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Polynomial–Time Recognition of Minimal Unsatisfiable Formulas with Fixed Clause–Variable Difference

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Abstract

A formula (in conjunctive normal form) is said to be *minimal unsatisfiable* if it is unsatisfiable and deleting any clause makes it satisfiable. Let $\mathcal{F}(k)$ be the class of formulas such that the number of clauses exceeds the number of variables exactly by k. Every minimal unsatisfiable formula belongs to $\mathcal{F}(k)$ for some $k \geq 1$. Polynomial–time algorithms are known to recognize minimal unsatisfiable formulas in $\mathcal{F}(1)$ and $\mathcal{F}(2)$, but not for $k \geq 3$. We state a polynomial–time algorithm that recognizes minimal unsatisfiable formulas in $\mathcal{F}(k)$ for any fixed $k \geq 1$, and we show that the running time of our algorithm is $\mathcal{O}(N^{k+3/2})$.

1 Introduction

A formula S (in conjunctive normal form, CNF for short) is minimal unsatisfiable, if S is unsatisfiable, but omitting any clause yields a satisfiable formula. Papadimitriou and Wolfe ([12]) showed that recognizing minimal unsatisfiable formulas is D^p–complete. D^p is the class of problems which can be considered as the difference of two NP–problems.

Let $\mathcal{F}(k)$ be the class of formulas where the number of clauses exceeds the number of variables (atoms) exactly by k. A result by Aharoni and Linial ([1]) states that every minimal unsatisfiable formula belongs to a class $\mathcal{F}(k)$ for $k \geq 1$. Davidov et al. ([5]) showed that, if $k \geq 1$ is fixed, then the recognition of minimal unsatisfiable formulas in $\mathcal{F}(k)$ is in NP. Moreover, Kleine Büning conjectured the following ([8], see also [9]).

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Conjecture 1 *For fixed $k \geq 1$, it can be decided in polynomial time whether a formula $S \in \mathcal{F}(k)$ is minimal unsatisfiable.*

The main result of this paper is a proof of this conjecture; we state an algorithm with running time $\mathcal{O}(N^{k+3/2})$ where N is the length of the input formula. It follows that $\mathcal{F}(k)$, $k = 1, 2, \ldots$ is a polynomial hierarchy containing all minimal unsatisfiable formulas.

So far, polynomial–time algorithms were only known for cases $k = 1$ and $k = 2$, with running time $\mathcal{O}(N^2)$ and $\mathcal{O}(N^3)$, respectively ([8,5]). Whence, in the cases $k = 1, 2$, our general algorithm is slightly slower than the quoted algorithms. Zhao and Ding [13] considered subclasses $\mathcal{F}'(k)$ of $\mathcal{F}(k)$ defined by a strong additional condition; the authors obtained algorithms to recognize minimal unsatisfiable formulas in $\mathcal{F}'(3)$ and $\mathcal{F}'(4)$ with running time $\mathcal{O}(N^5)$ and $\mathcal{O}(N^9)$, respectively.

However, comparing the time complexities of algorithms in terms of the number n of variables of the input formula S (instead of its length N), then our general algorithm is significantly slower than the quoted ones, since $N = \mathcal{O}(n^2)$.

2 Basic notations and results

2.1 Formulas

Let $\mathcal{A}$ be a finite alphabet of *atoms*; we will think of the elements of $\mathcal{A}$ as boolean variables. We define the *literals* to be elements of the form a or $-a$, where $a \in \mathcal{A}$. Literals which are atoms are called positive; the others are called negative.

A *clause* is a finite set of literals, and a *formula* is a finite set of clauses. For a clause C we let $\mathcal{A}(C)$ be the set of atoms a such that a or $-a$ is in C. For a formula S we put $\mathcal{A}(S) := \bigcup_{C \in S} \mathcal{A}(C)$.

The *length* of a formula S is given by $N := \sum_{C \in S} |C|$.

A *truth assignment* to a formula S is a map $f : \mathcal{A}(S) \to \{0, 1\}$. We define $f(-a) := 0$ if $f(a) = 1$ and $f(-a) := 1$ otherwise. Further, for $C \in S$ we define $f(C) := \max_{x \in C} f(x)$; and put $f(S) := \min_{C \in S} f(C)$. A formula S is *satisfied* by a truth assignment f if $f(S) = 1$. A formula S is called *satisfiable* if there exists a truth assignment which satisfies S; otherwise S is called *unsatisfiable.*

To decide whether a formula is satisfiable (the famous SAT problem) is the first problem which has been proved to be NP-complete ([4]).

For $X \subseteq \mathcal{A}(S)$ let $r_X(S)$ be the formula which is obtained by replacing in each clause of S every occurrence of a by $-a$ and every occurrence of $-a$ by a, for all $a \in X$. The formula $r_X(S)$ is called a *renaming* of S (c.f. [11]). The following lemma shows that satisfiability can be stated in terms of renamings.

Lemma 1 *A formula S is unsatisfiable if and only if for every renaming S' of S there is a clause $C' \in S'$ such that C' contains no positive literal.*

PROOF. For a truth assignment f to S let $X_f \subseteq \mathcal{A}(S)$ be given by

$$X_f := \{\, a \in \mathcal{A}(S) : f(a) = 0 \,\};$$

and for a set $X \subseteq \mathcal{A}(S)$ let f_X be the truth assignment to S characterized by

$$f_X(a) = 0 \quad \text{if and only if} \quad a \in X.$$

We observe that $X_{f_X} = X$, whence there is a one-to-one correspondence between subsets $X \subseteq \mathcal{A}(S)$ and truth assignments to S. It is easy to verify that S contains a clause C with $f(C) = 0$ if and only if $S' = r_{X_f}(S)$ contains a clause C' with $C' \cap \mathcal{A}(S) = \emptyset$. Whence the lemma follows. $\square$

Note that the binary relation on formulas of being a renaming of each other is an equivalence relation; moreover, a formula is unsatisfiable if and only if any renaming of it is unsatisfiable.

A formula is *minimal unsatisfiable* if it is unsatisfiable, but every proper subset of it is satisfiable.

Clearly, every unsatisfiable formula contains at least one subset which is minimal unsatisfiable.

Next, we state an easy consequence of this concept.

Lemma 2 *A formula S is minimal unsatisfiable if and only if S is unsatisfiable and for every $C \in S$ there is a truth assignment f to S such that $f(C) = 0$ but $f(S - \{C\}) = 1$; i.e., $f(C) = 0$ but $f(C') = 1$ for all $C' \in S - \{C\}$.*

The following construction is well known. There, a formula S is satisfiable if and only if S' derived from S, is satisfiable. We point out, however, that the same holds true with respect to minimal unsatisfiability (cf. [5, Lemmas 2 and 3]). Let S be a formula and assume there is a clause

$$C = \{x_1, \ldots, x_{d-1}, x_d, \ldots, x_r\} \in S$$

with $|C| \geq d+1 > 3$. Let $S' := (S - \{C\}) \cup \{C_1, C_2\}$ where $a \notin \mathcal{A}(S)$ is a new atom with

$$C_1 := \{a, x_1, \ldots, x_{d-1}\} \quad \text{and} \quad C_2 := \{-a, x_d, \ldots, x_r\}.$$

Now we can easily derive the following by multiple applications of the above construction.

Lemma 3 *Let $d \geq 3$ be an integer and S a formula of length N. Then we can obtain a formula S' of length N' in time $\mathcal{O}(N)$ such that*

(1) $|C| \leq d$ for all $C \in S'$;
(2) S is minimal unsatisfiable if and only if S' is minimal unsatisfiable;
(3) $|S| - |\mathcal{A}(S)| = |S'| - |\mathcal{A}(S')|$;
(4) $N' \leq 3N$.

2.2 Graphs and digraphs

For graph theoretic terminology not defined here, the reader is referred to [3]. All *graphs* considered are finite and simple. For a graph G, the sets of *vertices* and *edges* are denoted by $V(G)$ and $E(G)$, respectively. For $X, Y \subseteq V(G)$ we write $E(X, Y)$ for the set of edges $e = xy \in E(G)$ with $x \in X$ and $Y \in Y$. The set of *neighbors* of a vertex $v \in V(G)$ is denoted by $N_G(v)$; for $X \subseteq V(G)$ we put $N_G(X) := \left(\bigcup_{v \in X} N_G(v)\right) - X$. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is given by $|N_G(v)|$. The maximum degree of all vertices in $X \subseteq V(G)$ is denoted by $\Delta_G(X)$.

We use similar notation for *digraphs* (directed graphs). We consider digraphs D such that the graph G underlying D is simple (i.e., D contains neither loops, nor parallel arcs, nor directed cycles of length 2). Then we have $V(D) = V(G)$. We denote the set of *arcs* of D by $A(D)$. Further, we put $N_D(v) := N_G(v)$, $N_D(X) := N_G(X)$, $d_D(v) := d_G(v)$, and $\Delta_D(X) = \Delta_G(X)$ for $v \in V(D)$, $X \subseteq V(D)$. We say that D is *connected* if G is connected.

A (di)graph G is *bipartite* if its vertices can be partitioned into two classes U and W such that no vertices of the same class are adjacent. We write $G = (U, W)$ to denote a specific vertex–bipartition.

A set M of edges (or arcs) in G is a *matching* if no two elements of M have a vertex in common. A vertex is *matched by M* if it is incident with an element of M. Let X be a set of vertices in G. A matching of G is *X–perfect* if all vertices in X are matched by M. A $V(G)$–perfect matching is simply called *perfect matching*.

A *cover* of a graph G is a set C of vertices such that every edge of G is incident with at least one vertex in C. Note that if C is a cover of a bipartite graph $G = (U, W)$, then $E(U - C, W - C) = \emptyset$.

A vertex s of a digraph is a *sink* if it has no outgoing arcs. For a digraph $D = (U, W)$ we write $\sigma(D)$ for the set of sinks which belong to W.

3 Atom–clause digraphs

Bipartite digraphs can be used to represent formulas.

Definition 1 Let S be a formula and $D = (U, W)$ a bipartite digraph. We call D the *atom–clause digraph* of S if there exist bijective maps $g : U \to \mathcal{A}(S)$ and $h : W \to S$ such that

$$(w, u) \in A(D) \text{ if and only if } \quad g(u) \in h(w), \quad \text{and}$$
$$(u, w) \in A(D) \text{ if and only if } -g(u) \in h(w).$$

Clearly, such atom–clause digraph of S always exists for given S; and since all atom–clause digraphs of a formula S are isomorphic, it is admissible to call D *the* atom–clause digraph of S. Moreover, atom–clause digraphs contain no loops or parallel arcs. We may assume w.l.o.g. that no clause C contains both a and $-a$ for an atom a; otherwise, we may restrict our considerations to $S - \{C\}$ since S is satisfiable if and only if $S - \{C\}$ is satisfiable. Hence, atom–clause digraphs contain no directed cycles of length 2.

In the following we note some easy observations.

Lemma 4 *Let $D = (U, W)$ be a bipartite digraph.*

(1) D is the atom–clause digraph of some formula S if and only if U contains no isolates in D.
(2) If D is the atom–clause digraph of a formula S, $W' \subseteq W$, then the subdigraph of D induced by $W' \cup N_D(W')$ is the atom–clause digraph of a subset of S.
(3) If D is the atom–clause digraph of a minimal unsatisfiable formula, then D is connected. (This follows from (2) and the definition of minimal unsatisfiability.)
(4) If D is the atom–clause digraph of a formula S, then $|A(D)|$ equals the length of S.

Definition 2 Let $D = (U, W)$ be the atom–clause digraph of a formula S and let $X \subseteq U$. We obtain a digraph $r_X(D) = D'$ from D by reversing the orientations of all arcs incident with vertices in X. We call D' a *redirection* of D. For $u \in U$ we say that D and D' *agree* in u if $u \in U - X$.

Since renamings of formulas and redirections of their atom–clause digraphs correspond to each other, we obtain the following corollary to Lemma 1. We will use this graph theoretic characterization of satisfiability throughout this paper.

Corollary 1 *Let $D = (U, W)$ be the atom–clause digraph of a formula S. Then S is unsatisfiable if and only if for every redirection D' of D $\sigma(D') \neq \emptyset$.*

The following can be obtained from Lemma 2.

Corollary 2 *Let $D = (U, W)$ be the atom–clause digraph of an unsatisfiable formula S. Then S is minimal unsatisfiable if and only if for every $w \in W$ there is a redirection D' of D with $\sigma(D') = \{w\}$.*

4 X–elementary graphs

Definition 3 Let G be a graph and $X \subseteq V(G)$. An edge is called X*–allowed* if it lies in an X–perfect matching. Further, G is called X*–elementary* if all edges which lie in an X–perfect matching form a nontrivial connected subgraph (this concept can be viewed as a generalization of elementary graphs; see [10, p. 122].) Further, we say that a digraph is X–elementary if the underlying graph is X–elementary.

The proof of the following theorem on U–elementary bipartite graphs $G = (U, W)$ follows almost literally the proof of [10, Theorem 4.1.1.] on elementary bipartite graphs (the authors of [10] attribute this theorem 'mostly' to Hetyei). In our more general setting, however, we have to assume connectedness a priori.

Theorem 1 *For a connected bipartite graph $G = (U, W)$ with $|W| - |U| \geq 1$ the following statements are equivalent.*

(1) G is U–elementary;
(2) U is the only minimum cover of G;
(3) ("the strong Hall condition") *for every nonempty $X \subseteq U$,*

$$|N_G(X)| \geq |X| + 1;$$

(4) if $|U| \geq 2$ then $G - u - w$ has a $(U - \{u\})$–perfect matching for all $u \in U$, $w \in W$;
(5) all edges of G are U–allowed.

PROOF. *(1)⇒(2).* Clearly U is a cover. G being U–elementary implies that G has a U–perfect matching It follows that every cover has at least $|U| > 0$ elements. Thus U is a minimum cover.

Now suppose there is a minimum cover C with $C_U := U \cap C \neq \emptyset$ and $C_W := W \cap C \neq \emptyset$. First we want to show that $E(C_U, C_W)$ contains no U–allowed edges. Suppose to the contrary that for some $u \in C_U$, $w \in C_W$, the edge $e = uw$ is U–allowed. Let M be a U–perfect matching with $uw \in M$. Since $E(U - C_U, W - C_W) = \emptyset$ it follows that M matches $U - C_U$ into $C_W - w$, and therefore $|C_W| > |C_W - w| \geq |U - C_U|$. Hence $|C| = |C_U| + |C_W| > |C_U| + |U - C_U| = |U|$; thus, C cannot be a minimum cover, a contradiction. We conclude that $E(C_U, C_W)$ contains no U–allowed edges. However, $G - E(C_U, C_W)$ is obviously disconnected. Since every vertex of U is incident with some U–allowed edge, G cannot be U–elementary. However, by hypothesis W cannot be a minimum cover implying—together with G being bipartite—that $C = U$ or $C = W$. Thus $C_U = \emptyset$ or $C_W = \emptyset$; whence U is the only minimum cover.

(2)⇒(3). Since U is a minimum cover of G, $d(u) > 0$ for all $u \in U$. Thus, $N_G(X) \neq \emptyset$ for all $\emptyset \neq X \subseteq U$, and furthermore, $Z := N_G(X) \cup (U - X)$ is a cover. Since U is the only minimum cover, $|Z| \geq |U| + 1$. Hence $|N_G(X)| + |U| - |X| \geq |U| + 1$, i.e., $|N_G(X)| \geq |X| + 1$.

(3)⇒(4). Let $u \in U$, $w \in W$ and let $H := G - u - w$. For every $\emptyset \neq X \subset U - u$ we have $|N_H(X)| \geq |N_G(X)| - 1 \geq |X|$. Applying the Marriage Theorem we conclude that H has a $(U - \{u\})$–perfect matching.

(4)⇒(5). $U = \emptyset$ implies $E(G) = \emptyset$ by hypothesis, implying (5) trivially. If $|U| = 1$, then G is U–elementary, since G being connected and $|W| \geq 2$ implies $d(u) = |W| \geq 2$ $(U = \{u\})$. So, every $e \in E(G)$ contributes a U–perfect matching in this case. Whence assume $|U| \geq 2$. Choose $e = uw \in E(G)$ arbitrarily with $u \in U$, $w \in W$. Then $G - u - w$ has a $(U - \{u\})$–perfect matching M. Clearly $M \cup \{e\}$ is a U–perfect matching of G. Whence e is U–allowed for every $e \in E(G)$, in any case.

(5)⇒(1). Since every $e \in E(G)$ is U–allowed and G is connected, G is U–elementary by definition. □

In this paper we are faced several times by the problem of finding a matching of maximum cardinality in a bipartite graph G. Therefore we can apply the maximum cardinality matching algorithm of Hopcroft and Karp ([7]). Galil obtained the asymptotic bound $\mathcal{O}(|E(G)| \cdot |V(G)|^{1/2})$ for Hopcroft and Karp's algorithm, [6].

For the following lemma cf. [10, Exercise 4.1.2, p. 123].

Lemma 5 *A connected bipartite graph $G = (U, W)$ with $|W| - |U| \geq 1$, can be tested for U–elementarity in time $\mathcal{O}(|E(G)|^2 \cdot |V(G)|^{1/2})$.*

PROOF. Let $G = (U, W)$. Clearly, an edge $e = uw$ with $u \in U$, $w \in W$ is U–allowed, if and only if $G - u - w$ has a $(U - \{u\})$–perfect matching. Whence, the set of all U–allowed edges E' can be obtained by applying the maximum cardinality matching algorithm $|E(G)|$ times. By Theorem 1(5), G being U–elementary now reduces to deciding whether $E' = E$. □

5 Admissible matchings in bipartite digraphs

Definition 4 Let $D = (U, W)$ be a bipartite digraph. A matching M of D is called *admissible* if all arcs in M are directed from W to U. We denote by $\mathcal{DM}(D)$ the set of all pairs (D', M') such that D' is a redirection of D, and M' is an admissible matching of D'.

Definition 5 Let M be an admissible matching of a bipartite digraph $D = (U, W)$. A path P (in the undirected sense) of D is called (D, M)*–alternating* if

(1) the arcs in P are alternately in and out of M, and
(2) arcs in M are traversed according to their orientation.

If a (D, M)–alternating path P begins with an unmatched vertex in U and ends with an unmatched vertex in W, then we say that P is (D, M)*–augmenting.*

Lemma 6 *If $(D, M) \in \mathcal{DM}(D_0)$ contains a (D, M)–augmenting path then we can obtain a pair $(D', M') \in \mathcal{DM}(D)$ with*

$$|M'| > |M| \quad \textit{and} \quad \sigma(D') \subseteq \sigma(D).$$

PROOF. Let P be a (D, M)–augmenting path of minimal length joining $u \in U$ to $w \in W$, say. Let u' be the vertex of P preceding w, and set $W^- := \{\, w^- \in W : (w^-, u') \in A(D) \,\}$.

Note that the length $\ell(P)$ of a (D, M)–augmenting path P is odd; if $\ell(P) = 1$, then $u' = u$.

If $W^- \neq \emptyset$ we assume w.l.o.g. that $w \in W^-$ (observe that $W^- \cap (V(P) - \{w\}) = \emptyset$ by the minimality of $\ell(P)$, in any case). Set

$$M^* := (M - A(P)) \cup (A(P) - M), \tag{5.1}$$

and observe that M^* is a matching with $|M^*| = |M| + 1$. However, M^* is not necessarily admissible. Let X be the set of vertices in U which are tails of arcs in M^*. Clearly, $X \subseteq V(P)$. Moreover, M^* corresponds to an admissible matching M' in the redirection $D' = r_X(D)$. Now $(D', M') \in \mathcal{DM}(D)$ with $|M'| > |M|$; thus it remains to show that $\sigma(D') \subseteq \sigma(D)$.

Observe that every $y \in W$ which is matched by M, is also matched by M^* by (5.1), and is therefore matched by M'. Thus, if $s \in \sigma(D') - \sigma(D) \subseteq W$ exists, it cannot be matched by M', since M' is admissible, and thus cannot be matched by M. Since s became a sink through a redirection, we have $(s, x) \in A(D)$ for some $x \in X$. And since s is unmatched by M, $x = u'$ follows from the minimality of $\ell(P)$. Hence $s \in W^-$, and the redirection included the arcs incident with u'; i.e., $u' \in X$. Since every element of X is incident with precisely one element of M^*, $u' \in X$ implies $(u', w) \in A(P) \cap M^*$, whence $w \notin W^-$. Together with $s \in W^-$ this yields a contradiction to the choice of w. Thus, s cannot exist, hence $\sigma(D') \subseteq \sigma(D)$. □

An analogue to a famous theorem of Berge ([2]) holds for bipartite U–elementary digraphs.

Theorem 2 *Let $D_0 = (U, W)$ be a connected U–elementary bipartite digraph with $|W| - |U| \geq 1$ and let $(D, M) \in \mathcal{DM}(D_0)$. Then the following are equivalent.*

(1) M is U–perfect;
(2) D has no (D, M)–augmenting path.

PROOF. Obviously, if M is U–perfect, then D cannot have a (D, M)–augmenting path (since no unmatched $u \in U$ exists).

Conversely, assume that D has no (D, M)–augmenting path. Let Q be the set of all unmatched vertices of U. Suppose that M is not U–perfect, i.e., $Q \neq \emptyset$. Observe that no arc in D joins a pair of unmatched vertices.

Let R be the set of all vertices which can be reached from some vertex in Q by a (D, M)–alternating path of length > 0. We consider the subgraph $D^* = (U^*, W^*)$ of D induced by R.

Every $w \in W^*$ must be matched by M in D; otherwise D would have a (D, M)–augmenting path. On the other hand, by definition of a (D, M)–alternating path P, if P ends in a vertex $u \in U$, the last arc of P must lie in M. Whence, all vertices of D^* are matched by M in D.

Now we show that $M^* = M \cap A(D^*)$ is a perfect matching of D^*. Let P be a (D, M)–alternating path starting in some $q \in Q$ and ending in $u \in U$. If P does not contain some vertex $w \in N_D(u)$, then P can be extended to a (D, M)–alternating path which ends in w. Thus $w \in W^*$ follows. Hence $N_D(u) \subseteq W^*$ for every $u \in U^*$ and therefore $N_D(U^*) \subseteq W^*$. Consequently, the arc of M which is incident with $u \in U^*$ is in D^*; and this is also true for any $w^* \in W^*$, as follows. Let P be a (D, M)–alternating path from some $q \in Q$ to $w^* \in W^*$. Since w^* is matched by M (see the preceding paragraph), there is an arc $(w^*, u^*) \in M$. We observe that u^* cannot be an internal vertex of P, since w^* must be the predecessor of u^* in any (D, M)–augmenting path from some $q \in Q$. Whence, we can extend P by adding the arc (w^*, u^*), thus $u^* \in U^*$. It follows that M^* is a perfect matching of D^*. Consequently $|U^*| = |W^*|$. Since $N_{D_0}(U^*) = N_D(U^*) \subseteq W^*$, we have $|N_G(U^*)| \leq |U^*|$ (where G is the graph underlying D_0), a contradiction to Theorem 1(3). □

6 Minimal unsatisfiability and the parameter k

The following is an unpublished result of Tarsi (see [1]). It is an easy consequence of Theorem 4 below.

Theorem 3 *If S is a minimal unsatisfiable formula, then $|S| - |\mathcal{A}(S)| \geq 1$.*

Motivated by this theorem, one is lead to classify a formula S by the parameter $k = |S| - |\mathcal{A}(S)|$.

Definition 6 We write $\mathcal{F}(k)$ for the class of formulas S with $|S| - |\mathcal{A}(S)| = k$.

Obviously, if $S \in \mathcal{F}(k)$ and $D = (U, W)$ is the atom–clause digraph of S, then $|W| - |U| = k$. If S is minimal unsatisfiable, then $k \geq 1$ by Theorem 3.

Theorem 4 *[1, Theorem 3] Let $D = (U, W)$ be the atom–clause digraph of a formula S. Then the following hold.*

(1) If D has a W–perfect matching, then S is satisfiable.
(2) If S is minimal unsatisfiable, then D has a U–perfect matching.

The preceding theorem holds also for infinite formulas, which is irrelevant, however, for the following considerations.

Lemma 7 *The atom–clause digraph $D = (U, W)$ of a minimal unsatisfiable formula S is connected and U–elementary.*

PROOF. By Lemma 4(3), D is connected. We show that the strong Hall condition holds for D (see Theorem 1). Suppose there is a proper subset $X \neq \emptyset$ of U with $|X| \geq |N_D(X)|$; we may assume that X is the largest set with this property. By Theorem 4, there is a U–perfect matching M, hence $|X| = |N_D(X)|$. Let $D_1 = (X, N_D(X))$ be the subdigraph induced by $X \cup N_D(X)$. We obtain a redirection $D_1' = r_{X_1}(D_1)$, $X_1 \subseteq X$, such that M induces an admissible matching in D_1'. Clearly, $\sigma(D_1') = \emptyset$. Let $Y := W - N_D(X)$. By the maximality of X it follows that $N_D(Y) = U - X$. Consequently, by Lemma 4(2), the subdigraph D_2 of D induced by $V(D) - V(D_1) = Y \cup N_D(Y)$ is the atom–clause digraph of a proper subset S' of S. Since S is minimal unsatisfiable, there must be a redirection $D_2' = r_{X_2}(D_2)$, $X_2 \subseteq U - X$ with $\sigma(D_2') = \emptyset$. We combine D_1' and D_2' to a redirection $D' = r_{X_1 \cup X_2}(D)$ of D. Now $\sigma(D') = \emptyset$, a contradiction by Corollary 1. □

Definition 7 We consider a bipartite digraph $D = (U, W)$ with $|W| - |U| = k \geq 1$. Let $Z \subseteq W$ with $|Z| \leq k$. Consider $W_k \subseteq W$ with $|W_k| = k$ and $Z \subseteq W_k$.

(1) Denote by $\mathcal{R}_D(W_k, Z)$ the set of all redirections D' of D obtained by redirecting some (possibly none, possibly all) vertices in $N_D(W_k)$ such that

$$\sigma(D') \cap W_k = Z.$$

Possibly, $\mathcal{R}_D(W_k, Z) = \emptyset$.

(2) For $D' \in \mathcal{R}_D(W_k, Z) \neq \emptyset$ let $G_{D'}$ be the graph underlying $D' - W_k - A^*$, where A^* is the set of arcs in D' whose tails are in $N_D(W_k) = N_{D'}(W_k)$. Set

$$\mathcal{G}_D(W_k, Z) := \{\, G_{D'} : D' \in \mathcal{R}_D(W_k, Z) \,\}.$$

(3) Set

$$\mathcal{G}_D(Z) := \bigcup_{W_k} \mathcal{G}_D(W_k, Z);$$

note that this is a union of disjoint sets.

Observe that for some redirection D' of D we may have $G_{D'} \in \mathcal{G}_D(W_k, Z)$ although $D' \notin \mathcal{R}_D(W_k, Z)$ (cf. Lemma 9(1) below).

Lemma 8 *Let d, k be fixed positive integers. For all $D = (U, W)$ with $|W| - |U| = k$ and $\Delta_G(W) \leq d$, we have*

$$|\mathcal{G}_D(Z)| = \mathcal{O}\Big(\, |W|^{k-|Z|} \,\Big).$$

PROOF. Let $n := |W|$ and $z := |Z|$. Then we have $\binom{n-z}{k-z}$ different sets W_k ($\supseteq Z$) and $|N_D(W_k)| \leq kd$. So we have at most 2^{kd} redirections of D belonging to $\mathcal{R}_D(W_k, Z)$. $|\mathcal{G}_D(W_k, Z)| \leq |\mathcal{R}_D(W_k, Z)|$ follows from Definition 7(2). We have

$$|\mathcal{G}_D(Z)| \leq \binom{n-z}{k-z} 2^{kd} \leq \frac{2^{kd}}{(k-z)!}(n-z)^{k-z} \leq \frac{2^{kd}}{k!} n^{k-z}. \qquad \square$$

The next lemma is more technical in nature and helps to understand the use of the sets defined above in subsequent lemmas; it also shortens their proofs.

Lemma 9 *Let $D = (U, W)$ be a bipartite digraph, $|W| - |U| = k \geq 1$, $Z \subseteq W_k \subseteq W$, $|W_k| = k$. Then*

(1) $\mathcal{G}_D(W_k, Z) = \mathcal{G}_{D'}(W_k, Z)$ for every redirection D' of D;
(2) if $\mathcal{G}_D(W_k, Z)$ contains a graph with a perfect matching, then there is a redirection D' of D with $\sigma(D') = Z$.

PROOF. *(1).* Since D' is a redirection of D, $D' = r_{X'}(D)$ for some $X' \subseteq U$. Let $X^* = X' \cap (U - N_D(W_k))$. It follows that $D^* = r_{X^*}(D)$ agrees with D' in $U - N_D(W_k)$ which is sufficient to conclude $\mathcal{R}_{D'}(W_k, Z) = \mathcal{R}_{D^*}(W_k, Z)$ by Definition 7(1). This, in turn implies

$$\mathcal{G}_{D'}(W_k, Z) = \mathcal{G}_{D^*}(W_k, Z). \tag{6.2}$$

Since D^* agrees with D in $N_D(W_k) = N_{D^*}(W_k)$ the definition of G_{D^*} and G_D implies $\mathcal{G}_{D^*} = \mathcal{G}_D$, implying in turn the analogous equation

$$\mathcal{G}_{D^*}(W_k, Z) = \mathcal{G}_D(W_k, Z) \tag{6.3}$$

(although $\mathcal{R}_{D^*}(W_k, Z) \neq \mathcal{R}_D(W_k, Z)$ may hold). (1) now follows from (6.2) and (6.3).

(2). Let $G \in \mathcal{G}_D(W_k, Z)$ be chosen such that it has a perfect matching M. By Definition 7(2), there is a redirection D^* of D such that $G = G_{D^*}$ and $D^* \in \mathcal{R}_D(W_k, Z)$. Thus, by Definition 7(1), $D^* = r_{X^*}(D)$ for some $X^* \subseteq N_{D^*}(W_k) = N_D(W_k)$ and

$$\sigma(D^*) \cap W_k = Z. \tag{6.4}$$

M corresponds to a U–perfect matching M^* of D^*. By definition of G_{D^*} (see Definition 7(2)), no arc in M^* has its tail in $N_{D^*}(W_k)$. For $X' := \{ u \in U : (u, w) \in M^*,\ w \in W \}$ it thus follows of necessity that

$$X' \cap N_{D^*}(W_k) = \emptyset. \tag{6.5}$$

Set $D' := r_{X'}(D)$ and observe that M^* corresponds to a U–perfect admissible matching M' in D'. Since M' is admissible, we have $\sigma(D') \subseteq W_k$. Further, it follows from (6.5) that $\sigma(D^*) \cap W_k = \sigma(D') \cap (W_k)$. Together with (6.4) we obtain $\sigma(D') = Z$. $\square$

The next lemma provides a means for testing satisfiability in terms of $\mathcal{G}_D(\emptyset)$.

Lemma 10 *Let $D = (U, W)$ be bipartite with $|W| - |U| = k \geq 1$. If D is connected and U–elementary, then the following are equivalent.*

(1) There is a redirection D' of D with $\sigma(D') = \emptyset$;
(2) some graph in $\mathcal{G}_D(\emptyset)$ has a perfect matching.

PROOF. *(1)⇒(2).* Let M' be an admissible matching of D' of maximum cardinality, $(D, M) \in \mathcal{DM}(D)$. If $|M'| = |U|$ put $(D^*, M^*) = (D', M')$. Otherwise, by Theorem 2, there is a (D', M')–augmenting path; and by Lemma 6 we obtain a pair $(D'', M'') \in \mathcal{DM}(D)$ with $|M''| > |M'|$ and $\sigma(D'') \subseteq \sigma(D') = \emptyset$. By repeated application of this operation we obtain a pair $(D^*, M^*) \in \mathcal{DM}(D)$ such that M^* is an (admissible) U–perfect matching and $\sigma(D^*) = \emptyset$. In any case, an admissible U–perfect matching exists in some redirection D^* of D such that $\sigma(D^*) = \emptyset$.

Let $W_k \subseteq W$ be the set of vertices which are unmatched by M^*. Now $|W_k| = k$ and we have by Definition 7(2), Lemma 9(1), and Definition 7(3) that

$$G_{D^*} \in \mathcal{G}_D(W_k, \emptyset) = \mathcal{G}_{D^*}(W_k, \emptyset) \subseteq \mathcal{G}_D(\emptyset).$$

Since M^* is an admissible U–perfect matching of D^* in any case, therefore it corresponds to a perfect matching of G_{D^*}.

(2)⇒(1). Choose $G \in \mathcal{G}_D(\emptyset)$ having a perfect matching M, and let $W_k = V(D) - V(G)$; thus $G \in \mathcal{G}_D(W_k, \emptyset)$. By Lemma 9(2), there is a redirection D' of D such that $\sigma(D') = \emptyset$. $\square$

We note in passing that Lemmas 9 and 10 also hold for the case $k = 0$ (which is irrelevant, however, for our subsequent discussion).

Considering the case where Z is a singleton we can use the class $\mathcal{G}_D(Z)$ as the means for testing minimal unsatisfiability.

Lemma 11 *Let $D = (U, W)$ be bipartite with $|W| - |U| = k \geq 1$ and $w \in W$. If D is connected and U–elementary such that $\sigma(D') \neq \emptyset$ for every redirection D' of D, then the following are equivalent.*

(1) There is a redirection D' of D with $\sigma(D') = \{w\}$;
(2) some graph in $\mathcal{G}_D(\{w\})$ has a perfect matching.

PROOF. *(1)⇒(2).* Let M' be an admissible matching of D' of maximum cardinality. Analogously to the first part of the proof of Lemma 10 we obtain a pair (D^*, M^*) such that M^* is an admissible U–perfect matching of D^* and where D^* is a redirection of D' and thus also of D, such that $\sigma(D^*) \subseteq \sigma(D') = \{w\}$. That is, $(D^*, M^*) \in \mathcal{DM}(D)$. Since $\sigma(D^*) \neq \emptyset$ by assumption, $\sigma(D^*) = \{w\}$. Again, let $W_k \subseteq W$ be the set of vertices which are unmatched by M^*; consequently $|W_k| = k$ and $w \in W_k$ since M^* is admissible. Whence, by Lemma 9(1), $G_{D^*} \in \mathcal{G}_{D^*}(W_k, \{w\}) = \mathcal{G}_D(W_k, \{w\}) \subseteq \mathcal{G}_D(\{w\})$, and M^* corresponds to a perfect matching of G_{D^*}.

(2)⇒(1). Let $G \in \mathcal{G}_D(\{w\})$, $w \in W$, with a perfect matching M. It follows that $G \in \mathcal{G}_D(W_k, \{w\})$ for $W_k = V(D) - V(G)$. By Lemma 9(2), there is a redirection D' of D such that $\sigma(D') = \{w\}$. □

7 The Algorithm

We are now in the position to state a polynomial–time algorithm which computes whether a given formula $S \in \mathcal{F}(k)$ is minimal unsatisfiable.

Algorithm MU(k)

Input: A formula $S \in \mathcal{F}(k)$.

Output: **Yes** if S is minimal unsatisfiable; **No** otherwise.

Step 1. If S contains the empty clause and $S \neq \{\emptyset\}$ return **No**. Obtain a formula S_3 from S with $|C| \leq 3$ for all $C \in S_3$ according to Lemma 3. Let D_3 be the atom–clause digraph of S_3.

Step 2. Check whether D_3 is connected; if not, return **No**.

Step 3. Check whether D_3 is U–elementary; if not, return **No**.

Step 4. If an element of $\mathcal{G}_{D_3}(\emptyset)$ has a perfect matching, return **No**.

Step 5. For all $w \in W$, test whether $\mathcal{G}_{D_3}(\{w\})$ contains a graph which has a perfect matching. If such graph exists for all $w \in W$, return **Yes**; otherwise, return **No**.

Theorem 5 *The Algorithm* MU(k) *returns* **Yes** *if and only if $S \in \mathcal{F}(k)$ is minimal unsatisfiable.*

PROOF. Let $S \in \mathcal{F}(k)$. If the unsatisfiable formula $\{\emptyset\}$ is a proper subset of S, then S is not minimal unsatisfiable. By Lemma 3, S_3 is minimal unsatisfiable if and only if S is minimal unsatisfiable. Moreover, by Lemma 7, D_3 must be connected and U–elementary if S_3 is minimal unsatisfiable. Whence, if Step 4 is reached, we may assume that D_3 is connected and U–elementary. By Lemma 2 and Corollaries 1 and 2 it remains to show that

(1) $\sigma(D_3') \neq \emptyset$ for every redirection D_3' of D_3;
(2) for every $w \in W$, there is a redirection D_3' of D_3 with $\sigma(D_3') = \{w\}$.

By Lemma 10, (1) holds if and only if no graph in $\mathcal{G}_{D_3}(\emptyset)$ has a perfect matching (see Step 4). Likewise, by Lemma 11, (2) holds if and only if $\mathcal{G}_{D_3}(\{w\})$ contains a graph which has a perfect matching, for every $w \in W$ (see Step 5). $\square$

Observe that the algorithm and its justification (Theorem 5) make clear that for $j < i$, Step j will not be revisited once Step i has been reached. Moreover, the positive outcome of Step i constitutes a necessary condition for S to be minimal unsatisfiable, for $1 \leq i \leq 5$, provided the necessary conditions tested in Step j, $1 \leq j < i$, are fulfilled by S. Thus Step 5 constitutes also the sufficient condition for minimal unsatisfiability of S. These observations clarify that the complexity of the algorithm is determined by the largest complexity of the reached steps.

Theorem 6 *For a fixed $k \geq 1$, the running time of Algorithm* MU(k) *is $\mathcal{O}(N^{k+3/2})$ where N is the length of the input formula.*

PROOF. Let S be the input formula of length N. Clearly, searching for the empty clause in S and testing whether $S = \{\emptyset\}$ can be done in $\mathcal{O}(N)$ time. By Lemma 3, S_3 can be obtained in time $\mathcal{O}(N)$ and the length of S_3 is $\mathcal{O}(N)$, thus

$$|A(D_3)| = \mathcal{O}(N). \tag{7.6}$$

Since we now may assume D_3 contains no isolates (see Lemma 4(1)), we have $|V(D_3)| \leq 2\,|A(D_3)|$; whence

$$|V(D_3)| = \mathcal{O}(N). \tag{7.7}$$

Consequently, D can be tested for connectedness by depth–first–search in time $\mathcal{O}(N)$. Thus, the time required for Steps 1 and 2 is $\mathcal{O}(N)$.

By Lemma 5 and (7.6), (7.7), we obtain a maximum running time of $\mathcal{O}(N^{5/2})$ to process Step 3.

Considering Step 4 and applying Lemma 8 and (7.7), we first obtain $|\mathcal{G}_D(\emptyset)| = \mathcal{O}(N^k)$. For every $G \in \mathcal{G}_D(\emptyset)$ we have to find a maximum cardinality matching which requires at most $\mathcal{O}(N^{3/2})$ time each (see the discussion preceding Lemma 5). Thus, the running time of Step 4 is $\mathcal{O}(N^{k+3/2})$.

We proceed similarly with respect to Step 5: using, again, Lemma 8 and (7.7), we have $|\mathcal{G}_D(\{w\})| = \mathcal{O}(N^{k-1})$ for every $w \in W$. There are $|W| = \mathcal{O}(N)$ choices for $w \in W$; whence we perform $\mathcal{O}(N^k)$ steps of time complexity $O(N^{3/2})$ each. Thus, the running time of Step 5 is $\mathcal{O}(N^{k+3/2})$.

Since $k \geq 1$, we see that the time needed by each single step is asymptotically bounded by $N^{k+3/2}$; whence (see the observations preceding this theorem) the theorem now follows. □

Thus Conjecture 1 is shown to be true.

8 Concluding remarks

We have presented a polynomial–time algorithm to recognize minimal unsatisfiable formulas in $\mathcal{F}(k)$ for fixed $k \geq 1$. Since every minimal unsatisfiable formula is contained in $\mathcal{F}(k)$ for some $k \geq 1$, the sequence $\mathcal{F}(1), \mathcal{F}(2), \mathcal{F}(3), \ldots$ constitutes a polynomial hierarchy containing all minimal unsatisfiable formulas.

We obtained this algorithm by introducing the new concept of (D, M)–augmenting paths. It turned out, that the strong Hall condition is key for the application of (D, M)–augmenting paths. By our new concept of X–elementarity, we obtained a fast test for this condition.

The structure of minimal unsatisfiable formulas in $\mathcal{F}(1)$ and $\mathcal{F}(2)$ is well known ([8,5]). Interestingly enough, the authors of the quoted papers use the construction preceding Lemma 3 in the opposite direction, in the sense that they increase the cardinality of clauses (and the minimal number of occurrences of atoms).

One may expect that for fixed $k \geq 3$, a deeper analysis of the minimal unsatisfiable formulas in $\mathcal{F}(k)$, or the respective atom–clause digraphs, based on our concepts, will lead to a better understanding of the structure of minimal unsatisfiable formulas in $\mathcal{F}(k)$. Such knowledge might yield faster algorithms for the recognition of minimal unsatisfiable formulas in $\mathcal{F}(k)$, $k \geq 3$.

References

[1] R. Aharoni and N. Linial. Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas. *J. Combin. Theory Ser. A*, 43:196–204, 1986.

[2] C. Berge. Two theorems in graph theory. *Proc. Nat. Acad. Sci. U.S.A.*, 43:882–844, 1957.

[3] J. A. Bondy and U. S. R. Murty. *Graph Theory With Applications.* Macmillan, London, 1976.

[4] S. A. Cook. The complexity of theorem-proving procedures. In *Proc. 3rd Annual Symp. on Theory of Computing*, pages 151–158, Shaker Heights, Ohio, 1971.

[5] G. Davidov, I. Davydova, and H. Kleine Büning. An efficient algorithm for the minimal unsatisfiability problem for a subclass of CNF. *Annals of Mathematics and Artificial Intelligence*, 23:229–245, 1998.

[6] Z. Galil. Efficient algorithms for finding maximal matching in graphs. In *CAAP '83 (L'Aquila, 1983)*, volume 159 of *Lecture Notes in Computer Science*, pages 90–113. Springer Verlag, 1983.

[7] J. E. Hopcroft and R. M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *J. Comput.*, 2:225–231, 1973.

[8] H. Kleine Büning. On subclasses of minimal unsatisfiable formulas. Preprint, 1997.

[9] H. Kleine Büning. An upper bound for minimal resolution refutations. In G. Gottlob, E. Grandjean, and K. Seyr, editors, *CSL'98*, volume 1584 of *Lecture Notes in Computer Science*, pages 171–178. Springer Verlag, 1999.

[10] L. Lovász and M. D. Plummer. *Matching Theory*, volume 29 of *Annals of Discrete Mathematics.* Elsevier Science Publishers, North-Holland, 1986.

[11] B. Meltzer. Theorem-proving for computers: some results on resolution and renaming. *Comp. J.*, 8:341–343, 1966.

[12] C. H. Papadimitriou and D. Wolfe. The complexity of facets resolved. *J. of Computer and System Sciences*, 37:2–13, 1988.

[13] X. Zhao and D. Ding. Two tractable subclasses of minimal unsatisfiable formulas. *Science in China Ser. A*, 42(7):720–731, 1999.

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