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On the Power Assignment Problem in Radio Networks*

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Abstract

Given a finite set S of points (i.e. the stations of a radio network) on a d-dimensional Euclidean space and a positive integer $1 \leq h \leq |S| - 1$, the Min dD h-Range Assignment problem consists of assigning transmission ranges to the stations so as to minimize the total power consumption, provided that the transmission ranges of the stations ensure the communication beween any pair of stations in at most h hops.

Two main issues related to this problem are considered in this paper: the trade-off between the power consumption and the number of hops; the computational complexity of the Min dD h-Range Assignment problem.

As for the first question, we provide a lower bound on the minimum power consumption of stations on the plane for constant h. The lower bound is a function of $|S|$, h and the minimum distance over all the pairs of stations in S. Then, we derive a *constructive* upper bound as a function of $|S|$, h and the maximum distance over all pairs of stations in S (i.e. *the diameter* of S). It turns out that when the minimum distance between any two stations is "not too small" (i.e. well spread instances) the upper bound matches the lower bound. Previous results for this problem were known only for very special 1-dimensional configurations (i.e., when points are arranged on a line at unitary distance) [Kirousis, Kranakis, Krizanc, Pelc 1997].

As for the second question, we observe that the tightness of our upper bound implies that Min 2D h-Range Assignment restricted to well spread instances admits a polynomial time approximation algorithm. Then, we also show that the same approximation result can be obtained for random instances. On the other hand, we prove that for $h = |S| - 1$ (i.e. the *unbounded* case) Min 2D h-Range Assignment is NP-hard and Min 3D h-Range Assignment is APX-complete.

1 Introduction

There are important scenarious in which fixed wired infrastructure, such as the Internet, are not available either because it may not be economically practical or physically possible to provide the necessary infrastructure or because the expediency of the situation does not permit its installation (for instance networks formed by satellites, ships or airplains, or networks connecting rescue teams in case of earthquake or flood).

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In such situations a collection of hosts with wireless network interfaces may form a temporary network without the aid of any established infrastructure or centralized administration. Message communication in such kind of networks takes place by performing *multi-hop* transmissions. This type of wireless networks are known as *ad-hoc (multi-hop) radio networks* [11]. More formally, an ad-hoc radio network [17] (later on, radio network) is a finite set of radio stations located on a geographical region that are able to communicate by transmitting and receiving radio signals. A transmission range is assigned to each station s and any other station t within this range can directly (i.e. by one *hop*) receive messages from s. Communication between two stations that are not within their respective ranges can be achieved by *multi-hop* transmissions.

It is reasonably assumed [17] that the power $P(s)$ required by a station s to correctly transmit data to another station s' must satisfy the inequality

$$\frac{P_s}{d(s,s')^\beta} > \gamma \tag{1}$$

where $d(s,s')$ is the distance between s and s', $\beta \geq 1$ is the *distance-power gradient*, and $\gamma \geq 1$ is the *transmission-quality* parameter. In an ideal environment it holds that $\beta = 2$ but it may vary from 1 to more than 6 depending on the environment conditions of the place the network is located (see [17]).

Given a set S of radio stations, a *range assignment* is a function $r : S \to \mathcal{R}^+$. Any station $s \in S$ can directly transmit data to s' provided that $d(s,s') \leq r(s)$. In order to achieve a transmission range $r(s)$, a station s requires an amount of power determined by Eq. 1. In particular, by setting $\gamma = 1$ and $\beta = 2$, the power P_s must be at least $r(s)^2$. We then define the *cost* of a range assignment r as the overall power required by the network, that is

$$\mathsf{cost}(r) \;=\; \sum_{s\in S}(r(s))^2.$$

The Min dD h-Range Assignment problem consists of finding a minimum cost range assignment for a given set S of radio stations on the d-dimensional Euclidean space provided that the assignment ensures the communication beween any pair of stations in at most h hops. When $h = |S| - 1$ (i.e. the *unbounded case*), the problem will be simply denoted as Min dD Range Assignment. The cost of an optimal solution for Min dD h-Range Assignment for a given instance S is denoted as $\mathsf{opt}_h(S)$.
Though we have assumed $\gamma = 1$ and $\beta = 2$, all the results of this paper can be easily extended to any pair of constants $\gamma > 1$ and $\beta > 1$.

1.1 Previous works

Routing, broadcasting and scheduling problems on radio networks have been the subject of several papers over the last years [2, 7, 6, 12, 17, 22, 23]. Tradeoffs between connectivity and energy consumption have been obtained in [15, 16, 20, 24].

In particular, Kranakis *et al* provide the following

Theorem 1 (The Uniform Chain Case [15]) *Let N be a set of n colinear points at unit distance. It holds that*

- $\mathsf{opt}_h(N) = \Theta\left(n^{\frac{2^{h+1}-1}{2^h-1}}\right)$, *for any constant* h;
- $\mathsf{opt}_h(N) = \Theta\left(\frac{n^2}{h}\right)$, *for any* $h = \Omega(\log n)$.

Furthermore the two above (implicit) upper bounds are constructive.

Up to now this is the only known result for the function $\mathsf{opt}_h(\cdot)$. However, the same authors show that the complexity of MIN dD RANGE ASSIGNMENT depends on the number of dimensions d. They provide a polynomial-time (exact) algorithm for MIN 1D RANGE ASSIGNMENT based on dynamic programming and a polynomial-time approximation algorithm having worst-case performance ratio 2 that works for any dimension. On the other hand, they derive a polynomial-time reduction from MIN VERTEX COVER restricted to planar, 3-degree graphs thus showing that MIN 3D RANGE ASSIGNMENT is NP-hard. Finally, we emphasize that Kirousis *et al*'s reduction works only in the 3-dimensional case: they indeed left the complexity of MIN 2D RANGE ASSIGNMENT as an open problem.

1.2 Our results

1.2.1 General bounds on $\mathsf{opt}_h(\cdot)$

We provide a general lower bound on $\mathsf{opt}_h(\cdot)$, for *any* set of stations on the plane. Given a set of stations S, let us define

$$D(S) = \max\{d(s,s') \mid s,s' \in S\}; \quad \delta(S) = \{d(s,s') \mid s,s' \in S,\ s \neq s'\}$$

Theorem 2 *For any constant* $h > 0$*, and for any set* S *of stations on the plane, it holds that*

$$\mathsf{opt}_h(S) = \Omega(\delta(S)^2 |S|^{1+1/h}),$$

The second result of this paper is an efficient method to derive a solution for any instance of our problem for fixed values of h.

Theorem 3 *For any set of stations* S *on the plane, it is possible to construct in time*[1] $O(h|S|)$ *a feasible range assignment* $r_h(S)$ *such that*

$$\mathsf{cost}(r_h(S)) = O(D(S)^2 |S|^{1/h}),$$

for any constant $h > 0$.

Let us now consider the planar configuration G_n where n stations are placed on a square grid of side $\sqrt{n}$ and the distance between adjacent pairs of stations is 1 (notice that this is the 2-dimensional version of the unit chain case studied in [15] - see Theorem 1). Then, by combining Theorem 2 and Theorem 3, we easily obtain the following optimal bound

[1]The constant hidden by the O notation is linear in h.

$$\mathsf{opt}_h(G_n) = \Theta\left(n^{1+1/h}\right). \tag{2}$$

The square grid configuration is the most regular case of *well-spread* instances. In general, we say that a family $\mathcal{S}$ of planar instances is *well-spread* if, for any $S \in \mathcal{S}$, $\delta(S) \geq cD(S)/\sqrt{|S|}$ (for some positive constant $c > 0$). Notice that the above property is rather natural: informally speaking, in a well-spread instance, any two stations must be not "too close". Beacuse of interference problems, this is the typical situation in radio networks adopted in practice [16, 17]. It turns out that the optimal bound in Eq. 2 holds for any family of well-spread instances. The following corollary is thus an easy consequence of Theorems 2 and 3.

Corollary 1 *Let $\mathcal{S}$ be a family of well-spread instances. For any $S \in \mathcal{S}$, it holds that*

$$\mathsf{opt}_h(S) = \Theta\left(\delta(S)^2|S|^{1+1/h}\right),$$

for any positive constant h.

Beside being interesting in itself, the well-spread concept turns out to be useful to analyse another important family of instances: the *random instances.* It is not hard to show that a family $\mathcal{S}^R$ of uniformly distributed random instances, with high probability, does not satisfy the well-spread property. However, in Section 3.4, we will show that, given a family $\mathcal{S}^R$ of random instances, it is possible to construct a family $\mathcal{S}^W$ of well-spread instances having the following property. For any $S^r \in \mathcal{S}^R$, there is an $S^w \in \mathcal{S}^W$ such that $|S^w| = \Theta(|S^r|)$ and, with high probability, $\mathsf{opt}_h(S^r) = \Theta(\mathsf{opt}_h(S^w))$. This equivalence yields the following result.

Theorem 4 *Let l be any positive real. Let S^r be a set of n stations chosen uniformly and independently at random on a square of side l. Then, with high probability, it holds that*

$$\mathsf{opt}_h(S^r) = \Theta\left(l^2 n^{1/h}\right),$$

for any constant h.

1.2.2 The computational complexity of Min dD h-Range Assignment

As previously observed, Kranakis *et al*'s reduction [15] proving the NP-hardness of Min 3D Range Assignment does not work in the 2-dimensional case. In fact, the reduction starts from a planar orthogonal drawing of a (planar) cubic graph G and replace each edge by a *gadget* of stations drawn in the 3-dimensional space that "simulates" the connection between the two adjacent nodes. In order to preserve pairwise "independence" of the drawing of gadgets, their reduction strongly uses the third dimension left "free" by the planar drawing of G. Our technical contribution here is the construction of a new polynomial-time reduction that works for Min 2D Range Assignment.

Theorem 5 *The* Min 2D Range Assignment *problem is NP-hard.*

Then, we address the question whether the approximation algorithm given by Kirousis *et al* for the Min 3D Range Assignment problem can be significantly improved. More precisely, we ask whether or not the problem admit a *Polynomial-Time Approximation Scheme* (PTAS). We demonstrate the following result.

Theorem 6 *The* MIN 3D RANGE ASSIGNMENT *problem is* APX*-complete thus implying that it does not admit a* PTAS*, unless* P = NP *(see [18] for a formal definition of these concepts).*

The standard method to derive an APX-completeness result for a given optimization problem Π is: i) consider a problem Π' which is APX-hard and then ii) show an *approximation-preserving* reduction from Π' to Π [18]. We emphasize that Kirousis *et al*'s reduction does not satisfy any of these two requirements. In fact, as mentioned above, their reduction is from MIN VERTEX COVER restricted to *planar* cubic graphs which admits a PTAS and therefore cannot be APX-hard (unless P = NP) [4]. Furthermore, it is not hard to verify that their reduction is not approximation-preserving.

In order to achieve our hardness result, we instead consider the MIN VERTEX COVER problem restricted to (non-planar) cubic graphs which is known to be APX-complete [19, 1] and then we show an approximation-preserving reduction from this variant of MIN VERTEX COVER to MIN 3D RANGE ASSIGNMENT. Furthermore, our reduction is "efficient", so we obtain an interesting explicit relationship between the approximability behaviour of MIN VERTEX COVER and that of the MIN 3D RANGE ASSIGNMENT problem. In fact, we can state that if MIN VERTEX COVER on cubic graphs is not $\overline{\rho}$-approximable then MIN 3D RANGE ASSIGNMENT is not $\frac{\overline{\rho}+4}{5}$-approximable.

As for the bounded case, we emphasize that the lower bound obtained in Theorem 2 holds for any instance, so the constructive (and efficient) method of Theorem 3 and the equivalence yielding Theorem 4 easily imply the following result. Let Av-APX be the class of optimization problems (together with a probability function on the instance set) that admit a polynomial time algorithm that, with high probability, returns a feasible solution having performance ratio bounded by a fixed constant [3].

Corollary 2

- *Let $\mathcal{S}$ be any family of well-spread instances. Then, the* MIN 2D h-RANGE ASSIGNMENT *problem restricted to $\mathcal{S}$ admits a polynomial-time approximation algorithm with constant performance ratio (i.e. the restriction is in* APX*).*
- *The* MIN 2D h-RANGE ASSIGNMENT *problem (with uniform instance probability) is in* Av-APX*.*

The following table summarizes the complexity results obtained in this paper.

Problem version	**Previous results**	**Our results**
MIN 1D RANGE ASSIGNMENT	in P [15]	-
MIN 2D RANGE ASSIGNMENT	in APX [15]	NP-complete
MIN 2D h-RANGE ASSIGNMENT, well-spread	-	in APX
MIN 2D h-RANGE ASSIGNMENT	-	in Av-APX
MIN 3D RANGE ASSIGNMENT	NP-complete, in APX[15]	APX-complete

2 Preliminaries

Let $S = \{s_1, \ldots, s_n\}$ be a set of n points (representing stations) of an Euclidean space $\mathcal{E}$ with distance function $d : \mathcal{E}^2 \to \mathcal{R}^+$, where $\mathcal{R}^+$ denotes the set of non negative reals. We define

$$\delta(S) = \min\{d(s_i, s_j) \mid s_i, s_j \in S, i \neq j\}$$

and

$$D(S) = \max\{d(s_i, s_j) \mid s_i, s_j \in S\}.$$

A *range assignment* for S is a function $r : S \to \mathcal{R}^+$, and the *cost* of r is defined as

$$\mathsf{cost}(r) \;=\; \sum_{i=1}^{n} r(s_i)^2.$$

The *communication graph* of a range assignment r is the directed graph $G_r(S, E)$ where $(s_i, s_j) \in E$ if and only if $r(s_i) \geq d(s_i, s_j)$. We say that an assignment r for S is *feasible* if the corresponding communication graph is strongly connected. Given a set S of n points in a d-dimensional Euclidean space, the MIN dD RANGE ASSIGNMENT problem consists of finding a feasible range assignment r_{opt} for S of minimum cost.

We say that an assignment r for S is of diameter h ($1 \leq h \leq n-1$) if the corresponding communication graph is strongly connected and has diameter h (in short, an *h-assignment*). The cost of an optimal h-assignment for a given set S of stations in the plane is denoted as $\mathsf{opt}_h(S)$.

In the proof of our results we will make use of the well-known Hölder inequality. We thus present it in the following convenient form. Let x_i, $i = 1, \ldots, k$ be a set of k non negative reals and let $p, q \in \mathcal{R}$ such that $p \geq 1$ and $q \leq 1$. Then, it holds that:

$$\sum_{i=1}^{k} x_i^p \geq k \left(\frac{\sum_{i=1}^{k} x_i}{k} \right)^p; \tag{3}$$

$$\sum_{i=1}^{k} x_i^q \leq k \left(\frac{\sum_{i=1}^{k} x_i}{k} \right)^q. \tag{4}$$

The MIN VERTEX COVER problem is to find a subset K of the set V of vertices of a graph $G(V, E)$ such that K contains at least one endpoint of any edge in E and $|K|$ is as small as possible. MIN VERTEX COVER is known to be NP-hard even when restricted to planar cubic graphs [13]. Moreover, it is known to be APX-complete when restricted to cubic graphs [19, 1]. It follows that a constant $\overline{\rho} > 1$ exists such that MIN VERTEX COVER restricted to cubic graphs is not $\overline{\rho}$-approximable unless P = NP.

Given a graph G a *planar orthogonal grid drawing* is a drawing of G such that

1. Each vertex is represented as a point in the plane with integer coordinates;

2. Edges are represented as chains of horizontal and vertical segments (i.e. *polyline*) connecting the two endpoints and whose bends have integer coordinates;

3. Every polyline (representing an edge) crosses neither other polylines nor points representing vertices.

A drawing is said to be *straight-line* if all the edges are represented by one segment connecting the endpoints. Finally, a *3-dimensional orthogonal grid drawing* is a generalization to the 3-dimensional Euclidean space of planar orthogonal grid drawings.

3 The Bounded Hops Case

3.1 The Lower Bound

Given a set S of stations and a "base" station $b \in S$, we define $\mathsf{opt}_h(S,b)$ as the minimum cost of any range assignment ensuring that any station $s \in S$ can reach b in at most h hops. By the definition of the MIN 2D h-RANGE ASSIGNMENT problem, it should be clear that the cost required by any instance S of MIN 2D h-RANGE ASSIGNMENT is at least $\mathsf{opt}_h(S,b)$, for any $b \in S$. So, the main result of this section is an easy consequence of the following lemma.

Lemma 1 *Let S be any set of stations such that $\delta(S) = 1$. For every $b \in S$ and every positive constant integer h, it holds that*

$$\mathsf{opt}_h(S,b) = \Omega(|S|^{1+1/h}).$$

PROOF. We first observe that, since $\delta(S) = 1$, the maximum number of stations contained in a disk of radius $R = \sqrt{|S|/3}$ is at most $|S|/2$.
Let $r_h^{all-to-one}$ be a range assignment that ensures that all the stations in S can reach b in at most h hops. We prove that $\mathsf{cost}(r_h^{all-to-one}) = \Omega(|S|^{1+1/h})$ by induction on h.
For $h = 1$, consider the disk of radius R and centered in b. By the above observation, there are at least $|S|/2$ stations at distance greater than R from b. The cost required by such stations to reach b in one hop is at least

$$(|S|/2)R^2 = \Omega(|S|^2).$$

Let $h \geq 2$, we define

$$FAR = \{s \in S \mid d(s,b) > R\}.$$

Clearly, we have that $|FAR| \geq |S|/2$. Every station s in FAR must reach b in $k \leq h$ hops, it thus follows that there exist $k \leq h$ positive reals $x_1, \ldots, x_k$ (where x_i is the distance covered by the i-th hop of the communication from s to b) such that

$$x_1 + x_2 + \cdots + x_k \geq R.$$

So, at least one index j exists for which $x_j \geq R/k \geq R/h$. We can thus define the set of "bridge" stations

$$B = \{s \in S \mid r_h^{all-to-one}(s) \geq R/h)\}.$$

Two cases may arise.

Case $|B| \geq |S|^{\frac{1}{h}}$. In this case, since $|R| = \sqrt{|S|/3}$,

$$\begin{aligned}\sum_{s\in S}(r_h^{all-to-one}(s))^2 &\geq |B|(R/h)^2\\ &\geq \frac{1}{3h^2}|S|^{1+\frac{1}{h}}\\ &= \Omega\left(|S|^{1+\frac{1}{h}}\right).\end{aligned}$$

Case $|B| < |S|^{\frac{1}{h}}$. By means of the assignment $r_h^{all-to-one}$, every station in FAR reaches in at most $h-1$ hops some bridge station. Let $B = \{b_1, \ldots, b_{|B|}\}$. So, we can partition the set $FAR \cup B$ into $|B|$ subsets $A_1, \ldots, A_{|B|}$ such that all the stations in A_i reach b_i in at most $h-1$ hops. Notice that if a station reaches two or more bridge stations, we can put the station into any of the corresponding set A_i's. We must also guarantee that if a station s is put into A_i then *all* the stations in the path from s to B_i are put into A_i as well. We also assume that $b_i \in A_i$, for $1 \leq i \leq |B|$. So,

$$\begin{aligned}\sum_{s\in S}(r_h^{all-to-one}(s))^2 &\geq \sum_{i=1}^{|B|}\mathsf{opt}_{h-1}(A_i, b_i)\\ &= \Omega\left(\sum_{i=1}^{|B|}|A_i|^{1+\frac{1}{h-1}}\right)\end{aligned}$$

where the last bound is a consequence of the inductive hypothesis. Since

$$\sum_{i=1}^{|B|}|A_i| = |FAR \cup B| \geq |S|/2,$$

the Hölder inequality (see Eq. 3) implies that

$$\begin{aligned}\sum_{i=1}^{|B|}|A_i|^{1+\frac{1}{h-1}} &\geq |B|\left(\frac{|S|/2}{|B|}\right)^{1+\frac{1}{h-1}}\\ &= \Omega\left(\left(\frac{1}{|B|}\right)^{\frac{1}{h-1}}|S|^{1+\frac{1}{h-1}}\right)\\ &= \Omega\left(|S|^{1+\frac{1}{h}}\right)\end{aligned}$$

where the last equivalence is due to the condition $|B| < |S|^{\frac{1}{h}}$.

□

Proof of Theorem 2.
For $\delta(S) = 1$, the theorem is an immediate consequence of Lemma 1. The general case $\delta(S) > 0$ can be reduced to the previous case by simply rescaling the instance by a factor of $1/\delta(S)$.

3.2 The Upper Bound

Proof of Theorem 3.
The proof consists of a recursive construction of an h-assignment $r_h(S)$ having cost $O(D(S)^2|S|^{1/h})$. For $h = 1$, $r_1(S)$ assigns a range $D(S)$ to each station in S. Thus, $\mathsf{cost}(r_1(S)) = D(S)^2|S|$.

Let us consider the smallest square Q that contains all points in S. Notice that the side l of Q is at most $D(S)$. Let us consider a grid that subdivides Q into k^2 subsquares of the same size l/k (the choice of k will be given later).

Informally speaking, for every non empty subsquare we choose a "base" station and we give power sufficient to let it cover all the stations in S in one hop. Then, in every subsquare we complete the assigment by making any station able to reach the base station in $h-1$ hops. For this task we apply the recursive construction.

The cost of $r_h(S)$ is thus bounded by

$$\mathsf{cost}(r_h(S)) \leq k^2 D(S)^2 + \sum_{i=1}^{k^2} \mathsf{cost}(r_{h-1}(S_i)),$$

where S_i is the set of the stations in the i-th subsquare. Since $D(S_i) = O(D(S)/k)$ we apply the inductive hypothesis and we obtain

$$\begin{aligned}
\mathsf{cost}(r_h(S)) &= O\left(k^2 D(S)^2 + \sum_{i=1}^{k^2} |S_i|^{1/(h-1)} \left(\frac{D(S)}{k}\right)^2\right) \\
&= O\left(k^2 D(S)^2 + \left(\frac{D(S)}{k}\right)^2 \sum_{i=1}^{k^2} |S_i|^{1/(h-1)}\right) \\
&= O\left(k^2 D(S)^2 + \left(\frac{D(S)}{k}\right)^2 k^2 \left(\frac{|S|}{k^2}\right)^{1/(h-1)}\right),
\end{aligned}$$

where the last equality follows from the Hölder inequality (see Eq. 4) and from the fact that $\sum_{i=1}^{k^2} |S_i| = |S|$. Now we choose

$$k = |S|^{\frac{1}{2h}}$$

in order to equate the additive terms in the last part of the above equation. By replacing this value in the equation we obtain

$$\mathsf{cost}(r_h(S)) = O\left(D(S)^2 |S|^{1/h}\right).$$

It is easy to verify that the partition of Q into k^2 subsquares and the rest of the computation in each inductive step can be done in time $O(|S|)$. So, the overall time complexity is $O(h|S|)$.

□

3.3 Tight Bounds and Approximability

Let us consider the simple instance G_n of MIN 2D h-RANGE ASSIGNMENT in which n stations are placed on a square grid of side $\sqrt{n}$, and the distance between adjacent pairs of stations is 1.

By Combining Theorems 2 and 3, we easily obtain that

$$\mathsf{opt}_h(G_n) = \Theta\left(n^{1+1/h}\right).$$

This also implies that the range assignment constructed in the proof of Theorem 3 yields a constant-factor approximation.

It turns out that the above considerations can be extended to any "well-spread" instance.

Definition 1 *A family $\mathcal{S}$ of* well-spread *instances is a family of instances S such that $D(S) = O(\delta(S)\sqrt{|S|})$.*

The following two corollaries are easy consequences of Theorems 2 and 3.

Corollary 3 *Let $\mathcal{S}$ be a family of well-spread instances. For any $S \in \mathcal{S}$, it holds that*

$$\mathsf{opt}_h(S) = \Theta\left(\delta(S)^2|S|^{1+1/h}\right),$$

for any positive integer constant h.

Corollary 4 *Let $\mathcal{S}$ be any family of well-spread instances. Then, the* MIN 2D h-RANGE ASSIGNMENT *problem restricted to $\mathcal{S}$ is in* APX*, for any positive integer constant h.*

3.4 Random Instances

Theorem 7 *Let S^R be a set of n stations chosen uniformly and independently at random on a square of side l. Then, with high probability, it holds that*

$$\mathsf{opt}_h(S^R) = \Theta\left(l^2 n^{1/h}\right), \tag{5}$$

for any positive integer constant h.

PROOF. The upper bound is an immediate consequence of Theorem 3. As for the lower bound, given S^R, we construct an instance S^w as follows: consider a partition of the $l \times l$ square $\mathcal{Q}$ into a grid of $\sqrt{n} \times \sqrt{n}$ cells of side $u = l/\sqrt{N}$. The instance S^w consists of the set of stations located at the center of every non-empty cell. It is easy to verify that S^w is well-spread (indeed, $\delta(S^w) \geq u$). An easy application of the well known *occupancy problem* analysis [21] shows that there is a constant $\gamma > 0$ such that, w.h.p. $|S^w| \geq \gamma n$. Observe that Theorem 2 implies the lower bound in Eq. 5 for S^w. So, we will prove the theorem by showing that $\mathsf{opt}_h(S^R) = \Omega(\mathsf{opt}_h(S^w))$.

Let us consider a (feasible) h-assignment r^R for S^R. The corresponding h-assignment for S^w is defined as follows. For any $s \in S^w$,

$$r^w(s) = \sqrt{2}u + \max\{r^R(t) \mid t \in c_s\} \tag{6}$$

where c_s is the cell containing s. Since r^R is an h-assignment for S^R then it is easy to see that r^w is an h-assignment for S^w. We now need the following

Claim 1 *Two constants $\alpha > 0$ and $\beta > 0$ exist such that, with high probability (*w.h.p.*), at least $\beta|S^w|$ cells contain a station t of S^R with $r^R(t) \geq \alpha u$.*

PROOF OF CLAIM 1. In order to prove the above claim we consider *super-cells*, i.e. squares of 9 cells and of side $3u$. We then say that a supercell is *occupied* if it contains at least one station of S^R in the central cell;

We will now show that if there exist at least m occupied cells then it is possible to find at least $m/25$ pairwise disjoint supercells. In fact, assume on the contrary that the maximum number of occupied pairwise disjoint supercells that can be defined is less than $m/25$. Let us consider a set of pairwise disjoint supercells of maximum size and the set $\mathcal{M}$ of the central cells of such supercells. Then, the number of cells contained into the radius 2 neighbourhood (Manhattan distance) of some cell in $\mathcal{M}$ is less than m. So, at least one further disjoint occupied supercell can be found thus contraddicting the assumption. From the above fact, we can state that, w.h.p., there are at least $(\gamma/25)n$ pairwise disjoint occupied supercells.

We say that a supercell C is *bad* if it is occupied and no station t in C exists with $r^R(t) \geq \alpha u$ (the choice of α is given later) while a supercell is *good* if it is occupied and is not bad. We define the set BAD ($GOOD$) as the set of all the bad (good) supercells. Our next goal is to prove that $|BAD| \leq \alpha n$. We assume that there exist at least two distinct occupied supercells (this happens w.h.p.). Let C be a bad supercell and C' be another occupied supercell. Since the "central" station in C must communicate to any station in C', we have that

$$\sum_{s \in C} r^R(s) \geq u;$$

Since C is bad, we obtain that $|C| = |\{s \in S^R | s \in C\} \geq \frac{u}{\alpha u} = \frac{1}{\alpha}$. We thus have that $|BAD| \leq \alpha n$. By choosing $\beta = \alpha = \gamma/50$, we have that, w.h.p., $|GOOD| \geq \beta n$. □

For any $s \in S^w$, let $c(s)$ be the cell corresponding to s, and let $\max(s)$ be the maximum range $r^R(t)$ over all stations $t \in S^R$ such that $t \in c(s)$. From Claim 1, we have that:

$$\begin{aligned}
\mathsf{cost}(r^w(S^w)) &= \sum_{s \in S^w} r^w(s)^2 \\
&= \sum_{s \in S^w} \left(\sqrt{2}u + \max\left(c(s)\right)\right)^2 \\
&= \sum_{s \in S^w : \max(c(s)) < \alpha u} \left((1+\sqrt{2})u\right)^2 + \sum_{s \in S^w : \max(c(s)) \geq \alpha u} \left((1+\frac{\sqrt{2}}{\alpha})\max(c(s))\right)^2 \\
&\leq 2 \cdot \sum\nolimits_{s \in S^w : \max(c(s)) \geq \alpha u} \left((1+\frac{\sqrt{2}}{\alpha})\max(c(s)\right)^2 \\
&\leq 2\left(1+\frac{\sqrt{2}}{\alpha}\right)^2 \sum_{s \in S^R} r^R(s)^2
\end{aligned}$$

□

4 Min 2D Range Assignment is NP-hard

We will show a polynomial-time reduction from Min Vertex Cover restricted to planar, cubic graphs to Min 2D Range Assignment.

We first outline which step have to be performed in order to derive an instance $S(G)$ of Min 2D Range Assignment corresponding to a planar at most cubic graph G. To this aim, we will make use of an intermediate representation of G, by means of a planar orthogonal grid drawing $D(G)$ of it. This intermediate step will make the construction of $S(G)$ simpler. The whole construction will basically take these steps:

1. Construct a planar orthogonal grid drawing of G;
2. Add two new vertices for each bend of the drawing so to obtain a straight-line drawing $D(G)$;
3. Replace each straight-line (edge) in $D(G)$ with a suitable set of stations (gadget).

Notice that in order to obtain a polynomial time reduction we need to perform all the above steps in polynomial time. Moreover, in the second step, we have to preserve the optimality of the vertex cover solutions between G and the new graph represented by $D(G)$. As we will see in Section 4.2, if $2h$ is the number of vertices added by this operation, then G has a vertex cover of size k if and only if $D(G)$ has a vertex cover[2] of size $k + h$. Finally, in the third step, further vertices will be added in $D(G)$ still preserving the above relationship between the vertex covers of G and those of $D(G)$.

In the next section we provide the key properties of these gadgets and the reduction to Min 2D h-Range Assignment that relies on such properties. The detailed construction of the 2-dimensional gadgets is instead given in Section 4.2.

4.1 The Properties of the 2-Dimensional Gadgets and the Reduction

The type of gadget used to replace one edge of $D(G)$ depends on the local "situation" that occurs in the drawing (for example it depends on the degree of its endpoints). However, we can state the properties that characterize any of these gadgets.

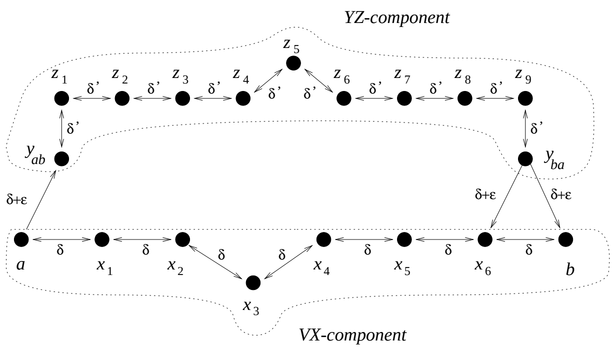

Figure 1: An example of a 2-dimensional gadget and a canonical assignment for it.

[2]In what follows, we will improperly use $D(G)$ to denote both the drawing and the graph it represents.

Definition 2 (Gadget Properties) *Let $\delta, \delta', \epsilon \geq 0$ such that $\delta + \epsilon > \delta'$ and $\alpha > 1$ (a suitable choice of such parameters will be given later). For any edge (a, b) the corresponding gadget g_{ab} contains the sets of points $X_{ab} = \{x_1, \ldots, x_{l_1}\}$, $Y_{ab} = \{y_{ab}, y_{ba}\}$, $Z_{ab} = \{z_1, \ldots, z_{l_2}\}$ and $V_{ab} = \{a, b\}$, where l_1 and l_2 depend on the length of the drawing of (a, b). These sets of points are drawn in $\mathcal{R}^2$ so that the following properties hold:*

1. *$d(a, y_{ab}) = d(b, y_{ba}) = \delta + \epsilon$.*
2. *X_{ab} is a chain of points drawn so that*
$$d(a, x_1) = d(x_1, x_2) = \cdots = d(x_{l_1-1}, x_{l_1}) = d(x_{l_1}, b) = \delta$$
and, for any $i \neq j$, $d(x_i, x_j) \geq \delta$.
3. *Z_{ab} is a chain of points drawn so that*
$$d(y_{ab}, z_1) = d(z_1, z_2) = \cdots = d(z_{l_2-1}, z_{l_2}) = d(z_{l_1}, y_{ba}) = \delta'$$
and, for any $i \neq j$, $d(z_i, z_j) \geq \delta'$.
4. *For any $x_i \in X_{ab}$ and $z_j \in Z_{ab}$, $d(x_i, z_j) > \delta + \epsilon$. Furthermore, for any $i = 1, \ldots, l_1$, $d(x_i, y_{ab}) \geq \delta + \epsilon$ and $d(x_i, y_{ba}) \geq \delta + \epsilon$.*
5. *Given any two different gadgets g_{ab} and g_{cd}, for any $v \in g_{ab} \setminus g_{cd}$ and $w \in g_{cd} \setminus g_{ab}$, we have that $d(v, w) \geq \delta$ and if $v \notin V_{ab} \cup X_{ab}$ or $w \notin V_{cd} \cup X_{cd}$ then $d(v, w) \geq \alpha\delta$.*

From the above definition, it turns out that the gadgets consist of two components whose relative distance is $\delta + \epsilon$: the *VX-component* consisting of the "chain" of points in $X_{ab} \cup V_{ab}$, and the *YZ-component* consisting of the chain of points in $Y_{ab} \cup Z_{ab}$.

Let $S(G)$ be the set of points obtained by replacing each edge of $D(G)$ by one gadget having the properties described above.

Note 1 *Let r^{min} be the range assignment of $S(G)$ in which every point in VX and in YZ have range δ and δ', respectively (notice that this assignment is not feasible). The corresponding communication graph consists of $m + 1$ strongly connected components, where m is the number of edges: the YZ-components of the m gadgets and the union $\mathcal{U}$ of all the VX-components of the gadgets. It thus follows that, in order to achieve a feasible assignment, we must define the "bridge-point" between $\mathcal{U}$ and every YZ-component.*

The above note leads us to define the following *canonical* (feasible) solutions for $S(G)$.

Definition 3 (Canonical Solutions for $S(G)$) *A range assignment r for $S(G)$ is* canonical *if, for every gadget g_{ab} of $S(G)$, the following properties hold.*

1. *Either $r(y_{ab}) = \delta + \epsilon$ and $r(y_{ba}) = \delta'$ (so, y_{ab} is a radio "bridge" from the YZ-component to the VX one) or vice versa.*
2. *For every $v \in \{a, b\}$, either $r(v) = \delta$ or $r(v) = \delta + \epsilon$. Furthermore, there exists $v \in \{a, b\}$ such that $r(v) = \delta + \epsilon$ (so, v is a radio "bridge" from the VX-component to the YZ one).*

3. *For every $x \in X_{ab}$, $r(x) = \delta$.*

4. *For every $z \in Z_{ab}$, $r(z) = \delta'$.*

Informally, our reduction is based on the following ideas:

1. If we minimize the number of "bridge" stations in the V-components then we minimize the overall cost of any canonical solution (observe that the cost of all the X- and YZ-components is fixed);

2. The graph $D(G)$ has a vertex cover of size k if and only if there exists a canonical solution for $S(G)$ with k "bridge" stations of type V;

3. Any non-canonical feasible solution can be transformed in polynomial time into a canonical one without paying any extra cost (notice that any canonical assignment is feasible).

In the remaining of this section we will formally prove the above statements.

Lemma 2 *Let us consider the construction $S(G)$ in which α, δ and ϵ are three positive constants such that*

$$\alpha^2\delta^2 \;>\; (m-1)[(\delta+\epsilon)^2 - \delta^2] + (\delta+\epsilon)^2. \tag{7}$$

Then, for any feasible range assignment r for $S(G)$, there is a canonical range assignment r^c such that $\mathsf{cost}(r^c) \leq \mathsf{cost}(r)$.

PROOF. Under the condition of the lemma, from any non canonical range assignment r, we will derive an iterative process that yields a canonical range assignment r^c such that

$$\mathsf{cost}(r^c) \leq \mathsf{cost}(r)\ .$$

The number of iterations is at most the number of points that have a non canonical range assignment in r. Let us describe the generic step of this iterative process.

Since r is non canonical, there exists a point u for which at least one property of Definition 3 is not satisfied. We distinguish two cases.

a) **(*Local Transformation.*)** The transmission range $r(u)$ is smaller than $\alpha\delta$ (this implies that u has power not sufficiently large to reach points of the YZ-component of other gadgets). Notice that if $r(u) < \delta + \epsilon$ then u cannot be the "bridge" between the VX-component and the YZ-component of the gadget: in this case, we can easily make $r(u)$ canonical without increasing the overall cost. We can thus assume that $r(u) \geq \delta + \epsilon$. In this case, we prove that the cost difference between r and r^c is at least

$$\mathsf{cost}(r) - \mathsf{cost}(r^c) \geq r(u)^2 - (\delta+\epsilon)^2.$$

So, the difference is non negative. In order to prove the above inequality, we analyze three subcases.

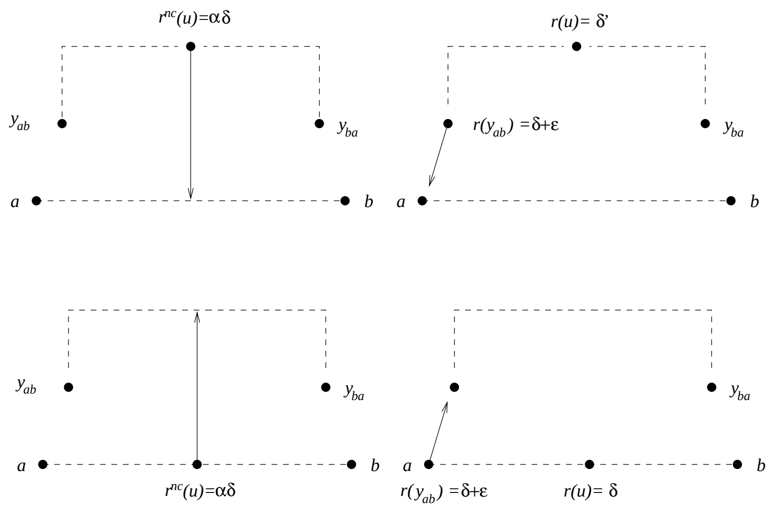

Figure 2: The local transformation.

1. $u \in Z_{ab}$. Set $r^c(u) = \delta'$ and set the range of one point from Y_{ab} (say y_{ab}) to $\delta + \epsilon$. Since in any feasible solution the range of y_{ab} is at least δ', we obtain

$$r(u)^2 + r(y_{ab})^2 - r^c(u)^2 - r^c(y_{ab})^2 \geq r(u)^2 - (\delta + \epsilon)^2 \ .$$

2. $u \in X_{ab}$. Set $r^c(u) = \delta$ and set the range of one point from V_{ab} (say a) to $\delta + \epsilon$. Since in any feasible solution the range of a is at least δ, we obtain

$$r(u)^2 + r(a)^2 - r^c(u)^2 - r^c(a)^2 \geq r(u)^2 - (\delta + \epsilon)^2 \ .$$

3. $u \in V_{ab} \cup Y_{ab}$. Simply set $r^c(u) = \delta + \epsilon$. Then

$$\mathsf{cost}(r) - \mathsf{cost}(r^c) = r(u)^2 - (\delta + \epsilon)^2 \ .$$

After this change, if it is the case that both y_{ab} and y_{ba} have range $\delta + \epsilon$, then we reduce one of them to δ'.

Other non canonical cases may arise but once a two-way transmission bridge is guaranteed by the assignment, there is no further reason to give larger transmission range to the stations of a fixed gadget unless they are used to reach other gadgets: this is the next case.

b) (***Global Transformation.***) Let $r(u) \geq \alpha\delta$. We transform r in two steps: i) Locally change the range assignment so that the range assignment of u is canonical; ii) Canonically assign the new ranges to the stations of those gadgets g' that were covered by u. The first step is made according to Case a). Thus this step always reduces the cost of r by

$$\mathsf{cost}(r) - \mathsf{cost}(r^c) \geq r(u)^2 - (\delta + \epsilon)^2 \geq (\alpha\delta)^2 - (\delta + \epsilon)^2. \tag{8}$$

As for the second step, it might happen that when we give the canonical assignment to u, there is some gadget g' (previously covered by $r(u)$) corresponding to the edge (a', b') whose $X_{a'b'}$ component is not anymore strongly connected to the $Z_{a'b'}$ component of g'. We then assign $r^c(a') = \delta + \epsilon$. Since $r(a') \geq \delta$ and since the number of gadgets covered by $r(u)$ is at most $m - 1$, the overall cost increment due to this change is bounded by

$$(m-1)[(\delta+\epsilon)^2 - \delta^2].$$

Eq.s 7 and 8 thus imply $\mathsf{cost}(r^c) \leq \mathsf{cost}(r)$.

□

We now assume that $S(G)$ satisfies the hypothesis of Lemma 2.

Lemma 3 *Given any planar cubic graph $G(V, E)$, assume that it is possible to construct the set of points $S(G)$ in the plane in time polynomial in the size of G. Then* Min Vertex Cover *is polynomial-time reducible to* Min 2D Range Assignment.

Proof. Let us consider the graph $D(G)$ and let us denote by V' and E' its set of vertices and edges, respectively. By construction, G has a vertex cover of size k if and only if $D(G)$ has a vertex cover of size $k + h$, where $2h$ is the number of new vertices added to G in the construction of $D(G)$. We can therefore consider the problem of finding an optimum vertex cover of $D(G)$. From Lemma 2, we can restrict ourselves to canonical solutions of Min 2D Range Assignment.

Given any vertex cover $K \subseteq V'$ for $D(G)$, we consider the canonical solution r_K for $S(G)$ where every $v \in K$ has range $\delta + \epsilon$ and every $w \in V' \setminus K$ has range δ. Further, the range assignment r_K for all the other points is made according to the definition of the canonical solution (notice that the cost of this part of the assignment is fixed). The cost $\mathsf{cost}(r_K)$ is given by

$$\mathsf{cost}(r_K) = |K|(\delta+\epsilon)^2 + (N_{S(G)} - |K|)\delta^2 + M_{S(G)} \,, \tag{9}$$

where

$$N_{S(G)} = |V'| + \left| \bigcup_{(a,b)\in E'} X_{ab} \right|$$

and

$$M_{S(G)} = \sum_{(a,b)\in E'} \left[(|YZ_{ab}| - 1)(\delta')^2 + (\delta+\epsilon)^2 \right] \,.$$

Notice that $M_{S(G)}$ is the overall power given to the points of type Z and Y in any canonical assignment for $S(G)$.

On the other hand, let r_K be any canonical solution for $S(G)$ and let κ be the number of points of type V whose range is $\delta + \epsilon$. Then, the cost of such a solution is

$$\kappa(\delta+\epsilon)^2 + (N_{S(G)} - \kappa)\delta^2 + M_{S(G)}.$$

Let K be the set of κ vertices of $D(G)$ corresponding to those stations whose solution r_K assigns range $\delta+\epsilon$. We now prove that K is a vertex cover. Suppose by contradiction that some edge (a, b) is not covered (i.e. both a and b are not in K). In the solution r_K, we should have $r_K(a) = r_K(b) = \delta$, thus contradicting the fact that r_K is canonical.

□

4.2 The Construction of the 2-Dimensional Gadgets

This section is devoted to the construction of the 2-dimensional gadgets that allow us to obtain the point set $S(G)$ corresponding to a given planar cubic graph G.

Definition 4 (Construction of $S(G)$) *Let $G(V, E)$ be a planar cubic graph, then the set of points $S(G)$ is constructed as follows:*

1. *Construct a planar orthogonal grid drawing of G with at most one bend per edge and polynomial area using the polynomial time algorithm in [14]* [3].
2. *For any edge represented by a polyline with one bend, add two new vertices so that any edge is represented with a straight line segment.*
3. *Starting from the obtained graph $D(G)$, replace its edges with the gadgets satisfying Definition 2 and Eq. 7. This step may require further vertices to be added to $D(G)$ while preserving the relationship between the vertex cover solutions.*

Let us first observe that G has a vertex cover of size k if and only if $D(G)$ has a vertex cover of size $k + h$, where $2h$ is the number of new vertices added in the last two steps. As we will see in the sequel h is polynomially bounded in the size of G. We can therefore consider the problem of finding a minimum vertex cover for $D(G)$.

During the third step of the construction, it is required to preserve Property 5 of Definition 2, i.e., points from different gadgets are required to be within distance at least $\alpha\delta$. Informally speaking, the main technical problem is drawing the Z-chains corresponding to incident edges so that the properties of Definition 2 hold. To this aim, we adopt a set of suitable construction rules that are described in the sequel. The correctness will be given in the next section.

R1) **(*Simple Chains.*)** We first consider the two simple cases in which a chain of three vertices a, b and c in $D(G)$ is drawn so that either: (i) the points lay on the same line; (ii) edge (a, b) is orthogonal to edge (b, c). It is easy to see that the two gadgets drawn as in Fig. 3 (dash lines represent X- and Z-chains) satisfy the properties of Definition 2.

R2) **(*Alternated Chains.*)** Let us now consider two slightly more complex situations. In the first one we have four vertices a, b, c and d as in Fig. 4(a). Notice that we cannot follow rule $R1$ to correctly place the gadget of (b, c). Indeed, the presence of gadget g_{ab} requires the Z_{bc}-chain to be placed to the right of X_{bc}. Similarly, gadget g_{cd} imposes the Z_{bc}-chain to be drawn to the left of X_{bc}. We solve this problem by adding four new vertices (namely b', b'', c'' and c') between b and c and by "splitting" the X_{bc}- and the Z_{bc}-chains into five chains

[3] We make use of the on-bend algorithm in [14] just because this will make the presentation easier. Actually any polyline orthogonal drawing algorithm such as the one in [25], would be suitable.

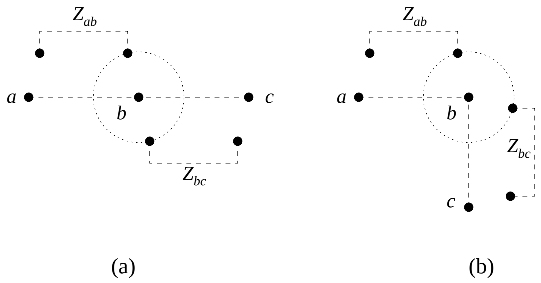

Figure 3: Two basic rules to place gadgets.

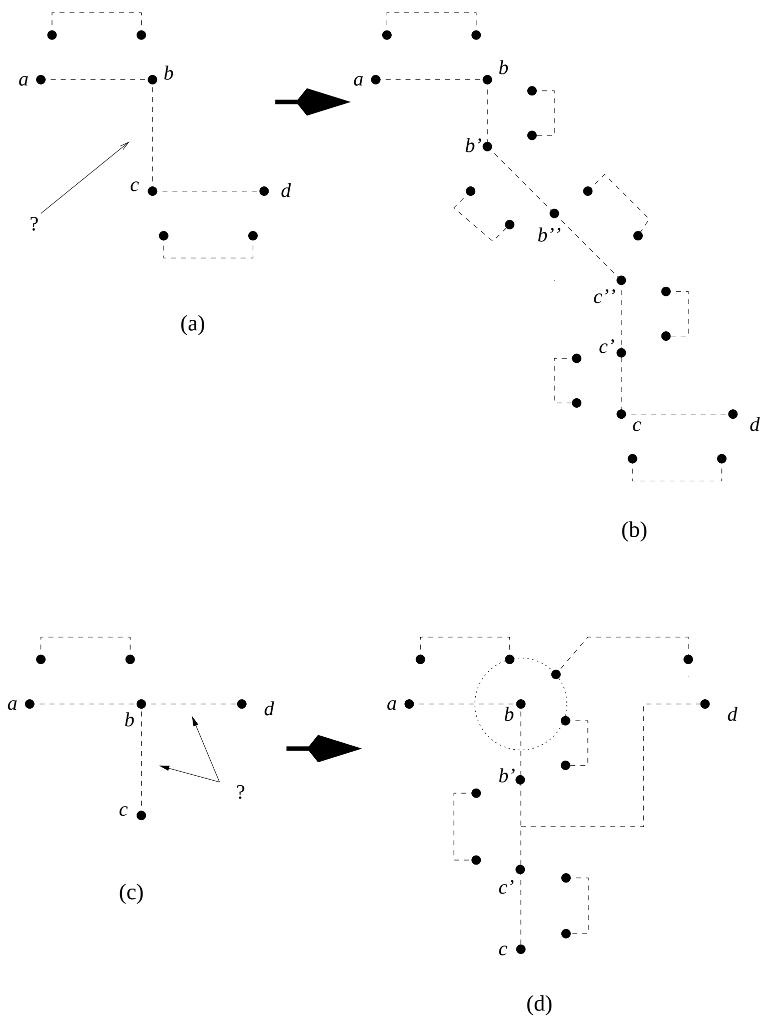

Figure 4: How to construct gadgets: (a) a chain of four vertices in $D(G)$; (b) the corresponding gadgets; (c) a node of degree three in $D(G)$; (d) the gadgets of nodes of degree three.

as shown in Fig. 4(b). Let us observe that this transformation is equivalent to perform the following steps:

- Add to $D(G)$ the four new vertices b', b'', c'' and c' between b and c.
- Modify $D(G)$ so that b' is moved to the bottom and to the right by one unit with respect to b. Similarly, move b'' to the right and to the bottom by two units. Finally, place all of the remaining points of $D(G)$ so that edges (b, b'), (c'', c') and (c', c) are drawn as vertical segments, and edges (b', b''), (c'', c') and (c', c) are represented as slanted segments. Notice that in this way we can also keep the grid requirement also for the new added vertices.
- Replace each vertical and horizontal segment with the corresponding gadget according to rule *R1* and slanted segments as in Fig. 4(b).

It is easy to verify that introducing these new points preserves the reduction from vertex cover since an even number of points is added. The above construction will be used every time the rule *R1* requires that the Z-chain of a certain gadget cannot lay on any of the two possible sides (left-right or above-below).

R3) **(*Degree Three Nodes.*)** Finally, let us consider a vertex of degree 3. By using simple rotations this situation can always reduced to that shown in Fig. 4(c). Notice that, similarly to the previous case, there is no way to place gadgets for two of the three edges incident to b. The main idea is to construct one of the three gadgets in a slightly different way from that of the previous cases. We first add two new points b' and c' between b and c. In particular, b' and c' are drawn respectively one and two units below b, and c' is moved to the bottom by two units. The resulting chain is then replaced with gadgets $g_{bb'}$, $g_{b'c'}$ and $g_{c'c}$ as shown in Fig. 4(d). Finally, we proceed in the construction of gadget g_{bd} as follows: (i) first place y_{bd} at distance $\delta + \epsilon$ from b in such a way that the angle $\angle aby_{bd} = 3\pi/4$ (see Figure 5); (ii) place the chain Z_{bd} as shown in Fig. 4(d) so to satisfy the gadget properties; (iii) construct a chain of X-nodes from d to one node in $X_{b'c'}$.

4.2.1 The Correctness.

In the sequel the term $S(G)$ will denote the network drawn from $D(G)$ according to the construction rules mentioned above. Let L_{min} be the minimum distance between any two V-points in $D(G)$. From the above construction it follows that $L_{\min}$ is also the minimum distance between any two V stations in $S(G)$. Finally, from the grid requirement of $D(G)$ we have that $L_{\min} \geq 1$.

Lemma 4 *Let $\delta = L_{min}/6$. Then, an $\epsilon > 0$ exists for which the corresponding network $S(G)$ satisfies Eq. 7, i.e.,*

$$\alpha^2\delta^2 \; > \; (m-1)[(\delta+\epsilon)^2 - \delta^2] + (\delta+\epsilon)^2$$

where

$$\alpha = \frac{1+\sqrt{2}}{2} \; .$$

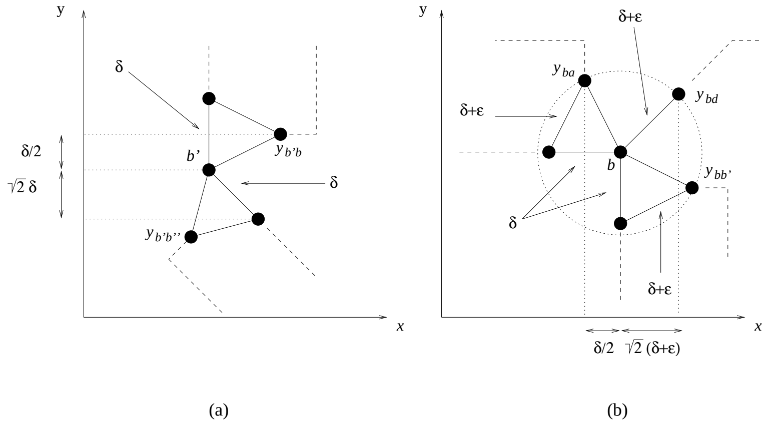

Figure 5: The proof of Lemma 4. (a) Rule *R2*. (b) Rule *R3*.

PROOF. We now show that if $\delta = L_{min}/6$ then the distance between two points from two different gadgets is at least $\alpha\delta$, where $\alpha = \frac{1+\sqrt{2}}{2}$.

From the drawing $D(G)$ and from the choice $\delta = L_{min}/6$, it should be clear that points from "non-adjacent" gadgets are within a distance larger than $2\delta > \alpha\delta$. So, we can focus on adjacent gadgets and distinguish the three rules ***R1-3*** used in the construction of such gadgets.

Rule R1. If the two adjacent gadgets g_{ab} and g_{bc} are placed on the same line as in Fig. 3(a) then their minimum distance is that between y_{bc} and its nearest X_{ab}-point. Such a distance is larger than $3\delta/2$. If g_{ab} and g_{bc} are drawn as in Fig. 3(b) then the minimum distance between the two gadgets is that between the X_{ab}-point nearest to b and the X_{bc}-point nearest to b. Such a distance equals $\sqrt{2}\delta$.

Rule R2. In this case, for a sufficiently small $\epsilon > 0$, the minimum distance is that between $y_{b'b}$ and its nearest $X_{b'b''}$-point (see Fig. 5(a)). Let $(p)_y$ be the y-projection of $p \in \mathcal{R}^2$. Then,

$$d(y_{b'b}, X_{b'b''}) \geq d((y_{b'b})_y, (X_{b'b''})_y) = \frac{1+\sqrt{2}}{2}\delta \ .$$

Rule R3. The two nearest points are y_{ba} and y_{bd} (see Fig. 5(b)). Similarly to the previous case we can derive the following lower bound for $d(y_{ba}, y_{bd})$

$$d(y_{ba}, y_{bd}) \geq d((y_{ba})_x, (y_{bd})_x) = \frac{\delta}{2} + \frac{\delta+\epsilon}{\sqrt{2}} > \frac{1+\sqrt{2}}{2}\delta.$$

Finally, let us observe that the right side of Eq. 7 tends to δ^2 as $\epsilon \to 0$ and that α is a constant larger than 1. This immediately proves the lemma.

□

Combining Lemma 3 with Lemma 4 we obtain the following result.

Theorem 8 MIN 2D RANGE ASSIGNMENT *is* NP-*hard.*

5 Min 3D Range Assignment is APX-complete

The APX-completeness of Min 3D Range Assignment is achieved by showing an approximation-preserving reduction from Min Vertex Cover restricted to cubic graphs, a restriction of Min Vertex Cover which is known to be APX-complete [19, 1]. The approximation-preserving reduction follows the same idea of the reduction shown in the previous section and thus requires a suitable 3-dimensional drawing of a cubic graph.

Theorem 9 ([10]) *There is a polynomial-time algorithm that, given any cubic graph $G(V, E)$, returns a 3-dimensional orthogonal drawing $D(G)$ of G such that:*

- *Every edge is represented as a polyline with at most three bends.*
- *Vertices are represented as points with integer coordinates, thus the minimum distance L_{min} between two vertices is at least 1.*
- *The maximum length L_{max} of an edge in $D(G)$ is polynomially bounded in $m = |E|$.*

5.1 The 3-Dimensional Gadgets

In what follows, we assume to have at hand the 3-dimensional, orthogonal drawing $D(G)$ of a cubic graph G that satisfies the properties of Theorem 9. Then the approximation-preserving reduction replaces each edge of $D(G)$ with a 3-dimensional gadget of stations having the following properties.

Definition 5 (Properties of 3-Dimensional Gadgets) *Let l and ϵ be positive constants (a suitable choice of such parameters will be given later). For any edge (a, b) the corresponding gadget contains the sets of points*
$X_{ab} = \{x_1, \dots, x_{l_1}\}$, $Y_{ab} = \{y_{ab}, y_{ba}\}$, $Z_{ab} = \{z_1, \dots, z_{l_2}\}$ *and* $V_{ab} = \{a, b\}$, *where l_1 and l_2 depend on the distance $d(a, b)$ and $d(y_{ab}, y_{ba})$, respectively. The above set of points is drawn in such a way that the following properties hold:*

1. $d(a, y_{ab}) = d(b, y_{ba}) = l$.
2. *X_{ab} and Z_{ab} are two chains of points drawn so that $d(a, x_1) = d(b, x_l) = \epsilon$ and $d(y_{ab}, z_1) = d(y_{ba}, z_m) = \epsilon$, respectively. Furthermore, for any $i = 1, \dots, l-1$, $d(x_i, x_{i+1}) = \epsilon$ and for any $j = 1, \dots, m-1$ $d(z_j, z_{j+1}) = \epsilon$.*
3. *For any $x_i \in X_{ab}$ and $z_j \in Z_{ab}$, $d(x_i, z_j) > l$. Furthermore $d(x_i, y_{ab}) \geq l$ and $d(x_i, y_{ba}) \geq l$.*
4. *Given any two different gadgets g_1 and g_2, for any $v \in g_1$ and $w \in g_2$ with $u \neq w$ of different type (for example, if u is a X-point then w is either a Y-point or a Z-point), we have that $d(v, w) > l$. Moreover, the minimum distance between the YZ-component[4] of g_1 and the YZ-component of g_2 is $2l$.*
5. *Given any two* non adjacent *gadgets g_1 and g_2, for any $v \in g_1$ and $w \in g_2$, $d(v, w) \geq L_{min}/2$.*

[4]Similarly to the 2-dimensional case, the sets of points $V_{ab} \cup X_{ab}$ and $Y_{ab} \cup Z_{ab}$ will be denoted as VX-component and YZ-component, respectively.

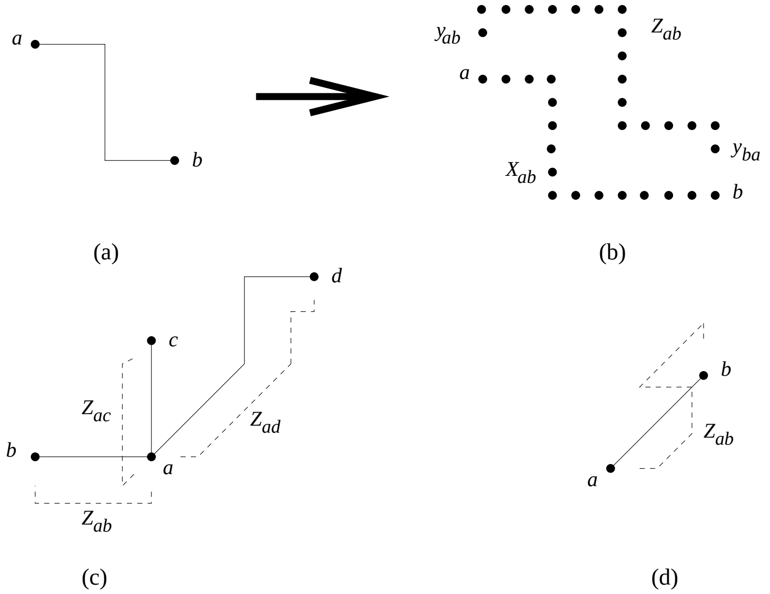

Figure 6: The 3-dimensional gadgets. (a) An edge represented with a polyline with 2 bends. (b) An example of a gadget corresponding to edge (a, b). (c) How to place Z-chains of gadgets whose nodes have degree three. (d) How to change the plane of YZ-chains.

5.2 The Construction of the 3-Dimensional Gadgets

Let l and ϵ two positive reals such that $l \leq L_{min}$ (this assumption guarantees Properties 4 and 5 of Definition 5) and $\epsilon < l$. The construction of the 3-dimensional gadgets can be obtained by adopting the same method of the 2-dimensional case.

Let l and ϵ two positive reals such that $l \leq L_{min}$ (this assumption guarantees Properties 4 and 5 of Definition 5) and $\epsilon < l$. The construction of the 3-dimensional gadgets can be obtained by adopting the same method of the 2-dimensional case (see Fig. 1). However, the presence of the third dimension makes the cases $R2$ and $R3$ (see Fig.s 7(a)-(b)) much easier: in fact, in order to keep the relative distance among YZ-components of adjacent gadgets we can locate such components on different planes in the space. Furthermore, the choice of the plane the YZ-component is placed on depends on the local situation of the two endpoints of the gadget; it could be the case that the plane required by one of these endpoint must be different from that required by the other one. This technical problem can be easily solved, *without using intermediate points*, by drawing the YZ-component over a polyline in the space around the corresponding VX-component (see Fig. 6(d)). More formally, each polyline representing an edge in $D(G)$ will be replaced with a gadget such that : (i) The X-points are drawn equally spaced on the polyline representing the edge. (ii) Z-points are drawn equally spaced and their distance from any X-point is a constant larger than l. In particular, the distance between the X-component and the Z one is achieved by drawing the Z-component on an orthogonal polyline as in Fig. 6. It thus follows that given any 3-dimensional drawing $D(G)$ satisfying Theorem 9, it is possible to replace all edges in $D(G)$ by the corresponding 3-dimensional gadgets in time polynomial in the number of edges.

We emphasize that the 3-dimensional gadgets have two further properties which will be strongly used to achieve an approximation-preserving reduction (see Theorem 10):

1. The set of V-points of $S(G)$ is the set of vertices of G, i.e. no new vertices will be added with respect to those of $D(G)$.
2. It is possible to make the overall range cost of both X and Z points of any gadget arbitrarily small by augmenting the number of equally spaced stations in these two chains.

Lemma 5 *Let L be the length of the polyline representing edge (a,b) in $D(G)$ and let k be the number of points in the X (or Z) component. Then the overall power needed for the X component is*

$$(k+2)\left(\frac{L}{k+1}\right)^2 \tag{10}$$

Moreover, it is possible to make the above value smaller than any fixed positive constant by considering a sufficiently high (but still polynomial) k.

PROOF. The proof easily follows from the fact that L is polynomially bounded in the size of G. □

5.3 The Approximation-Preserving Reduction

Definition 6 (Canonical Solutions for $S(G)$) *A range assignment r for $S(G)$ is* canonical *if, for every gadget g_{ab} of $S(G)$, the following properties hold.*

1. *Either $r(y_{ab}) = l$ and $r(y_{ba}) = \epsilon$ (so, y_{ab} is the radio "bridge" from the YZ-component to the VX one) or vice versa.*
2. *For every $v \in \{a,b\}$, either $r(v) = \epsilon$ or $r(v) = l$. Furthermore, there exists $v \in \{a,b\}$ such that $r(v) = l$ (so, v is a radio "bridge" from the VX-component to the YZ one).*
3. *For every $x \in X_{ab}$, $r(x) = \epsilon$.*
4. *For every $z \in Z_{ab}$, $r(z) = \epsilon$.*

Lemma 6 *For any graph G, let us consider the construction $S(G)$ in which l is a positive real that satisfies the following inequality*

$$l^2 \; < \; \frac{L_{min}^2}{m} \; . \tag{11}$$

Then, for any feasible range assignment r of $S(G)$, there is a canonical range assignment r^c such that $\mathsf{cost}(r^c) \leq \mathsf{cost}(r)$.

PROOF. We use the same method of Lemma 2. In particular, we describe the generic step of an iterative process that yields a canonical range assignment r^c such that $\mathsf{cost}(r) \geq \mathsf{cost}(r^c)$. The number of steps is bounded by the number of points having a non-canonical assignment.

By definition, at least one property of Definition 6 is not satisfied by r. The four cases can be easily reduced to the following two situations.

a) (*Local Transformation.*) We assume that any point in $S(G)$ has power range smaller than $L_{min}/2$ (this implies that at most two adjacent gadgets can be "covered" by a point). We now prove that the cost difference $\mathsf{cost}(r) - \mathsf{cost}(r)$ is non negative. Let u be the point having a non canonical assignment and let w be the V-point shared by the two covered gadgets. If $r(u) < l$ (i.e. u is not a "bridge" between the VX-component and the YZ one), then we can easily find a canonical assignment for u that does not increase the cost. So, in what follows, we assume that $r(u) \geq l$.

1. $u \in Z_{ab}$. Set $r^c(u) = \epsilon$ and set the range of w and one point from Y_{ab} (say y_{ab}) to l. Since in any feasible solution the transmission range of w and y_{ab} is at least ϵ, and from the 4-th property of Definition 5, i.e. $r(u) \geq 2l$, we obtain

$$\begin{aligned}\mathsf{cost}(r) - \mathsf{cost}(r^c) &= r(u)^2 + r(w)^2 + r(y_{ab})^2 - r^c(u)^2 - r^c(w)^2 - r^c(y_{ab})^2 \\ &\geq r(u)^2 + \epsilon^2 - 2l^2 \\ &\geq 2l^2 + \epsilon^2.\end{aligned}$$

2. $u \in X_{ab}$. Set $r^c(u) = \epsilon$ and $r^c(w) = l$. Since in any feasible solution the range of w is at least ϵ, we obtain

$$\begin{aligned}\mathsf{cost}(r) - \mathsf{cost}(r^c) &= r(u)^2 + r(w)^2 - r^c(u)^2 - r^c(w)^2 \\ &\geq r(u)^2 + \epsilon^2 - \epsilon^2 - l^2 \\ &\geq r(u)^2 - l^2,\end{aligned}$$

   which is non negative since $r(u) \geq l$.

3. $u \in \{a, b, y_{ab}, y_{ba}\}$. Simply set $r^c(u) = l$. Then the difference of the costs of the two solutions is $r(u)^2 - l^2 \geq 0$.

b) (*Global Transformation.*) Let u be a vertex whose range $r(u)$ is sufficient to cover two or more gadgets other than that containing u. We first set $r^c(u) = l$. We then canonically assign the new ranges to the stations of those gadgets $g_{a'b'}$ that were covered by u (i.e the VX-component of $g_{a'b'}$ is not anymore strongly connected to its YZ-component). In particular, if $g' = g_{a'b'}$, for some a' and b', we assign $r^c(a') = l$. Two cases may arise depending on the number of gadgets g' covered by u in the non canonical assignment.

1. At most two gadget g' and g'' (those adjacent to g - remind that we consider only graphs of maximum degree three) were covered by u. In this case the overall cost increment in r^c due to the gadgets g' and g'' is at most

$$2(l^2 - \epsilon^2)\ .$$

   From Property 4 of Definition 5, we have that $r(u) \geq 2l$, so,

$$\mathsf{cost}(r) - \mathsf{cost}(r^c) = r(u)^2 - l^2 + 2(l^2 - \epsilon^2) \geq l^2 + 2\epsilon^2 > 0\ .$$

2. At least three gadgets were covered by u in r. Since the number of gadgets of $S(G)$ is at most m, the overall cost increment in r^c due to such gadgets is at most

$$(m-1)[l^2 - \epsilon^2].$$

Since at least one gadget g' non adjacent to g was covered by $r(u)$, from the 5-th property of Definition 5, we have that $r(u) \geq L_{min}/2$. Thus

$$\begin{aligned} \mathsf{cost}(r) - \mathsf{cost}(r^c) &= r(u)^2 - l^2 - (m-1)[l^2 - \epsilon^2] \\ &\geq (L_{min}/2)^2 - l^2 - (m-1)[l^2 - \epsilon^2] \geq 0, \end{aligned}$$

where the last inequality follows from the assumption $L_{min}^2 > ml^2$.

□

Informally speaking, the presence of the third dimension in placing the gadgets allows to keep a polynomially large gap between the value of l (i.e. the minimum distance between the VX component and the YX one of a gadget) and that of ϵ (i.e. the minimum distance between points in the same chain component). This gap yields the significant weight of each bridge-point of type V in a canonical solution and it will be a key ingredient in proving the next theorem. Notice also that this gap cannot be smaller than a fixed positive constant in the 2-dimensional reduction shown in the previous section.

Theorem 10 MIN 3D RANGE ASSIGNMENT *is* APX*-complete.*

PROOF. The outline of the proof is the following. We assume that we have at hand a polynomial-time ρ-approximation algorithm $\mathcal{A}$ for MIN 3D RANGE ASSIGNMENT. Then, we show a polynomial-time method that transforms $\mathcal{A}$ into a ρ'-approximation algorithm for MIN VERTEX COVER on cubic graphs with $\rho' \leq 5\rho - 4$. Since a constant $\overline{\rho} > 1$ exists such that MIN VERTEX COVER restricted to cubic graphs is not $\overline{\rho}$-approximable unless P = NP [19, 1], the theorem follows.

Consider an at most cubic graph $G(V, E)$. Starting from the 3-dimensional orthogonal drawing $D(G)$ we construct in polynomial time the radio network $S(G)$ as described in Section 5.2 (see also Definition 5). Moreover, the construction of $S(G)$ is made so to satisfy the hypothesis of Lemma 6 (see Eq. 11). Using the same arguments in the proof of Lemma 3, we can show that any vertex cover $K \subseteq V$ of G yields a canonical assignment r_K whose cost is

$$\mathsf{cost}(r_K) = \kappa l^2 + ml^2 + \overline{\epsilon_K}, \tag{12}$$

where $\kappa = |K|$ and $\overline{\epsilon_K}$ is the overall cost due to all points v that have range ϵ. Since each gadget of $S(G)$ has at most $4L_{max}/\epsilon$ points, it holds that

$$\overline{\epsilon_K} \leq 4mL_{max}\epsilon\ . \tag{13}$$

On the other hand, from Lemma 6, we can consider only canonical solutions of $S(G)$. Thus, given a canonical solution r^c, we can consider the subset K of V-points whose range is l. It is easy to verify that K is a vertex cover of G. Furthermore, the cost of r^c can be written as follows

$$\mathsf{cost}(r^c) = |K|l^2 + ml^2 + \overline{\epsilon_K} \ .$$

Let K^{opt} be an optimum vertex cover for G, from the above equation we have that the optimum range assignment cost opt_r can be written as

$$\mathsf{opt}_r = |K^{opt}|l^2 + ml^2 + \overline{\epsilon_{K^{opt}}} \tag{14}$$

Since G has maximum degree 3 then $|K^{opt}| \geq m/3$; so, the above equation implies that

$$\mathsf{opt}_r \leq 4|K^{opt}|l^2 + \overline{\epsilon_{K^{opt}}}. \tag{15}$$

Let us now consider a ρ-approximation algorithm for MIN 3D RANGE ASSIGNMENT such that given $S(G)$ in input it returns a solution r^{apx} whose cost is less than $\rho \cdot \mathsf{opt}_r$. From Lemma 6, we can assume that r^{apx} is canonical. It thus follows that the cost $\mathsf{cost}(r^{apx})$ can be written as

$$\mathsf{cost}(r^{apx}) = |K^{apx}|l^2 + ml^2 + \overline{\epsilon_{K^{apx}}} \ .$$

From Eq.s 14 and 15 we obtain

$$\begin{aligned}
\frac{\mathsf{cost}(r^{apx})}{\mathsf{opt}_r} &= \frac{\mathsf{cost}(r^{apx}) - \mathsf{opt}_r}{\mathsf{opt}_r} + 1 && (16)\\
&= \frac{|K^{apx}|l^2 + ml^2 + \overline{\epsilon_{K^{apx}}} - |K^{opt}|l^2 - ml^2 - \overline{\epsilon_{K^{opt}}}}{\mathsf{opt}_r} + 1 && (17)\\
&\geq \frac{|K^{apx}|l^2 - |K^{opt}|l^2}{4|K^{opt}|l^2 + \overline{\epsilon_{K^{opt}}}} + 1 && (18)
\end{aligned}$$

Note that we can make $\overline{\epsilon_{K^{opt}}}$ arbitrarily small (independently from l) by reducing the parameter ϵ in the construction of $S(G)$: this is in turn obtained by increasing the number of X and Z points in the gadgets (see Lemma 5).

From Eq. 13, from the fact that L_{max} is polynomially bounded in the size of G and from the fact that l and L_{max} are polynomially related, we can ensure that $\overline{\epsilon_{K^{opt}}} \leq l^2$ by adding a polynomial number of points (see again Lemma 5). So, from Eq. 16 we obtain

$$\frac{\mathsf{cost}(r^{apx})}{\mathsf{opt}_r} \geq \frac{|K^{apx}|l^2 - |K^{opt}|l^2}{4|K^{opt}|l^2 + \overline{\epsilon_{K^{opt}}}} + 1 \geq \frac{|K^{apx}|}{5|K^{opt}|} + \frac{4}{5} \ .$$

Finally, it follows that the approximation ratio for MIN VERTEX COVER is bounded by

$$\frac{|K^{apx}|}{|K^{opt}|} \leq \frac{5\mathsf{cost}(r^{apx})}{\mathsf{opt}_r} - 4.$$

□

6 Open Problems

An interesting problem which is still open is whether MIN 2D RANGE ASSIGNMENT is APX-complete or admits a PTAS. Notice that a possible APX-completeness reduction should be from a different problem, since MIN VERTEX COVER restricted to planar graphs is in PTAS. As for the MIN 3D RANGE ASSIGNMENT problem it could be interesting to reduce the large gap between the factor 2 of the approximation algorithm and the inaproximability bound than can be derived by combining our reduction with the approximability lower bound of MIN VERTEX COVER on cubic graphs. As far as we know, there is no known significant explicit lower bound for the latter problem (an explicit 1.0029 lower bound for MIN VERTEX COVER on degree 5 graphs is given in [5] that – if it could be extended to cubic graphs and then combined with our reduction – would give a lower bound for MIN 3D RANGE ASSIGNMENT of 1.00059).

A crucial characteristic of the optimal solutions for the MIN 3D RANGE ASSIGNMENT instances given by our reduction is that stations that communicate directly have relative distance either equal to l or ϵ, where $l >> \epsilon$. It could be interesting to consider instances in which the above situation does not occur. Notice that this is the case of the MIN 2D RANGE ASSIGNMENT instances of our reduction. Thus, the problem on such restricted instances remains NP-hard. However, it is an open problem whether a better approximation factor or even a PTAS can be obtained.

Another interesting aspect concerns the maximum number of hops required by any two stations to communicate. This corresponds to the diameter h of the communication graph. Our constructions yield solutions whose communication graph has unbounded (i.e. linear in the number of stations) diameter. So, the complexity of MIN dD RANGE ASSIGNMENT with bounded diameter remains open also in the 1-dimensional case. A special case where stations are placed at uniform distance on a line and either h is constant or $h \in O(\log n)$ has been solved in [15].

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