this winning strategy we can, in turn, conclude that $\Phi_{uv}^k = \Phi_{u'v'}^k$. \square

Theorem 3.5. *A function $f: \Sigma^* \rightarrow \Gamma^*$ is defined by an FO translation if and only if it is defined by a single-valued aperiodic finite transducer.*

Proof. To simplify notation we assume that $\Gamma = (0, 1)$. The proof of the general case is a straightforward generalization.

(only if) Let $f: \Sigma^* \rightarrow \Gamma^*$ be defined by formula $\varphi(x)$ of quantifier-depth k (hence, for every $w \in \Sigma^*$ and every $i \leq |w|$, the i -th bit of $f(w)$ is 1 iff $w \models \varphi(i)$). We define a single-valued aperiodic finite transducer M with input alphabet Σ , output alphabet Γ , set $S^k \times S^k \cup \{q_0\}$ of states, initial state q_0 and accepting states $\{(\Phi, \Phi'_\epsilon) \mid \Phi \in S^k\}$. Informally, a state (Φ_1, Φ_2) of M represents a situation, in which M “knows” that Φ_1 contains exactly those formulas (of quantifier depth k) that hold in the prefix of the input string that was already read, and it “guesses” that Φ_2 contains exactly those formulas that hold in the remaining part of the string.

The transition relation δ of M is defined as follows. For every $\Phi_1, \Phi_2, \Phi'_1, \Phi'_2 \in S^k$, every $\sigma \in \Sigma$ and every $\tau \in \Gamma$ we let

$$((\Phi_1, \Phi_2), \sigma, \tau, (\Phi'_1, \Phi'_2)) \in \delta,$$

if there exist strings $u, v \in \Sigma^*$ such that $\Phi_1 = \Phi_u^k$, $\Phi'_1 = \Phi_{u\sigma}^k$, $\Phi_2 = \Phi_v^k$, $\Phi'_2 = \Phi_v^k$ and $\tau = 1 \iff u\sigma v \models \varphi(|u| + 1)$.

Analogously, for every $\Phi'_1, \Phi'_2 \in S^k$, every $\sigma \in \Sigma$ and every $\tau \in \Gamma$ we define

$$(q_0, \sigma, \tau, (\Phi'_1, \Phi'_2)) \in \delta,$$

if there exists a string $v \in \Sigma^*$ such that $\Phi'_1 = \Phi_\sigma^k$, $\Phi'_2 = \Phi_v^k$ and $\tau = 1 \iff \sigma v \models \varphi(1)$.

We first check that M is single-valued. Let $w = w_1 \cdots w_n$, and $f(w) = v_1 \cdots v_n$. We set $s_0 = q_0$ and, for every $i > 0$, $s_i = (\Phi_{w_1 \cdots w_i}^k, \Phi_{w_{i+1} \cdots w_n}^k)$. By using Lemma 3.4, it is easy to verify that $s_n \in F$ and, for every $i > 0$, we have $(s_{i-1}, w_i, v_i, s_i) \in \delta$. Hence $f(w) \in O_M(w)$.

We have to show now that no string $u = u_1 \cdots u_n \neq f(w)$ is in $O_M(w)$. Assume otherwise and let $s'_0 = q_0, s'_1, \dots, s'_n$ be a sequence of states that outputs u . Let, for every $i > 0$, $s'_i = (\Psi_i, \Theta_i)$. First, it is easy to observe that, for every $i > 0$, $\Psi_i = \Phi_{w_1 \cdots w_i}^k$. As u is different from v there must be a j such that $\Theta_j \neq \Phi_{w_{j+1} \cdots w_n}^k$ (Note that from the definition of δ it follows that $(s, \sigma, 1, s') \in \delta$ implies $(s, \sigma, 0, s') \notin \delta$). We conclude that for every $i > j$, $\Theta_i \neq \Phi_{w_{i+1} \cdots w_n}^k$: Assume, otherwise that $i > j$ is minimal, such that $\Theta_i = \Phi_{w_{i+1} \cdots w_n}^k$. By definition of δ and as $(s'_{i-1}, w_i, \tau, s'_i) \in \delta$ it

follows immediately that $\Theta_{i-1} = \Phi_{w_i w_{i+1} \dots w_n}^k$, a contradiction. Hence, in particular, $\Theta_n \neq \Phi_\epsilon^k$, i.e., $s'_n \notin F$. It follows that M is single-valued and $f_M = f$. It remains to show that M is aperiodic. First of all, it is well-known, and can be shown by an Ehrenfeucht game argument [EF95] that, for $n = 2^k$ and every $w \in \Sigma^*$ it holds $\Phi_w^k = \Phi_{w^{n+1}}^k$.

Let now $\Phi_1, \Phi_2, \Phi'_1, \Phi'_2 \in S^k$ and let $u, v \in \Sigma^*$ with $\Phi_1 = \Phi_u^k$ and $\Phi'_2 = \Phi_v^k$. From Lemma 3.4 and the definition of δ we can conclude that $(\Phi'_1, \Phi'_2) \in \delta^*((\Phi_1, \Phi_2), x)$ if and only if $\Phi_2 = \Phi_{xv}^k$ and $\Phi'_1 = \Phi_{ux}^k$. Hence, again with Lemma 3.4, we get for every w the following.

$$\begin{aligned} (\Phi'_1, \Phi'_2) \in \delta^*((\Phi_1, \Phi_2), w^n) &\iff \Phi_2 = \Phi_{w^n v}^k \text{ and } \Phi'_1 = \Phi_{u w^n}^k \\ &\iff \Phi_2 = \Phi_{w^{n+1} v}^k \text{ and } \Phi'_1 = \Phi_{u w^{n+1}}^k \\ &\iff (\Phi'_1, \Phi'_2) \in \delta^*((\Phi_1, \Phi_2), w^{n+1}) \end{aligned}$$

This implies that M is aperiodic.

(if) Let f be computed by a single-valued aperiodic finite transducer $M = (Q, \Sigma, \Gamma, q_0, \delta, F)$. It is easy to check that, for every $s, s' \subseteq Q$, the language

$$L(s, s') = \{u \mid s' \in \delta^*(s, u)\}$$

is accepted by an aperiodic finite automaton. Consequently, every $L(s, s')$ is characterized by a FO formula $\varphi^{s, s'}$. Let $\varphi(x)$ be the formula

$$\bigvee_{\substack{s, s', s'', \sigma \\ s'' \in F \wedge (s, \sigma, 1, s') \in \delta}} \varphi_{<}^{q_0, s}(x) \wedge P_\sigma(x) \wedge \varphi_{>}^{s', s''}(x).$$

Here, for every s and s' , $\varphi_{<}^{s, s'}(x)$ is the formula that is obtained by relativizing $\varphi^{s, s'}$ to all positions that are smaller than x and $\varphi_{>}^{s, s'}(x)$ is the formula that is obtained by relativizing $\varphi^{s, s'}$ to all positions that are greater than x (see for example [Str94, pp. 81f]).

Hence, for every position x , $\varphi(x)$ becomes true in a string w if and only if there are states s, s', s'' such that

- M can reach s from the initial state by reading the string left to x ,
- M can reach s' from s by reading the symbol at position x and output a 1, and
- M can reach the final state s'' from s' by reading the string to right to x .

As M is single-valued, $\varphi(x)$ defines $f_M(w)$, for every w . □

4 First-order with groupoidal quantifiers

4.1 The largest attainable class: LOGCFL

Theorem 4.1. *There is a fixed groupoid G such that*

$$Q_{G\text{FO}_{\text{bit}}} = \text{FO}_{\text{bit}}(Q_{\text{Grp}}) = \text{LOGCFL}.$$

Proof. $Q_G\text{FO}_{\text{bit}} \subseteq \text{FO}_{\text{bit}}(Q_{\text{Grp}})$ holds by definition for any groupoid G . To see that $\text{FO}_{\text{bit}}(Q_{\text{Grp}}) \subseteq \text{LOGCFL}$, note that [BIS90, Theorem 8.1] implies the existence of a logspace-uniform AC^0 -reduction, from any language in $\text{FO}_{\text{bit}}(Q_{\text{Grp}})$, to a set of groupoid word problems. The unbounded fan-in AND gates in the AC^0 reduction can be replaced by log depth bounded fan-in sub-circuits. Then the groupoid word problem oracle gates, of which no more than a constant number can appear on any path from circuit inputs to circuit output, can be expanded into SAC^1 sub-circuits, since groupoid word problems are context-free languages. There results a logspace-uniform SAC^1 circuit, proving membership in LOGCFL .

$\text{LOGCFL} \subseteq Q_G\text{FO}_{\text{bit}}$ is seen by appealing to the fixed G whose word problem is LOGCFL -complete under DLOGTIME reducibility [BLM93]. Since DLOGTIME was shown expressible in FO_{bit} by [BIS90], the inclusion follows. \square

4.2 Capturing LOGCFL without BIT

Theorem 4.2. *There is a fixed groupoid G such that $\text{LOGCFL} \subseteq Q_G\text{FO}$.*

Proof. We first show how to express plus and times and their negations as $\text{FO}^+(Q_{\text{Grp}})$ formulas (i. e. formulas which have outside of the groupoidal quantifier only a first-order quantifier prefix and in particular no negation).

Let us look at the predicate “ $a \cdot b = c$.” Define $L =_{\text{def}} \{ w \in (0, 1, \#)^* \mid |w|_0 = |w|_1 \}$ and

$$\phi(a, b, c) =_{\text{def}} Q_L(x, y, z) [(z = \min) \wedge (x \leq a) \wedge (y \leq b), (z = y = \max) \wedge (x \leq c)].$$

Given a word w of length n and assignments for a, b, c , the transformation $[z = \min \wedge x \leq a \wedge y \leq b, z = y = \max \wedge x \leq c]$ yields a string of length n^3 over the alphabet $(0, 1, \#)$ which contains $a \cdot b$ many 0s, c many 1s, and $n^3 - ab - c$ many #s. Thus this image is in L if and only if $a \cdot b = c$.

Observe that L is deterministic context-free, therefore its complement is context-free and we conclude that we can also express $a \cdot b \neq c$ by a $\text{FO}^+(Q_{\text{Grp}}\text{FO})$ formula (in fact even by a $Q_{\text{Grp}}\text{FO}$ formula).

In a similar way we can express $a + b = c$ and $a + b \neq c$ by $\text{FO}^+(Q_{\text{Grp}})$ formulas. All context-free languages involved in the definition of these predicates can be combined into one language L_0 , which is context-free and co-context-free. Now integer addition and multiplication are enough to simulate the BIT predicate. Indeed it can be shown that exponentiation can be defined from addition and multiplication (see e. g. [HP93, p. 301] and [Smo91, p. 192]), and from this it is not so hard to define the BIT predicate, as pointed out by [Lin94] (cf., [Imm98]). We conclude that there is a $\text{FO}^+(Q_{\text{Grp}})$ formula for the bit predicate. The only groupoid quantifiers needed in this definition are Q_{L_0} quantifiers, and they are applied to quantifier-free formulas.

From Theorem 4.1 we know that $\text{LOGCFL} = Q_{\text{Grp}}\text{FO}_{\text{bit}}$. Thus every set in $A \in \text{LOGCFL}$ can be defined by a formula

$$Q_L \bar{x} [\Phi_1, \dots, \Phi_s], \tag{1}$$

where each Φ_i is a FO_{bit} formula.

We will show how every such formula can be transformed into $Q_{L'}\text{FO}$ -formula, for some fixed context-free language L' .

Using the argument above we can replace each Φ_i in (1) by a formula without bit, but using the Q_{L_0} quantifier. This formula can then be transformed into the form

$$\exists \bar{x}_1 \forall \bar{x}_2 \exists \bar{x}_3 \cdots \bigvee_{i_1} \bigwedge_{i_2} \phi_{i_1, i_2}, \quad (2)$$

where each of the ϕ_{i_1, i_2} is either a positive atomic formula or a formula of the form $Q_{L_0}\chi$, where χ is quantifier-free.

Now we combine stepwise the inner quantifiers Q_{L_0} ($1 \leq j \leq m$) in formula (2) with the first-order connectives \bigvee , \bigwedge and the first-order quantifiers \exists , \forall . We give the construction for the case of an existential quantifier. Consider the formula $\exists x Q_{L_1} \bar{y} [\xi_1, \dots, \xi_{k-1}]$, where $L_1 \subseteq A^*$ is context-free and co-context-free. Suppose $A = (a_1, \dots, a_k)$, $\# \notin A$. Let $\bar{y} = (y_1, \dots, y_l)$. This formula is equivalent to $Q_{L_2}(x, z, y_1, \dots, y_l) [\xi_0, \xi'_1, \dots, \xi'_{k-1}]$ where

$$L_2 = \{ w \in (a_1, \dots, a_k, \#)^* \mid w = w_1 \#^+ w_2 \#^+ \cdots \#^+ w_n \#^+, w_i \in L_1 \text{ for some } i \},$$

ξ_0 is the formula $z > 1$ and each ξ'_i , $1 \leq i \leq k-1$, is the formula $z = 1 \wedge \xi_i$. The transformation f defined by $[\xi_0, \xi'_1, \dots, \xi'_{k-1}]$ maps a word w of length n to a word $f(w)$ of length n^{l+2} . $f(w)$ consists of n blocks u_1, \dots, u_n of length n^{l+1} each: $f(w) = u_1 \cdots u_n$. Here u_m corresponds to the assignment $x = m$. Each u_m consists of n blocks of length n^l , one block for each value of z . These blocks are all in $\#^*$ for $z > 1$, and consist of a word over A for $z = 1$. This word is exactly the word to which w is mapped under the transformation $[\xi_1, \dots, \xi_{k-1}]$, when $x = m$. Hence we see that $f(w) \in L_2$ if there is some value m such that $u_m \in L_1 \#^*$. This proves the correctness of the above construction. Certainly L_2 is context-free, and since the complement of L_1 is context-free, we see that the complement of L_2 is also context-free (the construction of appropriate PDAs is obvious).

The combinations of a Q_{L_j} with a universal quantifier, or with a first-order connective, are dealt with analogously.

We thus replaced the sub-formulas Φ_i in formula (1) above and obtained a formula of the form

$$Q_L \bar{x} [\Psi_1, \dots, \Psi_s], \quad (3)$$

where each Ψ_i is of the form $Q_{L_i} \psi_i$, ψ_i is quantifier-free, and L_i is context-free and co-context-free. Let $L \subseteq A_0^*$, where $A_0 = (a_1, \dots, a_s)$, $\#, \$ \notin A_0$. Let $B =_{\text{def}} (a_1, \dots, a_s, \#, \$)$. We now define a substitution h by

$$\begin{aligned} h(a_1) &= \$L_1\#B^* \\ h(a_2) &= \$\overline{L_1}\#^*L_2\#B^* \\ &\dots \\ h(a_i) &= \$\overline{L_1}\#^* \cdots \#^* \overline{L_{i-1}}\#^* L_i\#B^* \\ &\dots \\ h(a_s) &= \$\overline{L_1}\#^* \overline{L_2}\#^* \cdots \#^* \overline{L_{s-1}}\#^* \end{aligned}$$

and let $L' =_{\text{def}} h(L)$. Our formula replacing (1) then is

$$Q_{L'}\bar{z}[\Psi'_1, \dots, \Psi'_{s+1}], \quad (4)$$

where we have to construct the formulas Ψ'_i such that the following holds: Given a word w , suppose the transformation given by $[\Psi_1, \dots, \Psi_{s-1}]$ produces for a certain assignment of the variables \bar{x} the letter $a \in A$; more specifically: suppose that ψ_i produces w_i (for $1 \leq i \leq s-1$). Then $[\Psi'_1, \dots, \Psi'_{s+1}]$ has to produce a word $\$w_1\#^*w_2\#^*\dots\#^*w_{s-1}\#^*$. Certainly this can be done with quantifier-free formulas.

Thus we have shown that $\text{LOGCFL} \subseteq Q_{\text{Grp}}\text{FO}$. Now define H to be Greibach's hardest context-free language. Any cfl L reduces to H via a homomorphism (see [ABB97, p. 137]. This homomorphism is ϵ -free but not length-preserving. Applying a non-unary groupoidal quantifier to simple FO-formulas can realize this homomorphism, provided that a new padding or neutral symbol be introduced, to act as a filler in any word. Thus we see that any $Q_L\text{FO}$ formula can be transformed into an equivalent $Q_{\text{pad}(H)}\text{FO}$ formula. \square

A corollary to this proof is the following remarkable result:

Corollary 4.3. *Greibach's hardest context-free language with a neutral symbol is complete for LOGCFL under quantifier-free projections without BIT.*

A noteworthy strengthening of Theorem 4.1 thus follows from Theorem 4.2:

Corollary 4.4. $Q_{\text{Grp}}\text{FO} = \text{FO}(Q_{\text{Grp}}) = \text{LOGCFL}$.

4.3 Unary groupoidal quantifiers

In the previous subsection, we have shown that the situation with non-unary groupoidal quantifiers is clearcut, since a single such quantifier, even without the BIT predicate, captures all of LOGCFL. Here we examine the case of unary quantifiers. In this case, the presence or absence of the BIT predicate is once again relevant.

4.3.1 Unary groupoidal quantifiers without BIT

Theorem 4.5. $Q_{\text{Grp}}^{\text{un}}\text{FO} = \text{CFL}$.

Proof. The direction from right to left follows from [BLM93]: Every context-free language reduces via a length-preserving homomorphism to a groupoid word problem. We can even look at the letters in a given word as groupoid elements. This reduction can be expressed in FO.

The direction from left to right is proved by appealing to Theorem 2.3 and observing that the context-free languages have the required closure properties. \square

It follows immediately that nesting unary groupoidal quantifiers (in fact, merely taking the Boolean closure of $Q_{\text{Grp}}^{\text{un}}\text{FO}$) adds expressiveness:

Corollary 4.6.

$$\begin{aligned} Q_{\text{Grp}}^{\text{un}}\text{FO} = \text{CFL} &\subsetneq \text{BC}^+(Q_{\text{Grp}}^{\text{un}}\text{FO}) = \text{BC}^+(\text{CFL}) \\ &\subsetneq \text{BC}(Q_{\text{Grp}}^{\text{un}}\text{FO}) = \text{BC}(\text{CFL}) \\ &\subseteq \text{FO}(Q_{\text{Grp}}^{\text{un}}). \end{aligned}$$

Proof. All inclusions from left to right are clear. The first separation follows from the fact that CFLs are not closed under intersection. The second separation follows from considering the non-context-free language Y , consisting of all words of the form ww , the complement of which is context-free. \square

The inclusion $\text{CFL} \subseteq Q_{\text{Grp}}^{\text{un}}\text{FO}$ in Theorem 4.5 could have been proved alternatively by observing that the logic $\exists\text{MFO}$ capturing CFL (see [LST94]) is closed under FO translations. We note in the same vein:

Theorem 4.7. $Q_{\text{Grp}}\text{MSO} = \text{CFL}$.

Proof. In [LST94] it is in fact proved that $\text{CFL} = \exists\text{MMSO}$. This logic is closed under monadic second-order (MSO) transformations. Hence $\text{CFL} \subseteq Q_{\text{Grp}}\text{MSO} \subseteq \exists\text{MMSO} \subseteq \text{CFL}$. \square

Can we refine Theorem 4.5 and find a universal finite groupoid G which captures all the context-free languages as $Q_G^{\text{un}}\text{FO}$? Intuition from the world of monoids [BIS90, p. 303] suggests that the answer is no. Proving that this is indeed the case is the content of Theorem 4.9 below. We first make a definition and state a lemma.

Let D_t be the context-free one-sided Dyck language over $2t$ symbols, i. e. D_t consists of the well-bracketed words over an alphabet of t distinct types of parentheses. Recall that a PDA is a nondeterministic automaton which reads its input from left to right and has access to a pushdown store with a fixed pushdown alphabet. We say that a PDA A is *k -pushdown-limited*, for k a positive integer, iff

- the pushdown alphabet of A has size k , and
- A pushes no more than k symbols on its stack between any two successive input head motions.

Lemma 4.8. *No k -pushdown-limited PDA accepts D_t when $t \geq (k+1)^k + 1$.*

Proof. Suppose to the contrary that a k -pushdown-limited PDA A accepts D_t , where $t = (k+1)^k + 1$. A has a certain fixed number, s , of states. Consider A 's computation as it scans a length- n prefix of its input. Since A is k -pushdown-limited, no more than $(k+1)^{kn}$ different stack contents, hence no more than $s \cdot (k+1)^{kn}$ configurations, are encountered. But A must be able to distinguish between each pair of length- n prefixes consisting of left parentheses alone, because for any two such prefixes v_1 and v_2 , there is a Dyck word v_1w such that v_2w is not a Dyck word. Now, it is easy to see that t^n , the number of length- n words over an alphabet of t left parentheses, exceeds $s \cdot (k+1)^{kn}$ when n is large. Hence A cannot accept D_t . \square

Theorem 4.9. *Any finite groupoid G verifies $Q_G^{\text{un}}\text{FO} \subsetneq \text{CFL}$.*

Proof. Suppose to the contrary that G is a finite groupoid such that $Q_G^{\text{un}}\text{FO} = \text{CFL}$. Then there is a FO-translation from each context-free language to a word problem for G . This means that a finite set of PDAs (one for each word problem $\mathcal{W}(\cdot, G)$) can take care of answering each “oracle question” resulting from such a FO-translation. By Theorem 3.5, each FO-translation is computed by a single-valued NFA. Although the NFAs differ for different context-free languages (and this holds in particular when language alphabets differ), the NFAs do not bolster the “pushdown-limits” of the PDAs which answer all oracle questions. Hence if k is a fixed integer such that all word problems $\mathcal{W}(\cdot, G)$ for G are accepted by a k -pushdown-limited PDA, then for any positive integer t , D_t is accepted by a k -limited-pushdown PDA. This contradicts Lemma 4.8 when $t = (k + 1)^k + 1$. \square

In the next subsection we will see that the BIT-predicate provably adds expressive power to the logic $Q_{\text{Grp}}^{\text{un}}\text{FO}$. Since it is known that BIT can be expressed either by plus and times [Lin94] (cf., [Imm98]) or by the majority of pairs quantifier [BIS90], the following two simple observations about the power of $Q_{\text{Grp}}^{\text{un}}\text{FO}$ are of particular interest.

Theorem 4.10. *The majority quantifier is definable in $Q_{\text{Grp}}^{\text{un}}\text{FO}$.*

Proof. Majority is a context-free language. \square

Theorem 4.11. *Addition is definable in $Q_{\text{Grp}}^{\text{un}}\text{FO}$.*

Proof. Let i, j, k be positions in the input word. We want to express that $i + j = k$. We do this by using a quantifier for the context-free language $L =_{\text{def}} \{0^{i-1}a1^*b0^{j-1}c1^* \mid i \in \mathbb{N}\}$. Given a word $w \in L$, if symbol a is at position i and b is at position j , then c must be at position $i + j$. \square

4.3.2 Unary groupoidal quantifiers with BIT

What are $Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}}$ and $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$? It would seem plausible that $Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}} \subsetneq \text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}}) \subset \text{LOGCFL}$, but we are unable to prove $Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}} \subsetneq \text{LOGCFL}$, much less $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}}) \subsetneq \text{LOGCFL}$. The next lemma indicates that proving the latter would prove $\text{TC}^0 \neq \text{LOGCFL}$, settling a major open question in complexity theory.

Lemma 4.12. $\text{TC}^0 \subseteq \text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$.

Proof. TC^0 is captured by first-order logic with bit and majority quantifiers [BIS90]. \square

Hence the BIT predicate is expressive and will be difficult to defeat. The next lemma is not surprising, but it documents the provable expressiveness of BIT. Recall that $\text{CFL} = Q_{\text{Grp}}^{\text{un}}\text{FO}$ (Theorem 4.5).

Lemma 4.13. $\text{CFL} \subsetneq Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}}$.

Proof. The language of all words whose length is a power of two is in FO_{bit} hence in the difference of the two classes. \square

The remainder of this subsection is devoted to documenting a more complicated setting in which the BIT predicate provably adds expressiveness. We want to show that $\text{FO}(Q_{\text{Grp}}^{\text{un}}) \subset \text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$, i. e. that even when unary groupoidal quantifiers can be nested arbitrarily, the BIT predicate adds strength.

For this, we define, for strings u, w of equal length the operations \bar{u} , $u \wedge w$ and $u \vee w$ which denote the bitwise complementation of u , the bitwise AND of u and w and the bitwise OR of u and w . We say that a string w is (l, m) -bounded if it is in $u_1^* \cdots u_l^*$, for some strings u_i with $|u_i| \leq m$, for every i .

We are going to make use of the following Lemma.

Lemma 4.14. *Let u be an (l, m) -bounded 0-1-string and w an (l', m') -bounded 0-1-string, for some $l, m, l', m' \geq 1$, and $|u| = |w|$. Then the following hold.*

- (a) \bar{u} is (l, m) -bounded.
- (b) $u \wedge w$ and $u \vee w$ are $(5(l + l'), mm')$ -bounded.

Proof. (a) is trivial. We show (b) only for $u \wedge w$, the argument for $u \vee w$ being completely analogous.

We show the statement by induction on $l + l'$. The induction starts with the case $l = l' = 1$.

In this case, $u = u_1^i$ and $w = w_1^j$, for some i, j, u_1, w_1 , with $|u_1| \leq m$ and $|w_1| \leq m'$.

Let $u_1 \diamond w_1$ denote the string $u_1^{|w_1|} \wedge w_1^{|u_1|}$ of length $|u_1||w_1| \leq mm'$. Further let d and r be chosen such that $|u| = d|u_1||w_1| + r$. Then $u \wedge w = (u_1 \diamond w_1)^d v$ for some v with $|v| = r$. Hence $u \wedge w$ is $(2, mm')$ -bounded.

Now let $l + l' > 2$. W.l.o.g. we can assume that $u = u_1^{i_1} u_2^{i_2} u'$ and $w = w_1^j w'$ where $|u_1|, |u_2| \leq m$, $|w_1| \leq m'$, u' is $(l - 2, m)$ -bounded, w' is $(l' - 1, m')$ -bounded and $|w_1^j| \geq |u_1^{i_1}|$.

Let $0 \leq r < m'$ be such that $|u_1^{i_1}| + r$ is a multiple of $|w_1|$. Let $u_2^{\leftarrow r}$ be the word u_2 rotated r positions to the left. It should be clear that, from position $|u_1^{i_1}| + r$ in $u \wedge w$ onwards, the word $(u_2^{\leftarrow r} \diamond w_1)$ is repeated, as long as the $u_2^{i_2}$ portion of u and the w_1^j portion of w keep “overlapping”. We distinguish two cases.

Case 1: $|w_1^j| \leq |u_1^{i_1}| + |u_2^{i_2}|$, i.e. the “overlap” with $u_2^{i_2}$ runs out within w_1^j .

$$\begin{array}{l} u = \\ w = \end{array} \begin{array}{|c|c|c|} \hline u_1^{i_1} & u_2^{i_2} & u' \\ \hline w_1^j & & w' \\ \hline \end{array}$$

There are i_2', i_2'', u_3, u_4 with $|u_3|, |u_4| < m$, such that

$$\begin{array}{l} u = \\ w = \end{array} \begin{array}{|c|c|c|c|} \hline u_1^{i_1} & u_2^{i_2'} u_3 & u_4 u_2^{i_2''} & u' \\ \hline w_1^j & & & w' \\ \hline \end{array}$$

It is not hard to see that we can write $(u_1^{i_1} u_2^{i_2'} u_3) \wedge w_1^j$ as

$$(u_1 \diamond w_1)^{k_1} v_1 v_2 (u_2^{\leftarrow r} \diamond w_1)^{k_2} v_3,$$

for some v_2 of length r , some k_1, k_2 , and some v_1, v_3 of length at most mm' . As $u_4 u_2^{i_2''} u'$ is (l, m) -bounded and w' is $(l' - 1, m')$ -bounded it follows by induction that

$u_4 u_2^{i_2} u' \wedge w'$ is $(5(l + l' - 1), mm')$ -bounded. Altogether, $u \wedge w$ is $(5(l + l'), mm')$ -bounded, as required.

Case 2: $|w_1^j| \geq |u_1^{i_1}| + |u_2^{i_2}|$, i.e. $u_2^{i_2}$ runs out first.

$$\begin{array}{l} u = \\ w = \end{array} \begin{array}{|c|c|c|} \hline u_1^{i_1} & u_2^{i_2} & u' \\ \hline w_1^j & & w' \\ \hline \end{array}$$

Hence, there are j', j'' and w_2, w_3 with $|w_2|, |w_3| < m'$ such that

$$\begin{array}{l} u = \\ w = \end{array} \begin{array}{|c|c|c|c|} \hline u_1^{i_1} & u_2^{i_2} & u' & \\ \hline w_1^{j'} w_2 & & w_3 w_1^{j''} & w' \\ \hline \end{array}$$

Now, $u_1^{i_1} u_2^{i_2} \wedge (w_1^{j'} w_2)$ can be written as

$$(u_1 \diamond w_1)^{k_1} v_1 v_2 (u_2^{\leftarrow r} \diamond w_1)^{k_2} v_3,$$

where $|v_2| = r$ and $|v_1|, |v_3| < mm'$, hence this string is $(5, mm')$ -bounded. Again, by induction, it follows that the remaining part of $u \wedge w$ is $(5(l + l' - 1), mm')$ -bounded, which implies the statement of the lemma. \square

Let Σ be a fixed alphabet, and let σ denote the corresponding signature. Let φ be a FO(+)- σ -formula with free variables x and $\bar{y} = y_1, \dots, y_k$. For every string $w \in \Sigma^*$, we write $t_{\varphi}^{\bar{y}}(w)$ for the 0-1 string $v = v_1, \dots, v_{|w|}$ with $v_i = 1$ iff $\langle w, i, \bar{y} \rangle \models \varphi$.

Lemma 4.15. *Let $\Sigma = \{0\}$ and $\sigma_0 = \{P_0\}$. Let φ be a FO(+)- σ_0 -formula with free parameters x and $\bar{y} = y_1, \dots, y_k$. Then there are l and m such that for every n and y_1, \dots, y_k it holds that $t_{\varphi}^{\bar{y}}(0^n)$ is (l, m) -bounded.*

Proof. Let φ' be the FO(+)- \emptyset -formula which results from φ by replacing every sub-formula $P_0(t)$ by true, introducing a new free variable, n , and restricting all quantifiers relative to n . I. e., sub-formulas $\exists z \theta$ are replaced by $\exists z (z < n) \wedge \theta$ and $\forall z \theta$ is replaced by $\forall z (z < n) \rightarrow \theta$. Then we get

$$\langle 0^n, x, \bar{y} \rangle \models \varphi \iff \langle N, n, x, \bar{y} \rangle \models \varphi',$$

where N denotes the natural numbers. Using Presburger Quantifier Elimination (see [BJ89, pp. 220ff] or [Smo87, pp. 320ff]) we can transform φ' into an equivalent quantifier-free formula ψ which may additionally use the constants 0 and 1 and binary predicates $\cdot \equiv \cdot \pmod{c}$, for some constants c . The atomic formulas of ψ are of one of the following forms.

- $ax + bn + a_1 y_1 + \dots + a_k y_k = c$,
- $ax + bn + a_1 y_1 + \dots + a_k y_k < c$,
- $ax + bn + a_1 y_1 + \dots + a_k y_k > c$,
- $ax + bn + a_1 y_1 + \dots + a_k y_k \equiv c \pmod{d}$,

for some constants a, b, c, d, a_i . For every fixed n, y_1, \dots, y_k , the first formula defines, via the above equivalence, a $(3, 1)$ -bounded string in $0^* 1^* 0^*$, the second and third formula define a $(2, 1)$ -bounded string in $1^* 0^*$ and $0^* 1^*$ respectively, and the last formula defines a $(2, d)$ -bounded string in $0^* (10^{d-1})^*$. As ψ is fixed, by inductively applying Lemma 4.14 we get constants l and m , such that, for every n, \bar{y} , $t_{\varphi}^{\bar{y}}(0^n)$ is (l, m) -bounded. \square

Theorem 4.16. $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$ is not contained in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$.

Proof. We consider the language $\{0^{n^2} \mid n \in \mathbb{N}\}$, which is even expressible in FO_{bit} and show that it is not in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$.

In order to do so, we show that, for every unary language L in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$, the set $\{i \mid 0^i \in L\}$ is semi-linear (i. e. the finite union of some arithmetic progressions).

It is enough to show that, over a one-letter alphabet, every formula of the kind $Q_B x \varphi$ with CFL B and first-order φ (with addition) can be replaced by a first-order formula with addition.

Hence, let $\psi = Q_B x \varphi$, for some first-order ϕ (with addition) and CFL B .

Let, besides x , $\bar{y} = y_1, \dots, y_k$ be the free variables of φ .

By Lemma 4.15, there exist l and m such that, for every n and \bar{y} , $t_{\varphi}^{\bar{y}}(0^n)$ is (l, m) -bounded. Let u_1, \dots, u_p be an enumeration of all 0-1 strings of length at most m . Let L' denote the (regular) language, defined by $(u_1^* \cdots u_p^*)^l$. It follows that $t_{\varphi}^{\bar{y}}(0^n)$ is in L' , hence it can be written as $u_1^{i_{11}} \cdots u_p^{i_{1p}} u_1^{i_{21}} \cdots u_p^{i_{2p}} \cdots u_p^{i_{lp}}$ (where, for each $j = 1, \dots, l$, all but one of the i_{j1}, \dots, i_{jp} are 0). For a word $w \in L'$ we write $I(w)$ for the set of tuples (i_{11}, \dots, i_{lp}) with $u_1^{i_{11}} \cdots u_p^{i_{lp}} = w$. We show in the following that $I_B := \bigcup_{w \in B \cap L'} I(w)$ is a semi-linear set.

Let $\Gamma = a_{11}, \dots, a_{1p}, \dots, a_{lp}$ be a new (lp -letter) alphabet and let h be the homomorphism defined by $h(a_{ij}) = u_i$. Let τ denote the Parikh mapping for strings $a_{11}^* \cdots a_{lp}^*$. Then we have

$$I_B = \tau(h^{-1}(B \cap L') \cap a_{11}^* \cdots a_{lp}^*),$$

which is semi-linear by Parikh's theorem [Har78, Sect. 6.9].

Hence, ψ is equivalent to a $\text{FO}(+)$ formula. By induction, we get that every $\text{FO}(+)(Q_{\text{Grp}}^{\text{un}})$, hence also every $\text{FO}(Q_{\text{Grp}}^{\text{un}})$ formula, over a one-letter alphabet is equivalent to a $\text{FO}(+)$ formula. Hence $\{0^{n^2} \mid n \in \mathbb{N}\}$ is not in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$. \square

It is interesting to see that the proof makes use of quantifier elimination twice, first to get the bounded strings, and second to show that $\{0^{n^2} \mid n \in \mathbb{N}\}$ is not in $\text{FO}(+)$.

As a particular case we can now solve an open question of [BIS90], addressing the power of different arity for majority quantifiers.

Corollary 4.17. *Majority of pairs can not be expressed in first-order logic with unary majority quantifiers.*

Proof. In Theorem 4.10 it was observed that the unary majority quantifier can be simulated in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$. On the other hand in [BIS90] it is shown that majority of pairs is sufficient to simulate the BIT predicate. But as $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$ is not contained in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$ the BIT predicate and hence the majority of pairs is not definable in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$, hence it can not be simulated by unary majority quantifiers. \square

In the same way, this time relying on Theorem 4.11, we obtain:

Corollary 4.18. *Multiplication is not definable in $\text{FO}(Q_{\text{Grp}}^{\text{un}})$.*

5 Conclusion

Fig. 1 depicts the first-order groupoidal-quantifier-based classes studied in this paper. Together with the new characterization of FO-translations by means of aperiodic finite transducers, the relationships shown on Fig. 1 summarize our contribution.

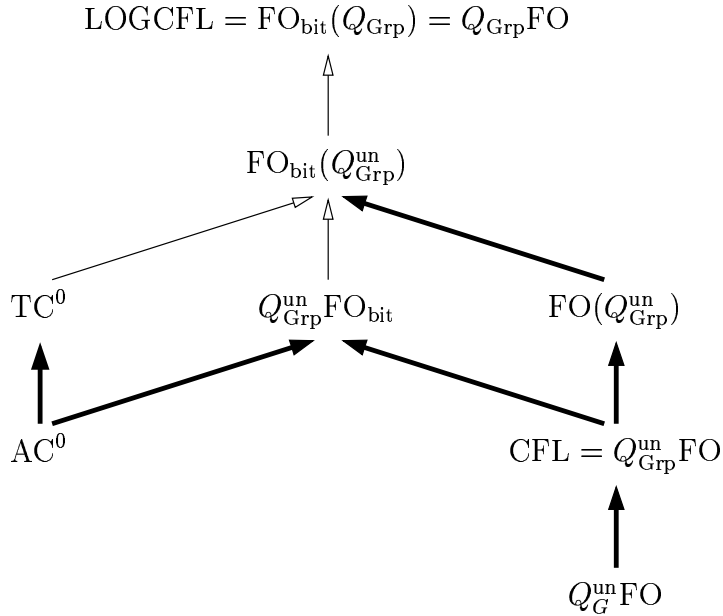


Figure 1: The new landscape. Here G stands for any fixed groupoid, and a thick line indicates strict inclusion.

A number of open questions are apparent from Figure 1. Clearly, it would be nice to separate the FO_{bit} -based classes, in particular $\text{FO}_{\text{bit}}(Q_{\text{Grp}}^{\text{un}})$ from $\text{FO}_{\text{bit}}(Q_{\text{Grp}})$, but this is a daunting task. A sensible approach then is to begin with $Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}}$. How does this compare with TC^0 for example? Can we at least separate $Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}}$ from LOGCFL ? We know that $Q_{\text{Grp}}^{\text{un}}\text{FO}_{\text{bit}} \not\subseteq \text{FO}(Q_{\text{Grp}}^{\text{un}})$; a witness for this is the set $\{0^{n^2} \mid n \in \mathbb{N}\}$, cf. the proof of Theorem 4.16.

Other natural questions prompted by our separation results concern extensions and refinements to Figure 1. For example, in the world with BIT, which specific groupoids G are powerful enough to express LOGCFL , and which are not? In the world without BIT, given the aperiodic transducer characterization of FO-translations, can we prove $\text{REG} \setminus (\text{REG} \cap Q_G^{\text{un}}\text{FO}) \neq \emptyset$ as easily as Lemma 4.8 implies $\text{CFL} \setminus Q_G^{\text{un}}\text{FO} \neq \emptyset$? More importantly, can we hope for an algebraic theory of groupoids to explain the detailed structure of CFL, much in the way that an elaborate theory of monoids is used in the extensive first-order parameterization of REG?

But perhaps the most fundamental (and hopefully tractable) question arising from our work is not apparent from Figure 1. It concerns the Boolean closure of the context-free languages. We have trivially used $\text{BC}(\text{CFL})$ (in fact $\text{BC}^+(\text{CFL})$)

sufficed) to witness the separation between $Q_{\text{Grp}}^{\text{un}}\text{FO}$ and $\text{FO}(Q_{\text{Grp}}^{\text{un}})$. But what is $\text{BC}(\text{CFL})$ exactly, and what techniques are available to prove that a language is not in $\text{BC}(\text{CFL})$? It is easy to prove that any non-regular language over a unary alphabet does not belong to $\text{BC}(\text{CFL})$, and a natural infinite hierarchy within $\text{BC}^+(\text{CFL})$ is known [LW73], but the full question seems to have fallen into the cracks. We have several good candidates for membership in $\text{FO}(Q_{\text{Grp}}^{\text{un}}) \setminus \text{BC}(\text{CFL})$, but so far have been unable to prove these two classes different.

Finally, ever since the regular languages in AC^0 and in ACC^0 were characterized (the latter modulo a natural conjecture [BCST92]), one might have wondered about a similar characterization for the context-free languages in these classes, and in NC^1 . A unified treatment of LOGCFL subclasses under the banner of first-order logic might constitute a useful step towards being able to answer these questions. Since circuit-based complexity classes are closed under Boolean operations however, a better understanding of the interaction between the complement operation and groupoidal quantifiers is required. This once again seems to highlight the importance of understanding $\text{BC}(\text{CFL})$.

Acknowledgments. We thank Dave Barrington, Gerhard Buntrock, Volker Diekert, Klaus-Jörn Lange, Ken Regan, Heinz Schmitz, Denis Thérien, Wolfgang Thomas, Klaus Wagner, and Detlef Wotschke for useful discussions at one stage or another in the course of this work.

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