Electronic Colloquium on Computational Complexity, Revision 01 of Report No. 077 (2000)

On the Power of Extra Queries to Selective Languages

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8th December 2000

Abstract

A language is *selective* if there exists a selection algorithm for it. Such an algorithm selects from any two words one, which is an element of the language whenever at least one of them is. Restricting the complexity of selection algorithms yields different *selectivity classes* like the P-selective or the semirecursive (i. e. recursively selective) languages. Generalising a concept introduced by Beigel et al. [4] we define the *selective query complexity* of a language as the minimum number of queries to any selective language needed to decide it. Our main result shows that the logspace k-truth-table equivalence closure of every selective, *non-cheatable* language has parallel selective query complexity exactly k. We rephrase this result in terms of the language $\mathrm{ODD}_k^N = \{\langle w_1, \dots, w_k\rangle \mid \sum_{i=1}^k \chi_N(w_i) \text{ is odd}\}$ and obtain the following generalisation of an important recursion theoretic result of Beigel et al. [4]: For selective, non-cheatable sets N the parallel selective query complexity of ODD_k^N is exactly k. Our proofs are based on a partial information analysis [20, 21] of the involved languages: We establish matching upper and lower bounds for the partial information complexity of the different equivalence and reduction closures of selective languages. From this we derive the main results as these bounds differ for different numbers of queries.

We give four applications of our main theorem and the proof technique. First, the relations $\mathrm{E}^{\mathrm{P}}_{k\text{-tt}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}}_{(k-1)\text{-tt}}(\mathrm{P}[\mathrm{SEL}])$ and $\mathrm{E}^{\mathrm{P}}_{\mathrm{tt}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}}_{\mathrm{btt}}(\mathrm{P}[\mathrm{SEL}])$ proven in [14] still hold *if we relativise only the right hand sides*. Second, we settle an open problem from [14]: Equivalence to a P-selective language with k *serial* queries cannot generally be replaced by a reduction using less than $2^k - 1$ *parallel* queries. Third, the k-truth-table reduction closures of selectivity classes are (m, n)-verbose [7] iff every walk on the n-dimensional hypercube with transition counts at most k visits at most m bitstrings. Lastly, these reduction closures are (m, n)-recursive [24] iff every such walk is contained in a closed ball of radius $n - m$.

Keywords

Bounded query complexity, selective query complexity, P-selective, semirecursive, cheatable, verbose, frequency computations.

ISSN 1433-8092

A *selector* for a language $L \subseteq \Sigma^*$ is a binary function $f\colon \Sigma^* \times \Sigma^* \to \Sigma^*$ such that $f(u,v) \in \{u,v\}$, and $f(u,v) \in L$ whenever $u \in L$ or $v \in L$. Jockusch [15] coined the term *semirecursive languages* for languages which have a recursive selector. The class of semirecursive languages, denoted by REC[SEL] in the following, plays a key role in the solution of Post's Problem [22].

Selman [25] introduced P-*selective languages*. Such languages have a polynomial time computable selector. The class of P-selective languages will be denoted by P[SEL], see Definitions 7 and 8 for the reasons for this notation. Selman proved that the satisfiability problem is not P-selective unless P = NP. This result has been considerably strengthened by extending it to *reduction closures* of the P-selective languages, see the following Facts 2 and 3.

Fact 1 ([25]). *If some* NP-*hard language is* P-*selective, then* NP = P.

Fact 2 ([6, 23, 1]). *If some* NP-*hard language is sublinear truth-table reducible to a* P-*selective language, then* NP = P.

Fact 3 ([25, 17]). *If some* NP-*hard language is Turing reducible to a* P-*selective language, then* NP $\subseteq$ P/poly.

Let us call a language L-*selective* if it has a logspace computable selector. An easy transfer of a result of Selman [25, Theorem 2] shows that L-selective languages are in L if they are logspace many-one reducible to their complements. Since the circuit value problem and the directed reachability problem *are* logspace many-one reducible to their complements, we can transfer Fact 1 to logspace. We do not know whether a logspace version of Fact 2 also holds, but conjecture that if the circuit value problem is logspace 1-truth-table reducible to an L-selective language, then L = P.

Observation 4. *If the circuit value problem is* L-*selective, then* L = P.

Observation 5. *If the directed reachability problem is* L-*selective, then* L = NL.

Facts 1 to 3 have been a major motivation for the study of the power of extra queries to P-selective languages. Observations 4 and 5 suggest that this study could be extended to L-selective languages. The underlying theme of this paper is that *extra queries to selective languages are helpful.* In order to state this more precisely we propose the notion of *selective query complexity* as a generalisation of the semirecursive query complexity defined in [4]. For 'reasonable' computational models C the (parallel) C-selective query complexity of a language L is the minimum number, taken over all C-selective sets N, of (parallel) queries to N needed in order to decide L. For example, by Fact 2 the P-selective query complexity of NP-hard problems is infinite unless P = NP.

Hemaspaandra et al. [14] have constructed for every k a language in the k-truth-table reduction closure of P[SEL] which has parallel P-selective query complexity exactly k. From this they derive that the polynomial time truth-table reduction closures of P[SEL] form a proper hierarchy. Using recursion theoretic methods Beigel et al. proved the following much more powerful fact:

Fact 6 ([4]). *Let N be semirecursive and non-recursive. Then* ODD_k^N *has parallel semirecursive query complexity exactly k.*

Here, $\text{ODD}_k^N := \{\langle w_1, \dots, w_k\rangle \mid \sum_{i=1}^{k} \chi_N(w_i) \text{ is odd}\}$ is similar to the parity function, which has been studied extensively in the contexts of circuit complexity [12, 9], pseudorandomness [13] and also quantum computing [11]. Note that Hemaspaandra et al.'s result follows from Fact 6 as there exist P-selective, non-recursive sets N and ODD_k^N is trivially reducible to N with k parallel queries.

Fact 6 does not hold for resource bounded computational models as there exist P-selective languages N outside P which are *cheatable* [2], see Definition 10. For cheatable languages ODD_k^N has parallel P-selective query complexity less than k for large k, see the remark following Definition 10. So the best result one could hope for is that for P-selective, *non-cheatable* languages N the parallel P-selective query complexity of ODD_k^N is exactly k for all k. This is exactly the claim of Theorem 23, which is an equivalent formulation of our main Theorem 20. It turns out that Fact 6 is a special case of Theorem 23 as 'recursively cheatable' languages are recursive by Beigel's Nonspeedup Theorem [2].

This paper is organised as follows. *Section 1* introduces basic notations and definitions. It includes a review of the main technical tool used in our proofs: partial information classes, a concept proposed in [5] and studied extensively by Nickelsen [20, 21]. In *Section 2* we establish matching upper and lower bounds on the partial information complexity of languages truth-table reducible to a partial information class. *Section 3* shows that for the special case of selectivity classes these bounds are walks on a hypercube with bounded transition counts. We deduce the main results by noting that for different transition counts these walks visit different numbers of bitstrings. In *Section 4* we reformulate our main Theorem 20 equivalently in terms of the selective query complexity of ODD_k^N, obtaining Theorem 23.

The concluding *Section 5* presents applications of the main result and its proof technique. First, we prove that $\text{E}^{\text{P}}_{k\text{-tt}}(\text{P}[\text{SEL}]) \not\subseteq \text{R}^{\text{P}^A}_{(k-1)\text{-tt}}(\text{P}^A[\text{SEL}])$ and $\text{E}^{\text{P}}_{\text{tt}}(\text{P}[\text{SEL}]) \not\subseteq \text{R}^{\text{P}^A}_{\text{btt}}(\text{P}^A[\text{SEL}])$ for all oracles A. These are powerful relativisations of the corresponding unrelativised relations proven in [14]. Second, we settle an open problem from [14]: There is a language which is k-Turing equivalent to a P-selective language, but not (2^k-2)-truth-table reducible to any P-selective language. Third, we quantify the amount of *verboseness* of reduction closures of selective languages and, lastly, we do the same for frequency computations. The verboseness of reduction closures of selective languages has been studied implicitly in [7]. We rephrase, in terms of walks and transition counts, the combinatorial characterisation established in [7]. This rephrasing shows that a conjecture from [7] seems hard to prove as it implies the existence of balanced Gray codes for arbitrary bitstring lengths.

1 Preliminaries

We start this section with basic notations and definitions. Then we review the concept of partial information as introduced in [20, 21]. Finally, we state some basic facts about truth-table reductions.

Basic Notations and Definitions. For every n, $\langle \cdot, \ldots, \cdot \rangle : (\Sigma^*)^n \to \Sigma^*$ is a tupling function computable in logspace with an inverse also computable in logspace. For a language $L \subseteq \Sigma^*$ and a word $w \in \Sigma^*$ the *characteristic value* $\chi_L(w)$ is defined by $\chi_L(w) := 1$ if $w \in L$, and $\chi_L(w) := 0$ otherwise. This is extended to tuples by setting $\chi_L(w_1, \ldots, w_n) := \chi_L(w_1) \ldots \chi_L(w_n)$. Let $\mathbb{B} := \{0, 1\}$. *Bitstrings of length* n are elements of $\mathbb{B}^n$. The *Cartesian product* $\phi_1 \times \cdots \times \phi_n \colon \mathbb{B}^{nk} \to \mathbb{B}^n$ of functions $\phi_1, \ldots, \phi_n \colon \mathbb{B}^k \to \mathbb{B}$ is defined by $(\phi_1 \times \cdots \times \phi_n)(b_1 \ldots b_n) := \phi_1(b_1) \ldots \phi_n(b_n)$ for $b_i \in \mathbb{B}^k$. For $\phi \colon \mathbb{B}^n \to \mathbb{B}^m$ and a set $P \subseteq \mathbb{B}^n$ the *ϕ-image of P* is $\phi(P) := \{\, \phi(b) \mid b \in P \,\}$. The *Hamming distance* $\mathrm{d}(b, c)$ of two bitstrings is the number of position indices where they differ. The *closed ball around $b \in \mathbb{B}^n$ of radius r* is $\mathrm{B}_r(b) := \{\, c \in \mathbb{B}^n \mid \mathrm{d}(b, c) \leq r \,\}$.

A *walk* is a sequence of bitstrings of the same length where consecutive bitstrings differ at exactly one position. A walk is *self-avoiding* if the sequence contains no duplicates except possibly for the first and last bitstring. If these are equal the walk is called a *cycle*. The *transition sequence* of a walk is the sequence of position indices where consecutive bitstrings differ. The *transition count* of a position index is its frequency in a transition sequence. An example of a walk is $001, 011, 010, 110, 111$; its transition sequence is $2, 3, 1, 3$; the transition count is 1 for the first two position indices and 2 for the last position index.

Partial Information Classes. Traditionally the complexity of a language L is measured by the amount of time or space needed to compute for words $w_1, \ldots, w_n$ the characteristic string $\chi_L(w_1, \ldots, w_n)$. In a *partial information analysis* [20, 21] we consider witness functions f such that $\chi_L(w_1, \ldots, w_n) \in f(w_1, \ldots, w_n)$ for all words. The function f may produce whole sets of bitstrings, called *n-pools* in the following. We now ask, which are the smallest and simplest pools computable in, say, polynomial time for a given language L?

We will say that Q *is an n-pool for the words* $w_1, \ldots, w_n$ if Q contains their characteristic string. For every non-trivial language every bitstring in $\mathbb{B}^n$ is the characteristic string of some appropriate words $w_1, \ldots, w_n$. Hence the set of all pools which a witness function outputs must necessarily form a covering of $\mathbb{B}^n$, called *n-covering* in the following. In this paper, we will use the following special n-coverings:

Definition 7. Let SEL_n be the set of all n-pools which can be written as $\{b_1, \ldots, b_m\}$ such that from each b_i to b_{i+1} only some 0's are changed into 1's. Let $m\text{-}\mathrm{SIZE}_n := \{\, P \subseteq \mathbb{B}^n \mid |P| \leq m \,\}$ for sizes $m \geq 1$. Let $\mathrm{CHEAT}_n := n\text{-}\mathrm{SIZE}_n$. Let $k\text{-}\mathrm{WALKS}_n$ be the set of all pools $P \subseteq \mathbb{B}^n$ which are contained in a walk with transition counts at most k. Let $r\text{-}\mathrm{FREQ}_n := \{\, P \mid \exists b \in \mathbb{B}^n : P \subseteq \mathrm{B}_r(b) \,\}$ for radii r.

A pool is *maximal* for a covering if there exists no proper superset of this pool in the covering. An n-covering $\mathcal{D}$ is *subset closed* if $Q \subseteq P \in \mathcal{D}$ implies $Q \in \mathcal{D}$. Note that all of the special n-coverings just defined are subset closed.

We define partial information classes for different computational models and C will be a variable for such a model. The arguments used in the following proofs are correct for all 'reasonable' C. In this paper a model will be called reasonable if it is L, P or

REC or a relativisation of one these models – see [27] for a much broader definition. The only property of a computational model C we will use and be interested in is the corresponding *function class FC* from which we can draw the witness functions. Although the functions in FC map words to words as is customary, in the following definition functions actually map tuples of words to pools. Since coding and decoding of both tuples of words and pools are trivial, we will not explicitly write down the appropriate conversions.

Definition 8. Let C denote a computational model and FC the corresponding function class. Let $\mathcal{D}$ be an n-covering. A language $L \subseteq \Sigma^*$ is in the *partial information class* $C[\mathcal{D}]$ (respectively $C_{\text{dist}}[\mathcal{D}]$) if there exists a witness function $f \in FC$ such that for all (distinct) words $w_1, \dots, w_n \in \Sigma^*$ we have

$$\chi_L(w_1, \dots, w_n) \in f(w_1, \dots, w_n) \in \mathcal{D}.$$

Note that we require witness machines to output complete pools. One can also consider the situation where a witness machine need only *enumerate* the output pool. We introduce a special notation for this situation. A language is in $\text{RE}[\mathcal{D}]$ if there exists a Turing machine that upon input of n words starts a possibly infinite computation. During the computation it writes bitstrings to an output tape such that the set of written bitstrings forms a pool $P \in \mathcal{D}$ for the input words. For example, $\text{RE}[m\text{-SIZE}_n]$ is the class of (m,n)*-verbose* [7] languages. If a theorem also holds for $\text{RE}[\mathcal{D}]$ we give a second version of it, like Theorem 20′.

The class $\text{P}[\text{SEL}_2]$ is exactly the class of P-selective languages: If $g \in \text{FP}$ is a selector for a language N and u and v are words, then $g(u,v) = u$ iff $\chi_N(u,v) \in \{00, 10, 11\}$ and $g(u,v) = v$ iff $\chi_N(u,v) \in \{00, 01, 11\}$. Thus we can trivially turn a P-selector into a witness function for $N \in \text{P}[\text{SEL}_2]$ and also the other way round. Likewise, $\text{L}[\text{SEL}_2]$ is the class of L-selective languages, $\text{REC}[\text{SEL}_2]$ the class of semirecursive languages and $\text{RE}[\text{SEL}_2]$ the class of *weakly semirecursive* [16] languages.

The following fact motivates our liberal notation $\text{P}[\text{SEL}]$ as it tells us that the index is not important for selective coverings.

Fact 9 ([21, 25]). *Let C be reasonable. Then $C[\text{SEL}_n] = C[\text{SEL}_2]$ for $n \geq 2$.*

Definition 10. Let C be reasonable. The languages in $C[\text{CHEAT}] := \bigcup_{n=1}^{\infty} C[\text{CHEAT}_n]$ will be called *C-cheatable.*

Cheatable languages have an important property [2]: For languages $L \in C[\text{CHEAT}_n]$ every truth-table reduction to L using more than n queries can be replaced by a truth-table reduction using only $n-1$ queries. In other words, starting from $n-1$ extra queries to n-cheatable languages do not help.

The following fact explains why neither the term 'r.e. cheatable' nor the term 'recursively cheatable' is commonly used.

Fact 11 (Nonspeedup Theorem [2]). $\text{RE}[\text{CHEAT}] = \text{REC}$.

Bounded Query Reductions. The following definition of truth-table reductions for reasonable computational models C is along the lines of [19].

Definition 12. Let C denote a computational model and FC the corresponding function class. A language L is *k-truth-table C-reducible* to a language N, written $L \leq^{C}_{k\text{-tt}} N$, if there exist a *generator* $g \in FC$ and an *evaluator* $e \in FC$ such that for all words $w \in \Sigma^*$ we have $\chi_L(w) = e\bigl(w, \chi_N(q_1, \dots, q_k)\bigr)$. Here, $\langle q_1, \dots, q_k \rangle = g(w)$ are the queries produced by the generator upon input w.
For a class X of languages we define $\mathrm{R}^{C}_{k\text{-tt}}(X) := \bigl\{ L \mid \exists N \in X \colon L \leq^{C}_{k\text{-tt}} N \bigr\}$ and $\mathrm{E}^{C}_{k\text{-tt}}(X) := \bigl\{ L \mid \exists N \in X \colon L \leq^{C}_{k\text{-tt}} N, N \leq^{C}_{k\text{-tt}} L \bigr\}$.

Note that for $C = \mathrm{REC}$ we require the evaluators to converge on all inputs. One can also consider *weak truth-table* reductions, written $L \leq_{k\text{-wtt}} N$, where an evaluator has to converge only when provided with the correct characteristic string of the queries. As this reduction makes sense only in the recursive setting we omit the superscript.

Definition 13. Let X be a class of languages and C a computational model. We define the *parallel C-selective query complexity* of X as the minimum number k such that $X \subseteq \mathrm{R}^{C}_{k\text{-tt}}\bigl(C[\mathrm{SEL}]\bigr)$ and as infinity if no such number exists.

The following fact shows that polynomial time truth-table and Turing reducibility to a P-selective language are in some sense interchangeable. The proof is easily transferred to most computational models, but note that Fact 14 does not hold for weak truth-table reductions.

Fact 14 ([14]). *Let N be* P*-selective. Then $L \leq^{\mathrm{P}}_{k\text{-T}} N$ iff $L \leq^{\mathrm{P}}_{(2^k-1)\text{-tt}} N$.*

2 Upper and Lower Bounds

We now prove matching upper and lower bounds for the partial information complexity of languages reducible to a partial information class. A special case are selectivity classes, which will be studied in the next section. There, we will use the matching bounds to prove our main result.
To fix notations, in this section C denotes a reasonable computational model and n, k, r are positive integers. For every nk-covering $\mathcal{E}$ the n-covering $\mathcal{E}_{k\text{-tt}}$ denotes the *image of $\mathcal{E}$ under products of k-ary Boolean functions*, defined by

$$\mathcal{E}_{k\text{-tt}} := \bigl\{ (\phi_1 \times \dots \times \phi_n)(Q) \mid Q \in \mathcal{E}, \phi_i \colon \mathbb{B}^k \to \mathbb{B} \bigr\}.$$

Theorem 15 (Upper Bound). *Let $\mathcal{E}$ be an nk-covering. Then $\mathrm{R}^{C}_{k\text{-tt}}\bigl(C[\mathcal{E}]\bigr) \subseteq C[\mathcal{E}_{k\text{-tt}}]$.*

Proof. Let $L \leq^{C}_{k\text{-tt}} N \in C[\mathcal{E}]$. To show $L \in C[\mathcal{E}_{k\text{-tt}}]$ we must compute partial information from $\mathcal{E}_{k\text{-tt}}$ for any n given words $w_1, \dots, w_n$. As L is k-truth-table reducible to N, for each word w_i a generator g yields queries $\bigl\langle q_i^1, \dots, q_i^k \bigr\rangle = g(w_i)$ and the corresponding evaluator e computes $\phi_i := e(w_i, \cdot)$ with $\phi_i\bigl(\chi_N(q_i^1, \dots, q_i^k)\bigr) = \chi_L(w_i)$. As $N \in C[\mathcal{E}]$, we can compute a pool $Q \in \mathcal{E}$ containing $\chi_N(q_1^1, \dots, q_1^k, \dots, q_n^1, \dots, q_n^k)$.

Let $Q = \{b_1, \dots, b_m\}$. One of these bitstrings, say b_j, is the correct value of $\chi_N(q_1^1, \dots, q_n^k)$. But then $(\phi_1 \times \cdots \times \phi_n)(b_j)$ is the correct value of the characteristic string $\chi_L(w_1, \dots, w_n)$. Hence, the $(\phi_1 \times \cdots \times \phi_n)$-image of Q is a pool for the input words and an element of $\mathcal{E}_{k\text{-tt}}$ by definition. □

Theorem 15′. *Let* $\mathcal{E}$ *be an nk-covering. Then* $\mathrm{R}_{k\text{-wtt}}\big(\mathrm{RE}[\mathcal{E}]\big) \subseteq \mathrm{RE}[\mathcal{E}_{k\text{-tt}}]$.

Proof. Same as above, except that pools are enumerated, not computed. □

Under certain conditions the upper bound just established is also a lower bound, i.e., there exists no smaller n-covering $\mathcal{D} \subsetneq \mathcal{E}_{k\text{-tt}}$ such that $\mathrm{R}^C_{k\text{-tt}}\big(C[\mathcal{E}]\big) \subseteq C[\mathcal{D}]$. The following theorem strengthens this claim in several ways. We show that the lower bound holds for certain individual languages in $C[\mathcal{E}]$. We consider logspace equivalence closures instead of arbitrary reduction closures on the left hand side. Finally we use the slightly larger class $C_{\mathrm{dist}}[\mathcal{D}]$ on the right hand side.

Theorem 16 (Lower Bound). *Let* $\mathcal{E}$ *be a subset closed nk-covering. Let* $N \in C[\mathcal{E}]$*, but* $N \notin C[\mathcal{E}']$ *for all subset closed coverings* $\mathcal{E}' \subsetneq \mathcal{E}$*. Then* $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \nsubseteq C_{\mathrm{dist}}[\mathcal{D}]$ *for all subset closed coverings* $\mathcal{D} \subsetneq \mathcal{E}_{k\text{-tt}}$.

Proof. We argue by contraposition. Let $\mathcal{D} \subsetneq \mathcal{E}_{k\text{-tt}}$ and $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \subseteq C_{\mathrm{dist}}[\mathcal{D}]$. We must show that there exists some $\mathcal{E}' \subsetneq \mathcal{E}$ such that $N \in C[\mathcal{E}']$. Let $P \in \mathcal{E}_{k\text{-tt}} \backslash \mathcal{D}$ be some maximal pool. By definition of $\mathcal{E}_{k\text{-tt}}$ there exists some maximal pool $Q \in \mathcal{E}$ together with functions $\phi_i \colon \mathbb{B}^k \to \mathbb{B}$ such that $(\phi_1 \times \cdots \times \phi_n)(Q) = P$. Define $\mathcal{E}' := \mathcal{E} \backslash \{Q\}$ and note that this is a subset closed covering.

We now show $N \in C[\mathcal{E}']$. If $N = \emptyset$ or $N = \Sigma^*$ this is trivial. Otherwise let $N \in C[\mathcal{E}]$ via a witness function $f \in FC$. Given any nk words $w_1^1, \dots, w_n^k$ we must compute a pool for these words from $\mathcal{E}$ *other than* Q. First, we compute the pool $T := f(w_1^1, \dots, w_n^k) \in \mathcal{E}$. Next, consider

$$N' := \big\{ \langle u_1, \dots, u_k, \phi, t \rangle \mid \phi \colon \mathbb{B}^k \to \mathbb{B}, \phi\big(\chi_N(u_1, \dots, u_k)\big) = 1, t \in \Sigma^* \big\},$$

which is logspace k-truth-table equivalent to N as $N \neq \emptyset, \Sigma^*$. We form n distinct words $z_i := \big\langle w_i^1, \dots, w_i^k, \phi_i, 0^i \big\rangle$. As $N' \in \mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \subseteq C_{\mathrm{dist}}[\mathcal{D}]$ we can now compute a pool $R \in \mathcal{D}$ such that $\chi_{N'}(z_1, \dots, z_n) \in R$. Note that $P \nsubseteq R$ and $R \in \mathcal{E}_{k\text{-tt}}$.

Let $S := \big\{ b \in \mathbb{B}^{nk} \mid (\phi_1 \times \cdots \times \phi_n)(b) \in R \big\}$ be the preimage of R. As

$$\begin{aligned} \chi_{N'}(z_1, \dots, z_n) &= \phi_1\big(\chi_N(w_1^1, \dots, w_1^k)\big) \cdots \phi_n\big(\chi_N(w_n^1, \dots, w_n^k)\big) \\ &= (\phi_1 \times \cdots \times \phi_n)\big(\chi_N(w_1^1, \dots, w_n^k)\big), \end{aligned}$$

the pool S must contain the bitstring $\chi_N(w_1^1, \dots, w_n^k)$ since R contains the bitstring $\chi_{N'}(z_1, \dots, z_n)$. Thus S is a pool for the original words, and so is $S \cap T$. If $S \cap T \neq Q$ we are done since we can then output $S \cap T \in \mathcal{E}'$. But $S \cap T = Q$ is impossible, since $Q \subseteq S$ implies $P = (\phi_1 \times \cdots \times \phi_n)(Q) \subseteq (\phi_1 \times \cdots \times \phi_n)(S) \subseteq R$, contradicting $P \nsubseteq R$. □

Theorem 16′. *Let* $\mathcal{E}$ *be a subset closed nk-covering. Let* $N \in \mathrm{RE}[\mathcal{E}]$*, but* $N \notin \mathrm{RE}[\mathcal{E}']$ *for all subset closed covering* $\mathcal{E}' \subsetneq \mathcal{E}$*. Then* $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \nsubseteq \mathrm{RE}_{\mathrm{dist}}[\mathcal{D}]$ *for all subset closed coverings* $\mathcal{D} \subsetneq \mathcal{E}_{k\text{-tt}}$.

Proof. Same as for Theorem 16 except that pools are enumerated instead of computed. Note that if R can be enumerated, so does S; and if S and T can be enumerated, so does $S \cap T$. $\square$

3 Selective Query Complexity of Equivalence Closures

This section applies the upper and lower bounds established in Section 2 to selectivity classes. First, in Lemmas 17 and 18 we study some combinatorial properties of *images of selective pools under products of Boolean functions.* In Lemma 19 we show that *the condition $N \notin C[\mathcal{E}']$ from Theorem 16 is met* for non-cheatable languages N. Put together these lemmas yield our main Theorem 20 on the selective query complexity of logspace equivalence closures of selective, non-cheatable languages.

Lemma 17. *The images of maximal pools in SEL_{nk} under products of k-ary Boolean functions are exactly the maximal elements of k-WALKS_n, i.e., $\left(\text{SEL}_{nk}\right)_{k\text{-tt}} = k\text{-WALKS}_n$.*

Proof. Let Q be maximal in SEL_{nk} and let $\phi_1, \dots, \phi_n \colon \mathbb{B}^k \to \mathbb{B}$. In Q from one bitstring to the next exactly one position is changed from 0 to 1. Hence, in the image $(\phi_1 \times \cdots \times \phi_n)(Q)$ from one bitstring to the next also at most one position can change. Leaving out consecutive duplicates the image forms a walk. Note that the walk need not be self-avoiding. Consider the transition count of some specific position index i. It is at most k, as a bitflip in the walk at index i can occur only if a bitflip occurs in the pool Q at one of the k positions used by the function ϕ_i. For the other direction it is easily seen that every walk with transition counts at most k is the image of some selective pool. $\square$

Lemma 18. *Let $nk + 1 < 2^n$. Then $k\text{-WALKS}_n \subsetneq (k+1)\text{-WALKS}_n$.*

Proof. Let $b_1, \dots, b_m \in \mathbb{B}^n$ be a walk with transition counts at most k visiting a maximal number of bitstrings. As $m \leq nk+1 < 2^n$ there must exist some bitstring $b \in \mathbb{B}^n$ not visited. We extend the walk from b_m as follows: From b_m to b_{m+1} we change the first position where b_m and b differ. Likewise from b_{m+1} to b_{m+2} for the second position and so on. This yields a walk $b_1, \dots, b_m, b_{m+1}, \dots, b_{m+\mathrm{d}(b_m, b)} = b$. This new walk visits *at least one bitstring more than the old walk*, namely b, and has transition counts *at most* $k+1$. Hence $\{b_1, \dots, b_{m+\mathrm{d}(b_m,b)}\} \in (k+1)\text{-WALKS}_n \setminus k\text{-WALKS}_n$. $\square$

Lemma 19. *Let C be reasonable and $\mathcal{E}' \subsetneq \text{SEL}_{nk}$ be subset closed. Then $C[\mathcal{E}']$ contains only C-cheatable languages.*

Proof. Let $L \in C[\mathcal{E}']$. For input words $w_1, \dots, w_{nk}$ compute pools $P_\sigma \in \mathcal{E}'$ for all *permutations* σ of these words. Each P_σ induces a pool Q_σ for the unpermuted input words. Since $\mathcal{E}'$ misses a maximal selective pool, one of the Q_σ has a size at most nk. Thus, the intersection of the Q_σ has size at most nk. This intersection is a pool for the input words as all Q_σ are. Thus, $L \in C[(nk)\text{-SIZE}_{nk}] = C[\text{CHEAT}_{nk}] \subseteq C[\text{CHEAT}]$. $\square$

Lemma 19′. *Let $\mathcal{E}' \subsetneq \text{SEL}_{nk}$ be subset closed. Then $\mathrm{RE}[\mathcal{E}'] \subseteq \mathrm{RE}[\text{CHEAT}]$.*

The following theorem is our main result.

Theorem 20. *Let C be reasonable. Let N be any C-selective, non-C-cheatable language. Then $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \nsubseteq \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\text{SEL}]\big)$, i.e., $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N)$ has parallel C-selective query complexity exactly k.*

Proof. Pick an n large enough such that $n(k-1)+1 < 2^n$. Because of Lemma 19 we can apply Theorem 16 to N. It states that the smallest subset closed n-covering $\mathcal{D}$ with $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \subseteq C[\mathcal{D}]$ is $\mathcal{D} = \big(\textsc{sel}_{nk}\big)_{k\text{-tt}}$, which is k-$\textsc{walks}_n$ by Lemma 17. By Theorem 15 the class $\mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$ is contained in $C[(k-1)\text{-}\textsc{walks}_n]$. Hence, if we had $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \subseteq \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$ then $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \subseteq C[(k-1)\text{-}\textsc{walks}_n]$ and as $\mathcal{D}$ was minimal we would get $(k-1)\text{-}\textsc{walks}_n = k\text{-}\textsc{walks}_n$, contradicting Lemma 18. □

Theorem 20′. *Let N be any weakly semirecursive, non-recursive language. Then* $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \not\subseteq \mathrm{R}_{(k-1)\text{-wtt}}\big(\mathrm{RE}[\textsc{sel}]\big)$.

Proof. The proof is the same as the proof of Theorem 20, except for the addition that by Fact 11 we have $\mathrm{RE}[\textsc{cheat}] = \mathrm{REC}$. □

Corollary 21. *Let C be reasonable and FC countable. Then*

$$C[\textsc{sel}] \subsetneq \mathrm{R}^{C}_{1\text{-tt}}\big(C[\textsc{sel}]\big) \subsetneq \mathrm{R}^{C}_{2\text{-tt}}\big(C[\textsc{sel}]\big) \subsetneq \dots$$

and likewise for the equivalence closures.

Proof. Provided there exists a C-selective, non-C-cheatable language N Theorem 20 states that for all k we have $\mathrm{E}^{\mathrm{L}}_{k\text{-tt}}(N) \not\subseteq \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$ and hence $\mathrm{R}^{C}_{k\text{-tt}}(C[\textsc{sel}]) \not\subseteq \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$ and $\mathrm{E}^{C}_{k\text{-tt}}(C[\textsc{sel}]) \not\subseteq \mathrm{E}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$. However, such a language N exists for countable FC: Using a dovetail argument, it is easily seen that $FC \subseteq \mathrm{FREC}^{A}$ for some oracle A. By a relativised version of the Nonspeedup Theorem the class $\mathrm{REC}^{A}[\textsc{cheat}]$ is countable. But even $\mathrm{L}[\textsc{sel}]$ is uncountable by a construction due to McLaughlin and Martin [15, 26]. □

Note that there also exist hierarchies of the *Turing* reduction closures for all computational models for which Fact 14 holds.

4 Selective Query Complexity of ODD^{N}_{k}

We now show that the selective query complexity of $\textsc{odd}^{N}_{k}$ is exactly k for selective, non-cheatable N, thus generalising Fact 6. The proof uses the following Fact 22, which is easily transferred to all reasonable computational models. For a proof of Fact 22 see for example the discussion after Theorem 4.1 in [3].

Fact 22. *Let N be* P*-selective. Then* $\mathrm{R}^{\mathrm{P}}_{k\text{-tt}}(N) = \mathrm{R}^{\mathrm{P}}_{1\text{-tt}}\big(\textsc{odd}^{N}_{k}\big)$.

Theorem 23. *Let C be reasonable. Let N be any C-selective, non-C-cheatable language. Then* $\textsc{odd}^{N}_{k} \in \mathrm{R}^{C}_{k\text{-tt}}\big(C[\textsc{sel}]\big) \setminus \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$.

Proof. The upper bound on the parallel C-selective query complexity of $\textsc{odd}^{N}_{k}$ is trivial. For the lower bound assume $\textsc{odd}^{N}_{k} \in \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$ for the sake of contradiction. Taking the 1-truth-table reduction closure on both sides yields the inclusion $\mathrm{R}^{C}_{1\text{-tt}}\big(\textsc{odd}^{N}_{k}\big) \subseteq \mathrm{R}^{C}_{1\text{-tt}}\big(\mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)\big) = \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$. By Fact 22 this implies $\mathrm{R}^{C}_{k\text{-tt}}(N) \subseteq \mathrm{R}^{C}_{(k-1)\text{-tt}}\big(C[\textsc{sel}]\big)$, contradicting Theorem 20. □

Theorem 23′. *Let N be any semirecursive, but non-recursive language. Then* $\textsc{odd}^{N}_{k} \in \mathrm{R}_{k\text{-wtt}}\big(\mathrm{RE}[\textsc{sel}]\big) \setminus \mathrm{R}_{(k-1)\text{-wtt}}\big(\mathrm{RE}[\textsc{sel}]\big)$.

Theorem 23′ has previously been proven by Beigel et al. [4]. Note that compared to the proof in [4] the proof in this paper is more elementary as it uses only the Non-speedup Theorem and not the more powerful and difficult Cardinality Theorem [18].

5 Applications

Two Powerful Relativisations. In [14] Hemaspaandra et al. prove the relation $\mathrm{E}^{\mathrm{P}}_{(k+1)\text{-tt}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}}_{k\text{-tt}}(\mathrm{P}[\mathrm{SEL}])$, from which one easily deduces that the k-truth-table reduction and equivalence closures of $\mathrm{P}[\mathrm{SEL}]$ form a proper hierarchy. We show that the relation still holds if we *relativise only the right hand side*. Likewise, we show that a one-sided relativisation is also possible for the relation $\mathrm{E}^{\mathrm{P}}_{\mathrm{tt}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}}_{\mathrm{btt}}(\mathrm{P}[\mathrm{SEL}])$, which is also proven in [14].

Theorem 24. *Let A be any oracle. Then* $\mathrm{E}^{\mathrm{P}}_{(k+1)\text{-tt}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}^A}_{k\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$.

Proof. The proof of Corollary 21 showed that there exists an $N \in \mathrm{P}[\mathrm{SEL}] \setminus \mathrm{P}^A[\mathrm{CHEAT}]$. For this N Theorem 20 states $\mathrm{E}^{\mathrm{L}}_{(k+1)\text{-tt}}(N) \not\subseteq \mathrm{R}^{\mathrm{P}^A}_{k\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$. Since $\mathrm{E}^{\mathrm{L}}_{(k+1)\text{-tt}}(N) \subseteq \mathrm{E}^{\mathrm{P}}_{(k+1)\text{-tt}}(\mathrm{P}[\mathrm{SEL}])$, we get the claim. □

Theorem 25. *Let A be any oracle. Then* $\mathrm{E}^{\mathrm{P}}_{\mathrm{tt}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}^A}_{\mathrm{btt}}(\mathrm{P}^A[\mathrm{SEL}])$.

Proof. Let $N \in \mathrm{P}[\mathrm{SEL}] \setminus \mathrm{P}^A[\mathrm{CHEAT}]$. Consider the language $\mathrm{ODD}^N_\omega := \bigcup_{k=1}^{\infty} \mathrm{ODD}^N_k$. It is linear truth-table equivalent to N and thus an element of $\mathrm{E}^{\mathrm{P}}_{\mathrm{tt}}(\mathrm{P}[\mathrm{SEL}])$. If we had $\mathrm{ODD}^N_\omega \in \mathrm{R}^{\mathrm{P}^A}_{k\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$ for some k, we would also have $\mathrm{ODD}^N_{k+1} \in \mathrm{R}^{\mathrm{P}^A}_{k\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$, contradicting Theorem 23 for $C = \mathrm{P}^A$. □

Serial Equivalence versus Parallel Reduction. By Fact 14 in a simulation of serial queries to a P-selective language by parallel ones we cannot avoid an exponential increase in the number of queries. In [14] Hemaspaandra et al. ask whether perhaps we have at least $\mathrm{E}^{\mathrm{P}}_{k\text{-T}}(\mathrm{P}[\mathrm{SEL}]) \subseteq \mathrm{R}^{\mathrm{P}}_{(2^k-2)\text{-tt}}(\mathrm{P}[\mathrm{SEL}])$. This is not the case.

Theorem 26. *Let A be any oracle. Then* $\mathrm{E}^{\mathrm{P}}_{k\text{-T}}(\mathrm{P}[\mathrm{SEL}]) \not\subseteq \mathrm{R}^{\mathrm{P}^A}_{(2^k-2)\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$.

Proof. Once more, let $N \in \mathrm{P}[\mathrm{SEL}] \setminus \mathrm{P}^A[\mathrm{CHEAT}]$ and let $\ell := 2^k - 1$. Theorem 23 states $\mathrm{ODD}^N_\ell \notin \mathrm{R}^{\mathrm{P}^A}_{(\ell-1)\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$. As ODD^N_ℓ is trivially polynomial time ℓ-truth-table reducible to N, by Fact 14 it is polynomial time k-Turing equivalent to N. Thus $\mathrm{E}^{\mathrm{P}}_{k\text{-T}}(N) \not\subseteq \mathrm{R}^{\mathrm{P}^A}_{(\ell-1)\text{-tt}}(\mathrm{P}^A[\mathrm{SEL}])$. Since $\mathrm{E}^{\mathrm{P}}_{k\text{-T}}(N) \subseteq \mathrm{E}^{\mathrm{P}}_{k\text{-T}}(\mathrm{P}[\mathrm{SEL}])$ and $\ell - 1 = 2^k - 2$, we get the claim. □

Verboseness of Reduction Closures. The languages in $\mathrm{RE}[m\text{-}\mathrm{SIZE}_n]$ are called *(m,n)-verbose* and those in $\mathrm{REC}[m\text{-}\mathrm{SIZE}_n]$ *strongly (m,n)-verbose* [7]. Every language is $(2^n, n)$-verbose. A language which is not $(2^n - 1, n)$-verbose is *n-superterse* [7].

Theorem 27. *Let C be reasonable, $FC \subseteq \mathrm{FREC}$ and let m, n, k be positive integers. Then the following statements are equivalent:*

1. *All languages in $\mathrm{E}^{C}_{k\text{-tt}}(C[\mathrm{SEL}])$ are (m,n)-verbose.*
2. *All languages in $\mathrm{R}^{C}_{k\text{-tt}}(C[\mathrm{SEL}])$ are (m,n)-verbose.*
3. *Every walk on $\mathbb{B}^n$ with transition counts at most k visits at most m bitstrings.*

Proof. Statement 2 trivially implies statement 1. Statement 3 implies statement 2, since $\mathrm{R}^{C}_{k\text{-tt}}\big(C[\text{SEL}]\big) \subseteq \mathrm{REC}[k\text{-WALKS}_n]$ by Theorem 15. Statement 1 implies statement 3, since by the same argument as in Theorem 20′ the covering $k\text{-WALKS}_n$ is the smallest covering $\mathcal{D}$ such that $\mathrm{E}^{C}_{k\text{-tt}}\big(C[\text{SEL}]\big) \subseteq \mathrm{RE}[\mathcal{D}]$. If we also have $\mathrm{E}^{C}_{k\text{-tt}}\big(C[\text{SEL}]\big) \subseteq \mathrm{RE}[m\text{-SIZE}_n]$, we get $k\text{-WALKS}_n \subseteq m\text{-SIZE}_n$. □

The above theorem reduces the problem of quantifying the amount of verboseness of reduction closures of selectivity classes to the purely combinatorial problem of finding walks visiting a maximum number of bitstrings. The following fact gives an easy upper bound on this number. Beigel et al. [7] conjecture that the bound is tight for all k for which $nk \leq 2^n$, and they prove this for $k < 7$. Lemma 29 shows that the bound is also tight for $k = 2^n/n$ and $n = 2^r$. The situation where n is not a power of two is discussed, but not fully solved, in [8].

Fact 28 ([7]). *Every walk on $\mathbb{B}^n$ with transition counts at most k visits at most $nk+1$ bitstrings for odd k and nk bitstrings for even k.*

Lemma 29. *Let $n = 2^r$. Then $(2^n/n)\text{-WALKS}_n \not\subseteq (2^n-1)\text{-SIZE}_n$.*

Proof. Wagner and West [28] proved that for $n = 2^r$ there exists a *balanced n-bit Gray code.* But such a code is a self-avoiding n-bit cycle that has transition counts exactly $2^n/n$. □

Corollary 30. *Let $n = 2^r$. Then $\mathrm{E}^{\mathrm{L}}_{2^n/n\text{-tt}}\big(\mathrm{L}[\text{SEL}]\big)$ contains an n-superterse language, while $\mathrm{R}_{(2^n/n-1)\text{-tt}}\big(\mathrm{REC}[\text{SEL}]\big)$ does not.*

Frequency Computations for Reduction Closures. The languages in the class $\mathrm{REC}_{\mathrm{dist}}[(n-m)\text{-FREQ}_n]$ are called *(m,n)-recursive* [24]. For them, for any n distinct words a bitstring can be computed that agrees with their characteristic string on at least m positions. Arguing as in the proof of Theorem 27 we get the following theorem.

Theorem 31. *Let C be reasonable and $FC \subseteq \mathrm{FREC}$. Then the following statements are equivalent for $0 < m < n$ and $k \geq 1$:*

1. *All languages in $\mathrm{E}^{C}_{k\text{-tt}}\big(C[\text{SEL}]\big)$ are (m,n)-recursive.*
2. *All languages in $\mathrm{R}^{C}_{k\text{-tt}}\big(C[\text{SEL}]\big)$ are (m,n)-recursive.*
3. *Every walk on $\mathbb{B}^n$ with transition counts at most k is contained in a closed ball of radius $n-m$.*

The following lemma, whose simple proof is given in the appendix, gives loose upper and lower bounds for k from the above theorem in terms of the *covering number* $\mathrm{k}(n,r)$. It is the smallest number of closed balls of radius r needed to cover $\mathbb{B}^n$. Except for some special cases [10] only upper and lower bounds are known for the covering number.

Lemma 32. *For $0 < m < n$ let $\kappa := \mathrm{k}(n,m-1)-1$. Then we have $\kappa\text{-WALKS}_n \not\subseteq (n-m)\text{-FREQ}_n$ and $\lfloor(\kappa-1)/n\rfloor\text{-WALKS}_n \subseteq (n-m)\text{-FREQ}_n$.*

Corollary 33. *There exist languages in $\mathrm{E}^{\mathrm{L}}_{(\mathrm{k}(n,m-1)-1)\text{-tt}}\big(\mathrm{L}[\text{SEL}]\big)$ that are not (m,n)-recursive, while all languages in $\mathrm{R}_{\lfloor(\mathrm{k}(n,m-1)-2)/n\rfloor\text{-tt}}\big(\mathrm{REC}[\text{SEL}]\big)$ are.*

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Technical Appendix

Proof of Lemma 32. For the first claim, we present a walk on $\mathbb{B}^n$ with transition counts at most κ whose complement does not contain a closed ball of radius $m-1$ and which is hence not contained in a closed ball of radius $n-m$. Let $b_1, \dots, b_{\mathrm{k}(n,m-1)}$ be bitstrings such that $\mathrm{B}_{m-1}(b_1), \dots, \mathrm{B}_{m-1}\big(b_{\mathrm{k}(n,m-1)}\big)$ cover $\mathbb{B}^n$. Consider a walk starting at b_1. To get from b_1 to b_2 we only need to change every position at most once. Likewise from b_2 to b_3 and so on. We get a walk visiting all b_i with transition counts at most κ. As every ball $\mathrm{B}_{m-1}(b)$ contains at least one b_i, the complement of the constructed walk does not contain any $\mathrm{B}_{m-1}(b)$.

For the second claim let $b_1, \dots, b_\ell$ be an arbitrary walk with transition counts at most $(\kappa-1)/n$. It visits at most $n(\kappa-1)/n+1=\kappa$ many bitstrings. We claim that the complement of the walk contains a ball of radius $m-1$. If this were not the case $\mathrm{B}_{m-1}(b_1), \dots, \mathrm{B}_{m-1}(b_\ell)$ would *cover* $\mathbb{B}^n$ which is impossible since $\ell \le \kappa < \mathrm{k}(n, m-1)$. ☐

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