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The Complexity of Minimal Satisfiability Problems*

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Abstract

A dichotomy theorem for a class of decision problems is a result asserting that certain problems in the class are solvable in polynomial time, while the rest are NP-complete. The first remarkable such dichotomy theorem was proved by T.J. Schaefer in 1978. It concerns the class of generalized satisfiability problems SAT(S), whose input is a CNF(S)-formula, i.e., a formula constructed from elements of a fixed set S of generalized connectives using conjunctions and substitutions by variables.

Here, we investigate the complexity of minimal satisfiability problems MIN SAT(S), where S is a fixed set of generalized connectives. The input to such a problem is a CNF(S)-formula and a satisfying truth assignment; the question is to decide whether there is another satisfying truth assignment that is strictly smaller than the given truth assignment with respect to the coordinate-wise partial order on truth assignments. Minimal satisfiability problems were first studied by researchers in artificial intelligence while investigating the computational complexity of propositional circumscription. The question of whether dichotomy theorems can be proved for these problems was raised at that time, but was left open. We settle this question affirmatively by establishing a dichotomy theorem for the class of all MIN SAT(S)-problems, where S is a finite set of generalized connectives. We also prove a dichotomy theorem for a variant of MIN SAT(S) in which the minimization is restricted to a subset of the variables, whereas the remaining variables may vary arbitrarily (this variant is related to extensions of propositional circumscription and was first studied by Cadoli). Moreover, we show that similar dichotomy theorems hold also when some of the variables are assigned constant values. Finally, we give simple criteria that tell apart the polynomial-time solvable cases of these minimal satisfiability problems from the NP-complete ones.

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1 Introduction and Summary of Results

Computational complexity strives to analyze important algorithmic problems by first placing them in suitable complexity classes and then attempting to determine whether they are complete for the class under consideration or they actually belong to a more restricted complexity class. This approach to analyzing algorithmic problems has borne fruit in numerous concrete cases and has led to the successful development of the theory of NP-completeness. In this vein, *dichotomy theorems* for classes of NP-problems are of particular interest, where a dichotomy theorem is a result that concerns an infinite class $\mathcal{F}$ of related decision problems and asserts that certain problems in $\mathcal{F}$ are solvable in polynomial time, while on the contrary all other problems in $\mathcal{F}$ are NP-complete. It should be pointed out that the a priori existence of dichotomy theorems cannot not be taken for granted. Indeed, Ladner [Lad75] showed that if P $\neq$ NP, then there are problems in NP that are neither NP-complete nor in P. Consequently, a given class $\mathcal{F}$ of NP-problems may contain such problems of intermediate complexity, which rules out the existence of a dichotomy theorem for $\mathcal{F}$.

The first remarkable (and highly non-trivial) dichotomy theorem was established by Schaefer [Sch78], who introduced and studied the class of Generalized Satisfiability problems (see also [GJ79, LO6, page 260]). A *logical relation* (or *generalized connective*) R is a non-empty subset of $\{0,1\}^k$, for some $k \geq 1$. If $S = \{R_1, \ldots, R_m\}$ is a finite set of logical relations, then a CNF(S)-formula is a conjunction of expressions (called *generalized clauses* or, simply, *clauses*) of the form $R'_i(x_1, \ldots, x_k)$, where each R'_i is a relation symbol representing the logical relation R_i in S and each x_j is a Boolean variable. Each finite set S of logical relations gives rise to the Generalized Satisfiability problem Sat(S): given a CNF(S)-formula φ, is φ satisfiable? Schaefer isolated six efficiently checkable conditions and proved the following dichotomy theorem for the class of all Generalized Satisfiability problems Sat(S): if the set S satisfies at least one of these six conditions, then Sat(S) is solvable in polynomial time; otherwise, Sat(S) is NP-complete. Since that time, only a handful of dichotomy theorems for other classes of decision problems have been established. Two notable ones are the dichotomy theorem for the class of Fixed Subgraph Homeomorphism problems on directed graphs, obtained by Fortune, Hopcroft and Wyllie [FHW80], and the dichotomy theorem for the class of H-Coloring problems on undirected graphs, obtained by Hell and Nešetřil [HN90]. The latter is a special case of Constraint Satisfaction, a rich class of problems that have been the object of systematic study in artificial intelligence. It should be noted that no dichotomy theorem for the entire class of Constraint Satisfaction problems has been established thus far, in spite of intensive efforts to this effect (see Feder and Vardi [FV99], Jeavons, Cooper and Gyssens [JCG97]).

In recent years, researchers have obtained dichotomy theorems for optimization problems, counting problems, enumeration problems, and decision problems that are variants of Generalized Satisfiability problems. Specifically, Creignou [Cre95], Khanna, Sudan, Trevisan and Williamson [KSTW01], and Zwick [Zwi98] obtained dichotomy theorems for certain classes of optimization problems related to propositional satisfiability and Boolean constraint satisfaction; Creignou and Hermann [CH96] proved a dichotomy theorem for the class of counting problems that ask for the number of satisfying assignments of a given CNF(S)-formula; Creignou and Hébrard [CH97] showed that a dichotomy theorem holds for the class of enumeration problems that ask whether there is a polynomial-delay algorithm that generates all satisfying assignments of a given CNF(S)-formula; also, Kavvadias and Sideri [KS98] established a dichotomy theorem for the class of decision problems Inverse Sat(S) that ask whether a given set of truth assignments is the set of all satisfying assignments of some CNF(S)-formula (in all these results S is a finite set of logical relations). Even more recently, Reith and Vollmer [RV00] proved a dichotomy theorem for the class of optimization problems LexMin Sat(S) and LexMax Sat(S) that ask for the lexicographically minimal (or maximal) truth assignment that satisfies a given CNF(S)-formula.

Researchers have also investigated the class of decision problems Min Sat(S) that ask whether a satisfying truth assignment of a CNF(S)-formula is minimal with respect to the coordinate-wise partial order. More precisely, if S is a finite set of logical relations, then Min Sat(S) is the following decision problem: given a CNF(S)-formula φ and a satisfying truth assignment α of φ, is there a satisfying truth assignment β of φ such that $\beta < \alpha$, where $<$ is the coordinate-wise partial order on truth assignments? These decision problems were introduced and studied by researchers in artificial intelligence while investigating *circumscription*, a well-developed formalism of common-sense reasoning introduced by McCarthy [McC80] about twenty years ago. The main question left open about Min Sat(S) was whether a dichotomy theorem holds for the class of all Min Sat(S) problems, where S is a finite set of logical relations. In the present paper, we settle this question in the affirmative and also provide easily checkable criteria that tell apart the polynomial-time solvable cases of Min Sat(S) from the NP-complete ones. Moreover, we obtain dichotomy theorems for classes of several related decision problems that have to do with powerful extensions of circumscription.

In circumscription, properties are specified using formulas of some logic, a natural partial order between models of each formula is considered, and preference is given to models that are minimal with respect to this partial order. McCarthy's key intuition was that minimal models should be preferred because they are the ones that have as few "exceptions" as possible and thus embody common-sense. A fundamental algorithmic problem about every logical formalism is *model checking*, the problem of deciding whether a finite structure satisfies a formula. As regards circumscription, model checking amounts to the problem of deciding whether a finite structure is a minimal model of a formula. The simplest case of circumscription is *propositional circumscription*, where properties are specified using formulas of propositional logic; thus, the model checking problem for propositional circumscription is precisely the problem of deciding whether a satisfying truth assignment of a propositional formula is minimal with respect to the coordinate-wise order. It is not hard to show that this problem is coNP-complete, when arbitrary propositional formulas are allowed as part of the input. For this reason, researchers in artificial intelligence embarked on the pursuit of tractable cases of the model checking problem for propositional circumscription. In particular, Cadoli [Cad92, Cad93] adopted Schaefer's approach, introduced the class of decision problems Min Sat(S), identified several tractable cases, and raised the question of the existence of a dichotomy theorem for this class (see [Cad93, page 132]). Moreover, Cadoli pointed out that if a dichotomy theorem for Min Sat(S) indeed exists, then the dividing line is going to be very different from the dividing line in Schaefer's dichotomy theorem for Sat(S). To see this, consider first the set $S = \{R_{1/3}\}$, where $R_{1/3} = \{(1,0,0),(0,1,0),(0,0,1)\}$. In this case, Sat($S$) is the well-known NP-complete problem Positive-1-In-3-Sat, while on the contrary Min Sat(S) is trivial, since it can be easily verified that every satisfying truth assignment of a given CNF(S)-formula is minimal. Thus, an intractable case of Sat(S) becomes a tractable (in fact, a trivial) case of Min Sat(S). In the opposite direction, Cadoli [Cad92, Cad93] showed that certain tractable (in fact, trivial) cases of Sat(S) become NP-complete cases of Min Sat(S). Specifically, one of the six tractable cases in Schaefer's dichotomy theorem is the case where S consists entirely of 1-*valid* logical relations, that is, every relation R in S contains the all-ones tuple $(1,\dots,1)$ (and, hence, every CNF(S)-formula is satisfied by the truth assignment that assigns 1 to every variable). In contrast, Cadoli [Cad92, Cad93] discovered a finite set S of 1-valid relations such that Min Sat(S) is NP-complete.

As it turns out, the collection of 1-valid relations holds the key to the dichotomy theorem for

MIN SAT(S). More precisely, we first establish a dichotomy theorem for the class of MIN SAT(S) problems, where S is a finite set of 1-valid relations. Using this restricted dichotomy theorem as a stepping stone, we derive the desired dichotomy theorem for the full class of MIN SAT(S) problems, where S is a finite set of arbitrary logical relations. It should be noted that our results differ from earlier dichotomy theorems in one important aspect. Specifically, in all earlier dichotomy theorems the tractable cases arise from conditions that are directly applied to the set S of logical relations under consideration; in our main dichotomy theorem, however, the tractable cases arise from conditions that are applied not to the set S of logical relations at hand, but to a certain set S^* of 1-valid logical relations obtained from S by projecting the relations in S in a particular way. This provides an a posteriori explanation for the earlier observation that the dividing line in any dichotomy theorem for MIN SAT(S) has to be different from the dividing line in Schaefer's dichotomy theorem for SAT(S).

All dichotomy theorems described thus far involve CNF(S)-formulas that do not contain the constant symbols **0** and **1**; Schaefer, however, [Sch78] also proved a dichotomy theorem for CNF(S)-formulas with constant symbols. Here, we derive dichotomy theorems for minimal satisfiability of CNF(S)-formulas with constant symbols as well. Researchers in artificial intelligence have also investigated various powerful extensions of circumscription in which the partial order among models of a formula is modified, so that some parts of the model are assigned fixed values and some other parts are allowed to vary arbitrarily [Lif85, McC85]. In the present paper, we also establish dichotomy theorems for the model checking problem for the main extensions of propositional circumscription; this answers another question left open by Cadoli [Cad92, Cad93].

2 Preliminaries and Background

This section contains the definitions of the main concepts used in this paper and a minimum amount of the necessary background material from Schaefer's work on the complexity of GENERALIZED SATISFIABILITY problems [Sch78].

Definition 2.1: Let $S = \{R_1, \ldots, R_m\}$ be a finite set of logical relations of various arities, let $S' = \{R'_1, \ldots, R'_m\}$ be a set of relation symbols whose arities match those of the relations in S, and let V be an infinite set of variables.

A CNF(S)-formula is a finite conjunction $C_1 \wedge \ldots \wedge C_n$ of clauses built using relation symbols from S' and variables from V, that is, each C_i is an atomic formula of the form $R'_j(x_1, \ldots, x_k)$, where R'_j is a relation symbol of arity k in S', and $x_1, \ldots, x_k$ are variables in V. A $\mathrm{CNF_C}(S)$-formula is a formula obtained from a CNF(S)-formula by substituting some of its variables by the constant symbols **0** or **1**. The semantics of CNF(S)-formulas and $\mathrm{CNF_C}(S)$-formulas are defined in a standard way by assuming that variables range over the set of bits $\{0, 1\}$, each relation symbol R'_j in S' is interpreted by the corresponding relation R_j in S, and the constant symbols **0** and **1** are interpreted by 0 and 1 respectively.

SAT(S) is the following decision problem: given a CNF(S)-formula φ, is it satisfiable? (i.e., is there a truth assignment to the variables of φ that makes every clause of φ true?) The decision problem $\mathrm{SAT_C}(S)$ is defined in a similar way. ∎

It is clear that, for each finite set S of logical relations, both SAT(S) and $\mathrm{SAT_C}(S)$ are problems in NP. Moreover, several well-known NP-complete problems and several important tractable cases of Boolean satisfiability can easily be cast as SAT(S) problems for particular sets S of logical relations. Indeed, we already saw in the previous section that the NP-complete problem POSITIVE-1-IN-3-SAT ([GJ79, LO4, page 259]) is precisely the problem SAT(S), where S is the singleton consisting of the relation $R_{1/3} = \{(1,0,0),(0,1,0),(0,0,1)\}$. Moreover, the prototypical NP-complete problem 3-SAT coincides with the problem SAT(S), where $S = \{R_0, R_1, R_2, R_3\}$ and $R_0 = \{0,1\}^3 - \{(0,0,0)\}$ (expressing the clause $(x \vee y \vee z)$), $R_1 = \{0,1\}^3 - \{(1,0,0)\}$ (expressing the clause $(\neg x \vee y \vee z)$), $R_2 = \{0,1\}^3 - \{(1,1,0)\}$ (expressing the clause $(\neg x \vee \neg y \vee z)$), and $R_3 = \{0,1\}^3 - \{(1,1,1)\}$ (expressing the clause $(\neg x \vee \neg y \vee \neg z)$). Similarly, but on the side of tractability, 2-SAT is precisely the problem SAT(S), where $S = \{R_0, R_1, R_2\}$ and $R_0 = \{0,1\}^2 - \{(0,0)\}$ (expressing the clause $(x \vee y)$), $R_1 = \{0,1\}^2 - \{(1,0)\}$ (expressing the clause $(\neg x \vee y)$), and $R_2 = \{0,1\}^2 - \{(1,1)\}$ (expressing the clause $(\neg x \vee \neg y)$).

The next two definitions introduce the key concepts needed to formulate Schaefer's dichotomy theorems.

Definition 2.2: Let φ be a propositional formula.

φ is 1-*valid* if it is satisfied by the truth assignment that assigns 1 to every variable. Similarly, φ is 0-*valid* if it is satisfied by the truth assignment that assigns 0 to every variable.

φ is *bijunctive* if it is a 2CNF-formula, i.e., it is a a conjunction of clauses each of which is a disjunction of at most two *literals* (variables or negated variables).

φ is *Horn* if it is the conjunction of clauses each of which is a disjunction of literals such that at most one of them is a variable. Similarly, φ is *dual Horn* if it is the conjunction of clauses each of which is disjunction of literals such that at most one of them is a negated variable.

φ is *affine* if it is the conjunction of subformulas each of which is an *exclusive disjunction* of literals or a negation of an exclusive disjunctions of literals (by definition, an exclusive disjunction of literals is satisfied exactly when an odd number of these literals are true; we will use $\oplus$ as the symbol of the exclusive disjunction). Note that a formula φ is affine precisely when the set of its satisfying assignments is the set of solutions of a system of linear equations over the field $\{0,1\}$. ∎

Definition 2.3: Let R be a logical relation and S a finite set of logical relations.

R is 1-*valid* if it contains the tuple $(1,1,\ldots,1)$, whereas R is 0-*valid* if it contains the tuple $(0,0,\ldots,0)$. We say that S is 1-*valid* (0-*valid*) if every member of S is 1-valid (0-valid).

R is *bijunctive* (*Horn*, *dual Horn*, or *affine*, respectively) if there is a propositional formula φ which is bijunctive (Horn, dual Horn, or affine, respectively) and such that R coincides with the set of truth assignments satisfying φ.

S is *Schaefer* if at least one of the following four conditions hold: every member of S is bijunctive; every member of S is Horn; every member of S is dual Horn; every member of S is affine. Otherwise, we say that S is *non-Schaefer*. ∎

There are simple criteria to determine whether a logical relation is bijunctive, Horn, dual Horn, or affine. In fact, a set of such criteria was already provided by Schaefer [Sch78]; moreover, Dechter and Pearl [DP92] gave even simpler criteria for a relation to be Horn or dual Horn. Each of these criteria involves a *closure property* of the logical relations at hand under a certain function. Specifically, a relation R is bijunctive if and only if for all $t_1, t_2, t_3 \in R$, we have that $(t_1 \vee t_2) \wedge (t_2 \vee t_3) \wedge (t_1 \vee t_3) \in R$, where the operators $\vee$ and $\wedge$ are applied coordinate-wise to the bit-tuples. This means that the i-th coordinate of the tuple $(t_1 \vee t_2) \wedge (t_2 \vee t_3) \wedge (t_1 \vee t_3)$ is equal to 1 exactly when the majority of the i-th coordinates of t_1, t_2, t_3 is equal to 1. Thus, this criterion states that R is bijunctive exactly when it is closed under coordinate-wise applications of the ternary *majority* function. R is Horn (respectively, dual Horn) if and only if for all $t_1, t_2 \in R$, we have that $t_1 \wedge t_2 \in R$ (respectively, $t_1 \vee t_2 \in R$). Finally, R is affine if and only if for all $t_1, t_2, t_3 \in R$, we have that $t_1 \oplus t_2 \oplus t_3 \in R$. As an example, it is easy to apply these criteria to the ternary relation $R_{1/3} = \{(1,0,0),(0,1,0),(0,0,1)\}$ and verify that $R_{1/3}$ is neither bijunctive, nor Horn, nor dual Horn, nor affine; moreover, it is obvious that $R_{1/3}$ is neither 1-valid nor 0-valid. Finally, there are polynomial-time algorithms that given a logical relation that is bijunctive (Horn, dual Horn, or affine, respectively), produce a defining propositional formula which is bijunctive (Horn, dual Horn, or affine, respectively). See [DP92, KV00] for descriptions of these algorithms.

If S is a 0-valid or a 1-valid set of logical relations, then $\textsc{Sat}(S)$ is a trivial decision problem (the answer is always "yes"). If S is an affine set of logical relations, then $\textsc{Sat}(S)$ can easily be solved in polynomial time using Gaussian elimination. Moreover, there are well-known polynomial-time algorithms for the satisfiability problem for the class of all bijunctive formulas (2-SAT), the class of all Horn formulas, and the class of all dual Horn formulas. Schaefer's seminal discovery was that the above six cases are the *only* ones that give rise to tractable cases of $\textsc{Sat}(S)$; furthermore, the last four are the *only* ones that give rise to tractable cases of $\textsc{Sat}_{\mathrm{C}}(S)$.

Theorem 2.4: [Schaefer's Dichotomy Theorems, [Sch78]] *Let S be a finite set of logical relations.*

If S is 0-valid or 1-valid or Schaefer, then $\textsc{Sat}(S)$ is solvable in polynomial time; otherwise, it is NP-complete.

If S is Schaefer, then $\textsc{Sat}_{\mathrm{C}}(S)$ is solvable in polynomial time; otherwise, it is is NP-complete.

As an application, Theorem 2.4 immediately implies that POSITIVE-1-IN-3-SAT is NP-complete, since this is the same problem as $\textsc{Sat}(R_{1/3})$, and $R_{1/3}$ is neither 0-valid, nor 1-valid, nor Schaefer.

To obtain the above dichotomy theorems, Schaefer had to first establish a result concerning the expressive power of $\mathrm{CNF}_{\mathrm{C}}(S)$ formulas. Informally, this result asserts that if S is a non-Schaefer set of logical relations, then $\mathrm{CNF}_{\mathrm{C}}(S)$-formulas have extremely highy expressive power, in the sense that every logical relation can be defined from a $\mathrm{CNF}_{\mathrm{C}}(S)$-formula using existential quantification.

Theorem 2.5: [Schaefer's Expressibility Theorem, [Sch78]] *Let S be a finite set of logical relations. If S is non-Schaefer, then for every k-ary logical relation R there is a $\mathrm{CNF}_{\mathrm{C}}(S)$-formula $\varphi(x_1, \ldots, x_k, z_1, \ldots, z_m)$ such that R coincides with the set of all truth assignments to the variables $x_1, \ldots, x_k$ that satisfy the formula $(\exists z_1) \cdots (\exists z_m)\varphi(x_1, \ldots, x_k, z_1, \ldots, z_m)$.*

3 Dichotomy Theorems for Minimal Satisfiability

In this section, we present our main dichotomy theorem for the class of all minimal satisfiability problems $\textsc{Min Sat}(S)$. We begin with the precise definition of $\textsc{Min Sat}(S)$, as well as of certain variants of it that will play an important role in the sequel.

Definition 3.1: Let $\leq$ denote the standard total order on $\{0, 1\}$, which means that $0 \leq 1$.

Let k be a positive integer and let $\alpha = (a_1, \ldots, a_k)$, $\beta = (b_1, \ldots, b_k)$ be two k-tuples in $\{0, 1\}^k$. We write $\beta \leq \alpha$ to denote that $b_i \leq a_i$, for every $i \leq k$. Also, $\beta < \alpha$ denotes that $\beta \leq \alpha$ and $\beta \neq \alpha$.

Let S be a finite set of logical relations. $\textsc{Min Sat}(S)$ is the following decision problem: given a $\mathrm{CNF}(S)$-formula φ and a satisfying truth assignment α of φ, is there a satisfying truth assignment β of φ such that $\beta < \alpha$? In other words, $\textsc{Min Sat}(S)$ is the problem to decide whether or not a given truth assignment of a given $\mathrm{CNF}(S)$-formula is minimal. The decision problem $\textsc{Min Sat}_{\mathrm{C}}(S)$ is defined in a similar way by allowing $\mathrm{CNF}_{\mathrm{C}}(S)$-formulas as part of the input.

Let S be a 1-valid set of logical relations. 1-$\textsc{Min Sat}(S)$ is the following decision problem: given a $\mathrm{CNF}(S)$-formula φ (note that φ is necessarily 1-valid), is there a satisfying truth assignment of φ that is different (and, hence, smaller) from the all-ones truth assignment $(1, \ldots, 1)$?

A $\mathrm{CNF}_1(S)$-formula is obtained from a $\mathrm{CNF}(S)$-formula by replacing some of its variable by the constant symbol **1**. The decision problem 1-$\textsc{Min Sat}_1(S)$ is defined the same way as

1-MIN SAT(S), except that CNF$_1$(S)-formulas are allowed as part of the input (arbitrary CNF$_C$(S)-formulas are not allowed, since substituting variables by $\mathbf{0}$ may destroy 1-validity). ∎

As mentioned in the introduction, Cadoli [Cad92, Cad93] raised the question of whether a dichotomy theorem for the class of all MIN SAT(S) problems exists. Note that if S is a 0-valid set of logical relations, then MIN SAT(S) is a trivial decision problem. Moreover, Cadoli showed that if S is a Schaefer set, then MIN SAT(S) is solvable in polynomial time. To see this, let φ be a CNF(S)-formula and α be a k-tuple in $\{0,1\}^k$ that satisfies φ. By reordering the variables, we may assume without loss of generality that for some l such that $1 \leq l \leq k+1$, each component a_j, for $1 \leq j < l$, is equal to 0, while each of the remaining components is equal to 1. For each i such that $l \leq i \leq k$, let φ_i be the CNF$_C$(S)-formula obtained from φ by substituting the variables $x_1, \ldots, x_{l-1}$ *and* the variable x_i with $\mathbf{0}$. It is easy to see that φ has a satisfying truth assignment strictly less than α if and only if at least one of the formulas φ_i for $l \leq i \leq k$ is satisfiable. This argument shows that MIN SAT(S) is polynomially reducible to SAT$_C$(S); consequently, if S is Schaefer, then MIN SAT(S) is solvable in polynomial time. This argument also shows that if S is Schaefer, then MIN SAT$_C$(S) is solvable in polynomial time. Actually, if S is Schaefer then the complexities of both MIN SAT(S) and MIN SAT$_C$(S) are at most n times the complexity of SAT$_C$(S), where n is the size of the input. Moreover, if, in addition, SAT$_C$(S) is in NC, then so are MIN SAT(S) and MIN SAT$_C$(S) (for a characterization of the polynomial-time cases of SAT(S) that are in NC, see [CKS01]). On the intractability side, however, Cadoli [Cad92, Cad93] showed that there is a 1-valid set of logical relations such that MIN SAT(S) is NP-complete. Consequently, any dichotomy theorem for MIN SAT(S) will be substantially different from Schaefer's dichotomy theorem for SAT(S).

Clearly, a dichotomy theorem for MIN SAT(S) should also yield a dichotomy theorem for the special case of MIN SAT(S) in which S is restricted to be a 1-valid set of logical relations. In what follows, we first establish a dichotomy theorem for this restricted case of MIN SAT(S) and then use it to derive the desired dichotomy theorem for MIN SAT(S), where S is an arbitrary finite set of logical relations. In fact, the first step in establishing the dichotomy of MIN SAT(S) for 1-valid S is to prove an NP-hardness result for a special case of 1-MIN SAT$_1$(S) (see Step 1 in the proof of Theorem 3.2 below). At this point, we should mention that Creignou and Hébrard [CH97] have established a dichotomy theorem for the family of the following decision problems SAT*(S), where S is an arbitrary finite set of logical relations: given a CNF(S)-formula φ, does φ have a satisfying truth assignment that differs from *both* the all-ones truth assignment $(1, \ldots, 1)$ and the all-zeros assignment $(0, \ldots, 0)$? While SAT*(S) bears a similarity to 1-MIN SAT(S), Creignou and Hébrard's results for SAT*(S) do not imply the NP-hardness for the special case of 1-MIN SAT(S) that is need in our proof; furthermore, they do not imply the dichotomy theorem for MIN SAT(S), when S a 1-valid set of logical relations. This is because an affirmative answer to an instance φ of 1-MIN SAT(S) does not imply an affirmative answer to the same instance of SAT*(S) (for example, if φ is satisfied only by the all-ones and the all-zeros assignment, then φ is a "yes" instance of 1-MIN SAT(S), but a "no" instance of SAT*(S)). Therefore, we cannot directly deduce that a hard case of SAT*(S) is a hard case of 1-MIN SAT(S) or of MIN SAT(S), even for 1-valid sets S of logical relations. Moreover, the proof of the dichotomy of SAT*(S) in [CH97] cannot be used to simplify the proof of our result, because of a certain "symmetry" between zero and one which is exploited in establishing the NP-hard cases of SAT*(S) in [CH97]. The problems we study in this paper, however, lack this kind of "symmetry" and, consequently, the proofs of our dichotomy results turn out to be different. The dichotomy theorem in [CH97], however, uses a certain technical lemma, which is also used in our proof (see Lemma 3.3 below).

Theorem 3.2: [Dichotomy of MIN SAT(S) for 1-valid S] *Let S be a 1-valid set of logical relations.*

If S is 0-valid or Schaefer, then MIN SAT(S) *is solvable in polynomial time; otherwise, it is NP-complete.*

If S is Schaefer, then MIN SAT$_C$(S) *is solvable in polynomial time; otherwise, it is NP-complete.*

Proof: Let S be a 1-valid set of logical relations. In view of the remarks preceding the statement of the theorem, it remains to establish the intractable cases of the two dichotomies. The proof involves three main steps; the first step uses Schaefer's Expressibility Theorem 2.5, whereas the second step requires the development of additional technical machinery concerning the expressibility of the binary logical relation $\{(0,0),(0,1),(1,1)\}$, which represents the *implication* connective $\to$.

Step 1: If S is 1-valid and non-Schaefer, then SAT($R_{1/3}$) is log-space reducible to 1-MIN SAT$_1$($S \cup \{\to\}$). Consequently, if S is 1-valid and non-Schaefer, then 1-MIN SAT$_1$($S \cup \{\to\}$) is NP-complete.

Step 2: If S is 1-valid and non-Schaefer, then 1-MIN SAT$_1$($S \cup \{\to\}$) is log-space reducible to MIN SAT$_C$(S). Consequently, if S is 1-valid and non-Schaefer, then MIN SAT$_C$(S) is NP-complete.

Step 3: If S is 1-valid but neither 0-valid nor Schaefer, then MIN SAT$_C$(S) is log-space reducible to MIN SAT(S). Consequently, if S is 1-valid but neither 0-valid nor Schaefer, then MIN SAT(S) is NP-complete.

Proof of Step 1: Assuming that S is 1-valid and non-Schaefer, we will exhibit a log-space reduction of SAT($R_{1/3}$) to 1-MIN SAT$_1$($S \cup \{\to\}$). According to Definition 3.1, the latter problem asks: given a CNF$_1$($S \cup \{\to\}$)-formula, is it satisfied by a truth assignment that is different from the all-ones truth assignment $(1,\ldots,1)$?

Let $\varphi(\bar{x})$ be a given CNF($\{R_{1/3}\}$)-formula, where $\bar{x} = (x_1,\ldots,x_n)$ is the list of its variables. By applying Schaefer's Expressibility Theorem 2.5 to the occurrences of $R_{1/3}$ in $\varphi(\bar{x})$, we can construct in log-space a CNF(S)-formula $\chi(\bar{x},\bar{z},w_0,w_1)$, such that $\varphi(\bar{x}) \equiv \exists\bar{z}\chi(\bar{x},\bar{z},\mathbf{0}/w_0,\mathbf{1}/w_1)$, where $\bar{z} = (z_1,\ldots,z_m), w_0, w_1$ are new variables different from $\bar{x}$ (substitutions of different variables by the same constant can be easily consolidated to substitutions of the occurrences of a single variable by that constant). Notice that the formula $\chi(\bar{x},\bar{z},w_0,\mathbf{1}/w_1)$, whose variables are $\bar{x}, \bar{z}$, and w_0, is a CNF$_1$(S)-formula, since it is obtained from a CNF(S)-formula by substitutions by $\mathbf{1}$ only. Let $\psi(\bar{x},\bar{z},w_0)$ be the following formula:

$$\chi(\bar{x},\bar{z},w_0,\mathbf{1}/w_1) \wedge \left(\bigwedge_{i=1}^{n}(w_0 \to x_i)\right) \wedge \left(\bigwedge_{j=1}^{m}(w_0 \to z_j)\right).$$

It is clear that $\psi(\bar{x},\bar{z},w_0)$ is a CNF$_1$($S \cup \{\to\}$)-formula (hence, 1-valid, because S is 1-valid) and that the following two logical equivalences hold:

$$\psi(\bar{x},\bar{z},\mathbf{0}/w_0) \equiv \chi(\bar{x},\bar{z},\mathbf{0}/w_0,\mathbf{1}/w_1) \quad \text{and} \quad \varphi(\bar{x}) \equiv \exists\bar{z}\chi(\bar{x},\bar{z},\mathbf{0}/w_0,\mathbf{1}/w_1) \equiv \exists\bar{z}\psi(\bar{x},\bar{z},\mathbf{0}/w_0).$$

It is now easy to verify that the given CNF($\{R_{1/3}\}$)-formula $\varphi(\bar{x})$ is satisfiable if and only if the CNF$_1$($S \cup \{\to\}$)-formula $\psi(\bar{x},\bar{z},w_0)$ is satisfied by a truth assignment different from the all-ones truth assignment $(1,\ldots,1)$. This completes the proof of Step 1. ∎

To motivate the proof of Step 2, let us consider the combined effect of Steps 1 and 2. Once both these steps have been established, it will follow that SAT($\{R_{1/3}\}$) is log-space reducible to MIN SAT$_C$(S), which means that an NP-complete satisfiability problem will have been reduced to a minimal satisfiability problem. Note that the only information we have about S is that it is a 1-valid, non-Schaefer set of logical relations. Therefore, it is natural to try to use Schaefer's Expressibility

Theorem 2.5 in the desired reduction, since it tells us that $R_{1/3}$ is definable from some $\text{CNF}_\text{C}(S)$-formula using existential quantification. The presence of existential quantifiers, however, introduces a new difficulty in our context, because this way we reduce the satisfiability of a $\text{CNF}(\{R_{1/3}\})$-formula $\varphi(\bar{x})$ to the minimal satisfiability of a $\text{CNF}_\text{C}(S)$-formula $\psi(\bar{x}, \bar{z})$, where $\bar{z}$ are additional variables. It is the presence of these additional variables that creates a serious difficulty for minimal satisfiability, unlike the case of satisfiability in Schaefer's Dichotomy Theorem 2.4. Specifically, it is conceivable that, while we toil to preserve the minimality of truth assignments to the variables $\bar{x}$, the witnesses to the existentially quantified variables $\bar{z}$ may very well destroy the minimality of truth assignments to the entire list of variable $\bar{x}, \bar{z}$. Note that this difficulty was bypassed in Step 1 by augmenting S with the implication connective $\rightarrow$, which made it possible to produce formulas in which we control the witnesses to the variables $\bar{z}$. The proof of Step 2, however, hinges on the following technical result that provides precise information about the definability of the implication connective $\rightarrow$ from an arbitrary 1-valid, non-Schaefer set S of logical relations. This result is an immediate corollary of Lemma 4.9 in [CH97].

Lemma 3.3: [Creignou and Hébrard, [CH97]] *Let S be a 1-valid, non-Schaefer set of logical relations. Then at least one of the following two statements is true about the implication connective.*

1. *There exists a $\text{CNF}_\text{C}(S)$-formula $\varepsilon(x, y)$ such that $(x \rightarrow y) \equiv \varepsilon(x, y)$.*
2. *There exists in $\text{CNF}_\text{C}(S)$-formula $\eta(x, y, z)$ such that*

   (i) $(x \rightarrow y) \equiv (\exists z)\eta(x, y, z)$; (ii) $\eta(x, y, z)$ is satisfied by the truth assignment $(1, 1, 1)$; (iii) $\eta(x, y, z)$ is not satisfied by the truth assignment $(1, 1, 0)$.

   In other words, the formula $(\exists z)\eta(x, y, z)$ is logically equivalent to $(x \rightarrow y)$ and has the additional property that 1 is the only witness for the variable z under the truth assignment $(1, 1)$ to the variables (x, y).

To make this paper as self-contained as possible, we include in the Appendix a proof of the above Lemma 3.3. We are now ready to embark on the proof of Step 2.

Proof of Step 2: Assuming that S is 1-valid and non-Schaefer, we will exhibit a log-space reduction of 1-$\text{Min Sat}_1(S \cup \{\rightarrow\})$ to $\text{Min Sat}_\text{C}(S)$. According to Lemma 3.3, either there is a $\text{CNF}_\text{C}(S)$-formula $\varepsilon(x, y)$ that is logically equivalent to $(x \rightarrow y)$ or there is a $\text{CNF}_\text{C}(S)$-formula $\eta(x, y, z)$ such that $(\exists z)\eta(x, y, z)$ is logically equivalent to $(x \rightarrow y)$ and conditions (i)-(iii) above hold. In what follows, we assume that the latter case holds, since the former is similar and actually much easier to handle.

Given a $\text{CNF}_1(S \cup \{\rightarrow\})$-formula $\varphi(\bar{x})$, construct in log-space a $\text{CNF}_\text{C}(S)$-formula $\chi(\bar{x}, \bar{z})$ by removing each occurrence of the implication connective in $\varphi(\bar{x})$ as follows: (i) if the occurrence is the form $(\mathbf{1} \rightarrow \mathbf{1})$ or of the form $(x_i \rightarrow \mathbf{1})$, then this occurrence is simply deleted; (ii) if the occurrence is of the form $(\mathbf{1} \rightarrow x_i)$, then it is substituted by the formula $\eta(1, x_i, z_p)$, where a different new variable z_p is used in each such substitution; and (iii) if the occurrence is of the form $(x_i \rightarrow x_j)$, then it is substituted by the formula $\eta(x_i, x_j, z_p)$, where a different new variable z_p is used in each such substitution. Note that the variables of χ are the original variables $\bar{x}$ and the new variables $\bar{z} = (z_1, \ldots, z_p, \ldots)$. Using Lemma 3.3, it is not hard to show that $\chi(\bar{x}, \bar{z})$ is 1-valid and that $\varphi(\bar{x}) \equiv \exists \bar{z} \chi(\bar{x}, \bar{z})$. Finally, using Lemma 3.3 again, one can show that $\varphi(\bar{x})$ is satisfied by a truth assignment that is different from the all-ones truth assignment $(1, \ldots, 1)$ to the variables $\bar{x}$ if and only if $\chi(\bar{x}, \bar{z})$ is satisfied by a truth assignment that is different from the all-ones truth assignment $(1, \ldots, 1)$ to the variables $\bar{x}$ and $\bar{z}$. Indeed, first assume that we have a different from the all-ones truth assignment α to the variables $\bar{x}$ that satisfies $\varphi(\bar{x})$. Extend α to a different from the all-ones truth assignment β that satisfies $\chi(\bar{x},\bar{z})$ by letting the variables $\bar{z}$ be assigned the truth values that witness the fact that $\exists\bar{z}\chi(\bar{x},\bar{z})$ is satisfied by α. Conversely, assume that a different from the all-ones truth assignment β of $\chi(\bar{x},\bar{z})$ is given. If we restrict β to the variables $\bar{x}$, we obviously get a truth assignment α that satisfies $\varphi(\bar{x})$. It remains to show that α is different from the all-ones assignment. Suppose it were not. Then by the property (iii) of Lemma 3.3 and by the way χ was constructed, we would conclude that the truth values corresponding to the variables $\bar{z}$ would also be all equal to one, contradicting the hypothesis that β was different from the all-ones assignment. This completes the proof of Step 2. ∎

Proof of Step 3: Assuming that S is 1-valid but neither 0-valid nor Schaefer, we will exhibit a log-space reduction of MIN SAT$_C(S)$ to MIN SAT(S). In constructing the reduction, we will need the following fact: For any set of relations T that is not 0-valid, there is a CNF(T)-formula $\tau(w_0,w_1)$ such that $\tau(\mathbf{0}/w_0,w_1)$ is satisfied by a truth assignment if and only if w_1 takes the value 1 under this assignment. To see this, let R be a non-0-valid relation in T, and say that R is of arity k. Let $\alpha=(a_1,\dots,a_k)$ be an element of R which is different from the all-zeros k-tuple. Let R' be a relation symbol corresponding to R. The desired formula $\tau(w_0,w_1)$ is then obtained from the CNF(T)-formula $R'(x_1,\dots,x_k)$ as follows: if $a_i=1$, replace x_i by w_1; otherwise, replace x_i by w_0.

Consider an instance of MIN SAT$_C(S)$ consisting of a CNF$_C(S)$-formula $\varphi(\bar{x},\mathbf{0}/w_0,\mathbf{1}/w_1)$ and a satisfying truth assignment α for the variables $\bar{x}$. Let $\chi(\bar{x},w_0,w_1)$ be the CNF(S)-formula $\varphi(\bar{x},w_0,w_1)\wedge\tau(w_0,w_1)$, where $\tau(w_0,w_1)$ is as above. The satisfying truth assignment α of φ can be extended to a satisfying truth assignment α' of χ by letting $w_0=0$ and $w_1=1$. It is then easy to verify that $\varphi(\bar{x},\mathbf{0}/w_0,\mathbf{1}/w_1)$ has a satisfying truth assignment $\beta<\alpha$ if and only if $\chi(\bar{x},w_0,w_1)$ has a satisfying truth assignment $\beta'<\alpha'$. This holds true because any satisfying truth assingment of χ that is smaller than α' must assign the value 0 to w_0; consequently, it must also assign the value 1 to w_1 (by the way $\tau(\mathbf{0}/w_0,w_1)$ is defined). This completes the proof of Step 3, as well as the proof of Theorem 3.2. ∎

The following three examples illustrate the preceding Theorem 3.2.

Example 3.4: Consider the ternary logical relation

$$K=\{(1,1,1),(0,1,0),(0,0,1)\}.$$

Since K is 1-valid, the satisfiability problem SAT$(\{K\})$ is trivial (the answer is always "yes"). In contrast, Theorem 3.2 implies that the minimal satisfiability problems MIN SAT$(\{K\})$ and MIN SAT$_C(\{K\})$ are NP-complete. Indeed, it is obvious that K is not 0-valid. Moreover, using the criteria mentioned after Definition 2.3, it is easy to verify that K is neither bijunctive, nor Horn, nor dual Horn, nor affine (for instance, K is not Horn because $(0,1,0)\wedge(0,0,1)=(0,0,0)\notin K$).

Note that the logical relation K can also be used to illustrate Lemma 3.3. Specifically, it is clear that $(x\to y)$ is logically equivalent to the formula $(\exists z)K(x,y,z)$; moreover, 1 is the only witness for the variable z such that $(\exists z)K(1,1,z)$ holds. ∎

Example 3.5: Consider the 1-valid set

$$S=\{R_0,R_1,R_2\},$$

where $R_0=\{0,1\}^3-\{(0,0,0)\}$ (expressing the clause $(x\vee y\vee z)$), $R_1=\{0,1\}^3-\{(1,0,0)\}$ (expressing the clause $(\neg x\vee y\vee z)$), $R_2=\{0,1\}^3-\{(1,1,0)\}$ (expressing the clause $(\neg x\vee\neg y\vee z)$).

Since S is a 1-valid set, $\textsc{Sat}(S)$ is trivial. In contrast, Theorem 3.2 implies that $\textsc{Min Sat}(S)$ and $\textsc{Min Sat}_C(S)$ are NP-complete. Indeed, S is not 0-valid, since R_0 is not a 0-valid logical relation. Moreover, it is not hard to verify that S is not Schaefer. For this, observe that R_1 is not Horn (since $(1,1,0) \wedge (1,0,1) = (1,0,0) \notin R_1$), R_1 is not bijunctive (since the coordinate-wise majority of $(1,1,0)$, $(1,0,1)$, $(0,0,0)$ is $(1,0,0) \notin R_1$), and R_1 is not affine (since $(1,1,1) \oplus (1,1,0) \oplus (1,0,1) = (1,0,0) \notin R_1$). Furthermore, R_2 is not dual Horn (since $(1,0,0) \vee (0,1,0) = (1,1,0) \notin R_2$). ∎

Example 3.6: Consider the 1-valid set

$$S = \{R_1, R_2\},$$

where R_1 and R_2 are as in the preceding Example 3.5. Clearly, $\textsc{Min Sat}(S)$ is trivial, since S is a 0-valid set. Theorem 3.2, however, implies that $\textsc{Min Sat}_C(S)$ is NP-complete, since S is not Schaefer (as seen in the preceding example, the relations R_1 and R_2 form a non-Schaefer set). ∎

Theorem 3.2 yields a dichotomy for $\textsc{Min Sat}(S)$, where S is a 1-valid set of logical relations. In what follows, we will use this result to establish a dichotomy for $\textsc{Min Sat}(S)$, where S is an arbitrary set of logical relations. Before doing so, however, we need to introduce the following crucial concept.

Definition 3.7: Let R be a k-ary logical relation and R' be a corresponding relation symbol. We say that a logical relation T is a 0-*section* of R if either T is the relation R itself or T can be defined from the formula $R'(x_1, \ldots, x_k)$ by replacing at least one, but not all, of the variables $x_1, \ldots, x_k$ by the constant symbol $\mathbf{0}$. ∎

To illustrate this concept, observe that the 1-valid logical relation $\{(1)\}$ is a 0-section of $R_{1/3} = \{(1,0,0), (0,1,0), (0,0,1)\}$, since it is definable by $R'_{1/3}(x_1, \mathbf{0}, \mathbf{0})$, where $R'_{1/3}$ is a relation symbol corresponding to $R_{1/3}$. Note that the logical relation $\{(1,0), (0,1)\}$ is also a 0-section of $R_{1/3}$, since it is definable by the formula $R'_{1/3}(\mathbf{0}, x_2, x_3)$, but it is not 1-valid. In fact, it is easy to verify that $\{(1)\}$ is the *only* 0-section of $R_{1/3}$ that is 1-valid.

Theorem 3.8: [Dichotomy of $\textsc{Min Sat}(S)$] *Let S be a set of logical relations and let S^* be the set of all logical relations P such that P is both 1-valid and a 0-section of some relation in S.*

If S^* is 0-valid or Schaefer, then $\textsc{Min Sat}(S)$ *is solvable in polynomial time; otherwise, it is NP-complete.*

If S^* is Schaefer, then $\textsc{Min Sat}_C(S)$ *is solvable in polynomial time; otherwise, it is NP-complete.*

Moreover, each of these two dichotomies can be decided in polynomial time; that is to say, there is a polynomial-time algorithm to decide whether, given a finite set S of logical relations, $\textsc{Min Sat}(S)$ *is solvable in polynomial time or NP-complete (and similarly for* $\textsc{Min Sat}_C(S)$*).*

Proof: We only prove the theorem for $\textsc{Min Sat}(S)$; the case with constants is analogous and, in fact, easier. We first show that if the 1-valid set S^* is neither 0-valid nor Schaefer, then $\textsc{Min Sat}(S)$ is NP-complete. In this case, by Theorem 3.2, $\textsc{Min Sat}(S^*)$ is NP-complete. We will produce a polynomial-time reduction of $\textsc{Min Sat}(S^*)$ to $\textsc{Min Sat}(S)$. Let φ be a $\mathrm{CNF}(S^*)$-formula and let α be a satisfying truth assignment of φ. For every relation symbol P'_i in φ, let P_i be the corresponding logical relation in S^*, let R_i be a relation in S such that P_i is a 0-section of R_i, and let R'_i be the corresponding relation symbol. Let also w_0 be a new variable. We now construct a

CNF(S)-formula χ by transforming each occurrence of P_i' in φ to an occurrence of R_i' as follows: (i) we put the variable w_0 in all variable positions that correspond to coordinates of R_i that were set to 0 to obtain the 0-section P_i; (ii) for the remaining variables of R_i', we use the variables in the occurrence of P_i' under consideration and we put them in the same arrangement they appear in this occurrence of P_i'. Also, we extend α to a satisfying truth assignment α' of χ by assigning the value 0 to w_0. The formula χ and the assignment α' is the instance of MIN SAT(S) to which the reduction is made. It is not hard to prove that there is a truth assignment $\beta < \alpha$ that satisfies φ if and only if there is a truth assignment $\beta' < \alpha'$ that satisfies χ. This completes the reduction that establishes the hardness part of the dichotomy for MIN SAT(S).

For the other direction, suppose that the 1-valid set S^* is 0-valid or Schaefer. In this case, by Theorem 3.2, MIN SAT(S^*) is solvable in polynomial time. We will now reduce MIN SAT(S) to MIN SAT(S^*) in polynomial time. Suppose we are given a CNF(S)-formula χ and a satisfying truth assignment α of it. We now construct in polynomial time a CNF(S^*)-formula φ as follows. Consider an occurrence $R_i'(y_1, \ldots, y_m)$ of a relation symbol R_i' in χ, and let R_i be the corresponding logical relation in S. Depending on the values assigned by α to the variables $y_1, \ldots, y_m$, we either eliminate this occurrence of R_i' or transform it to an occurrence of a relation symbol P_i' corresponding to a relation P_i in S^*. Specifically, there are three cases to consider:

1. If α assigns value 0 to each variable $y_1 \ldots, y_m$, then we eliminate the occurrence $R_i'(y_1, \ldots, y_m)$.

2. If α assigns value 1 to each variable $y_1, \ldots, y_m$, then we keep the occurrence $R_i'(y_1, \ldots, y_m)$. Note that, since α satisfies χ, the relation R_i must be 1-valid and so it is a member of S^*.

3. If α assigns value 0 to some variable y_r and value 1 to some other variable y_s in the occurrence $R_i'(y_1, \ldots, y_m)$, then let P_i be the 0-section of R_i obtained by setting equal to 0 each variable y_r to which α assigns value 0. Note that, since α satisfies χ, the relation P_i must be 1-valid and so it is a member of S^*. We now replace the occurrence $R_i'(y_1, \ldots, y_m)$ by the occurrence $P_i'(y_{i_1}, \ldots, y_{i_t})$, where $(y_{i_1}, \ldots, y_{y_t})$ is the subsequence of the sequence $(y_1, \ldots, y_m)$ consisting of all variables among $y_1, \ldots, y_m$ to which α assigns value 1.

   It should be pointed out that in this construction the same relation symbol R_i' in χ may give rise to several different relation symbols in φ, corresponding to different occurrences of R_i' in χ; the reason for this is that, depending on α, different occurrences of the same relation symbol may correspond to different 0-sections.

It is easy to show that χ has a satisfying assignment $\beta < \alpha$ if and only if φ has a satisfying assignment that is different from the all-ones assignment. This completes the proof of the tractability part of the dichotomy.

Note that the size of S^* is polynomial in the size of S, since every relation in S^* is either a relation in S or is determined by a relation R in S and a tuple α in R that is different from the all-ones tuple $(1, \ldots, 1)$ (the positions of the zeros in α determine the variables that are replaced by 0 to obtain a 1-valid 0-section of R). The existence of a polynomial-time algorithm for deciding between tractability and NP-completeness in the dichotomy is established by combining this fact with the existence of a uniform polynomial-time algorithm for deciding whether a given set of logical relations is Schaefer (see [KV00]).∎

We now present several different examples that illustrate the power of Theorem 3.8.

Example 3.9: If m and n are two positive integers with $m < n$, then $R_{m/n}$ is the n-ary logical relation consisting of all n-tuples that have m ones and $n-m$ zeros. Clearly, $R_{m/n}$ is neither 0-valid nor 1-valid. Moreover, it is not hard to verify that $R_{m/n}$ is not Schaefer. To see this consider the following three n-tuples in $R_{m/n}$:

$$\alpha = (\underbrace{1,\ldots,1}_{m},0,\ldots,0),\ \beta = (0,\underbrace{1,\ldots,1}_{m},0,\ldots,0),\ \gamma = (1,0,\underbrace{1,\ldots,1}_{m-1},0,\ldots,0).$$

$R_{m/n}$ is neither Horn nor dual Horn, because $\alpha \wedge \beta$ has $m-1$ ones and $\alpha \vee \beta$ has $m+1$ ones. Moreover, $R_{m/n}$ is not bijunctive, because the coordinate-wise majority of α, β and γ has $m+1$ ones. Finally, $R_{m/n}$ is not affine, because $\alpha \oplus \beta \oplus \gamma$ has $m-2$ ones.

Let S be a set of logical relations each of which is a relation $R_{m/n}$ for some m and n with $m < n$. The preceding remarks and Schaefer's Dichotomy Theorem 2.4 imply that SAT(S) is NP-complete. In contrast, the Dichotomy Theorem 3.8 implies that MIN SAT(S) and MIN SAT$_C$(S) are solvable in polynomial time. Indeed, S^* is easily seen to be Horn (and, hence, Schaefer), since every relation P in S^* is a singleton $P = \{(1,\ldots,1)\}$ consisting of the m-ary all-ones tuple for some m.

This family of examples contains POSITIVE-1-IN-3-SAT as the special case where $S = \{R_{1/3}\}$; thus, Theorem 3.8 provides an explanation for the difference in complexity between the satisfiability problem and the minimal satisfiability problem for POSITIVE-1-IN-3-SAT. ∎

Example 3.10: Consider the 3-ary logical relation

$$T = \{0,1\}^3 - \{(0,0,0),(1,1,1)\}.$$

SAT($\{T\}$) is the well-known problem POSITIVE-NOT-ALL-EQUAL-3-SAT: given a 3CNF-formula φ with clauses of the form $(x \vee y \vee z)$, is there a truth assignment such that in each clause of φ at least one variable is assigned value 1 and at least one variable is assigned value 0? Using Schaefer's Dichotomy Theorem 2.4, it is easy to see that this problem is NP-complete. To begin with, it is obvious that T is neither 0-valid nor 1-valid. Moreover, T is neither Horn nor dual Horn, because $(1,1,0) \wedge (0,0,1) = (0,0,0) \notin T$ and $(1,1,0) \vee (0,0,1) = (1,1,1) \notin T$. Finally, T is neither bijunctive nor affine, because the coordinate-wise majority of $(1,1,0)$, $(0,1,1)$ and $(1,0,1)$ is $(1,1,1) \notin T$, whereas their coordinate-wise $\oplus$ is $(0,0,0) \notin T$.

In contrast, the Dichotomy Theorem 3.8 easily implies that MIN SAT($\{T\}$) and MIN SAT$_C$($\{T\}$) are solvable in polynomial time. To see this, observe that

$$\{T\}^* = \{\{(1)\},\{(0,1),(1,0),(1,1)\}\},$$

where the logical relation $\{(1)\}$ is the 0-section of T obtained from T by setting any two variable to 0 (for instance, it is definable by the formula $T'(x,0,0)$) and the logical relation $\{(0,1),(1,0),(1,1)\}$ is the 0-section of T obtained from T by setting any one variable to 0 (for instance, it is definable by the formula $T'(x,y,0)$). It is clear that each of these two logical relations is bijunctive (actually, each is also dual Horn), hence $\{T\}^*$ is Schaefer.

This provides another example of a natural NP-complete satisfiability problem whose associated minimal satisfiability problem is tractable. ∎

Example 3.11: As seen earlier, 3-SAT coincides with SAT(S), where $S = \{R_0, R_1, R_2, R_3\}$ and $R_0 = \{0,1\}^3 - \{(0,0,0)\}$ (expressing the clause $(x \vee y \vee z)$), $R_1 = \{0,1\}^3 - \{(1,0,0)\}$ (expressing the clause $(\neg x \vee y \vee z)$), $R_2 = \{0,1\}^3 - \{(1,1,0)\}$ (expressing the clause $(\neg x \vee \neg y \vee z)$), and $R_3 = \{0,1\}^3 - \{(1,1,1)\}$ (expressing the clause $(\neg x \vee \neg y \vee \neg z)$).

Since the logical relations R_0, R_1, R_2 are 1-valid, they are members of S^*. It follows that S^* is not 0-valid, since it contains R_0. Moreover, as seen in Example 3.5, the logical relation R_1 is not Horn, it is not bijunctive, and it is not affine, whereas the logical relation R_2 is not dual Horn. Consequently, S^* is not Schaefer. We can now apply Theorem 3.8 and immediately conclude that MIN SAT(S) (i.e., MIN 3-SAT) is NP-complete.

This example illustrates a fine point in the concept of a 0-section of a logical relation. Specifically, it is crucial to allow each logical relation to be a 0-section of itself (see Definition 3.7). Indeed, it is easy to see that every 0-section of R_i other than R_i itself is bijunctive, $0 \leq i \leq 3$. Consequently, if a logical relation were not allowed to be a 0-section of itself, then S^* would consist entirely of bijunctive relations and, hence, it would be Schaefer. ∎

Example 3.12: Consider the set $S = \{R_0, R_3\}$, where R_0 and R_3 are as in the preceding Example 3.11. In this case, SAT(S) is the problem MONOTONE 3-SAT, that is to say, the restriction of 3-SAT to 3CNF-formulas in which every clause is either the disjunction of positive literals or the disjunction of negative literals. Clearly, S is neither 0-valid nor 1-valid. Moreover, it is easy to verify that S is not Schaefer. Consequently, Schaefer's Dichotomy Theorem 2.4 implies that SAT(S) is NP-complete.

In contrast, the Dichotomy Theorem 3.8 implies that MIN SAT(S) and MIN $\text{SAT}_\text{C}(S)$ are solvable in polynomial time. For this, it suffices to verify that S^* is Schaefer. Note that the relation R_0 is dual Horn, because it is definable by the formula $(x \vee y \vee z)$. Since dual Horn formulas are closed under substitutions by constants, it follows that every 0-section of R_0 is dual Horn as well. Let us now consider those 0-sections of R_3 that are also 1-valid relations. Observe that R_3 is not such a relation, since it is not 1-valid. Thus, every 1-valid 0-section of R_3 must be obtained from R_3 either by setting one variable to 0 or by setting two variables to 0. The first case gives rise to the trivial binary relation $\{(0,0),(0,1),(1,0),(1,1)\}$, whereas the second case gives rise to the trivial unary relation $\{(0),(1)\}$. Since each of these relations is dual Horn, it follows that S^* is dual Horn and, hence, S^* is Schaefer. ∎

4 Dichotomy Theorems for Extensions of Minimal Satisfiability

In this section, we establish a dichotomy theorem for minimal satisfiability problems with respect to a modified partial order between truth assignments. This modified partial order allows for a part of the assignment to be kept fixed, while another part of it may vary arbitrarily. As mentioned earlier, the existence of a dichotomy theorem for these minimal satisfiability problems was raised by Cadoli [Cad92, Cad93] while investigating propositional circumscription and its extensions.

Definition 4.1: Let k be a positive integer, let $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_k)$ be two k-tuples in $\{0,1\}^k$, and let (P, Q, Z) be a partition of the set $\{1, 2, \ldots, k\}$ in which P is non-empty, while at least one of Q and Z may be empty.

We write α/P to denote the tuple that results from α by keeping only those coordinates a_j such that $j \in P$. We also write $\beta <_{(P,Q,Z)} \alpha$ to denote that $\beta/P < \alpha/P$ and $\beta/Q = \alpha/Q$.

Let S be a finite set of logical relations. $(P;Q;Z)$-MIN SAT(S) is the following decision problem: given a CNF(S)-formula φ, a satisfying truth assignment α of φ, and a partition (P, Q, Z) of the set of variables of φ (in which P is non-empty, while at least one of Q and Z may be empty), is there a satisfying truth assignment β of φ such that $\beta <_{(P,Q,Z)} \alpha$? (here the partition (P, Q, Z) of the indices of α is the one induced by the partition (P, Q, Z) of the variables of φ). ∎

Observe that $(P;Q;Z)$-MIN SAT(S) contains both MIN SAT(S) and MIN $\text{SAT}_\text{C}(S)$ as restricted cases. Indeed, MIN SAT(S) is the same problem as $(P;Q;Z)$-MIN SAT(S) with $Q = \emptyset$ and $Z = \emptyset$, while $\textsc{Min Sat}_{\mathrm{C}}(S)$ is the same problem as $(P;Q;Z)$-$\textsc{Min Sat}(S)$ with $Z = \emptyset$. The last result of this paper establishes a dichotomy for $(P;Q;Z)$-$\textsc{Min Sat}(S)$ (with no restrictions on Q or Z), as well as for the restricted case of $(P;Q;Z)$-$\textsc{Min Sat}(S)$ with $Q = \emptyset$.

Theorem 4.2: [Dichotomy of $(P;Q;Z)$-$\textsc{Min Sat}(S)$] Let S be a set of logical relations.

If S is Schaefer, then $(P;Q;Z)$-$\textsc{Min Sat}(S)$ is solvable in polynomial time; otherwise, it is NP-complete.

If S is 0-valid or Schaefer, then $(P;Q;Z)$-$\textsc{Min Sat}(S)$ with $Q = \emptyset$ is solvable in polynomial time; otherwise, it is NP-complete.

Proof: Let K be a logical relation such that $\textsc{Min Sat}(\{K\})$ is NP-complete (for instance, the logical relation K introduced in Example 3.4 has this property). To show the NP-completeness of $(P;Q;Z)$-$\textsc{Min Sat}(S)$, when S is not Schaefer, we will exhibit a polynomial-time reduction of $\textsc{Min Sat}(\{K\})$ to $(P;Q;Z)$-$\textsc{Min Sat}(S)$. Suppose we are given a $\mathrm{CNF}(\{K\})$-formula $\varphi(\bar{x})$, where $\bar{x} = (x_1, \ldots, x_n)$ is the list of its variables, and a satisfying truth assignment α of φ. By repeatedly applying Schaefer's Expressibility Theorem 2.5 to the occurrences of K in $\varphi(\bar{x})$, we can construct in log-space a $\mathrm{CNF}(S)$-formula $\chi(\bar{x}, \bar{z}, w_0, w_1)$, such that $\varphi(\bar{x}) \equiv \exists \bar{z} \chi(\bar{x}, \bar{z}, \mathbf{0}/w_0, \mathbf{1}/w_1)$, where $\bar{z} = (z_1, \ldots, z_m), w_0, w_1$ are new variables different from $\bar{x}$ (substitutions of different variables by the same constant can be easily consolidated to substitutions of the occurrences of a single variable by that constant). Let P be the set of variables $\overline{x}$ of φ, let Q be the two-element set $\{w_0, w_1\}$, and let Z be the set of variables $\overline{z}$. We now construct the following truth assignment β that satisfies $\chi(\bar{x}, \bar{z}, w_0, w_1)$: to the variables $\overline{x}$, it assigns α; to the variable w_0, it assigns 0; to the variable w_1, it assigns 1; finally, to the variables $\overline{z}$, it assigns a tuple that witnesses the fact that the formula $\exists \bar{z} \chi(\bar{x}, \bar{z}, \mathbf{0}/w_0, \mathbf{1}/w_1) \equiv \varphi(\bar{x})$ is satisfied by α. The $\mathrm{CNF}(S)$-formula $\chi(\bar{x}, \bar{z}, w_0, w_1)$, the partition (P, Q, Z), and the satisfying truth assignment β constitute the instance of $(P;Q;Z)$-$\textsc{Min Sat}(S)$ to which the instance $\varphi(\bar{x})$ and α of $\textsc{Min Sat}(S)$ is being reduced. It is now immediate that there is a truth assignment β' satisfying $\chi(\bar{x}, \bar{z}, w_0, w_1)$ and such that $\beta' <_{(P,Q,Z)} \beta$ if and only if $\varphi(\bar{x}) \equiv \exists \bar{z} \chi(\bar{x}, \bar{z}, \mathbf{0}/w_0, \mathbf{1}/w_1)$ has a satisfying truth assignment $\alpha' < \alpha$. This completes the NP-hardness proof of the first part of the theorem.

Next, assume that S is neither 0-valid nor Schaefer. To show that $(P;Q;Z)$-$\textsc{Min Sat}(S)$ is NP-complete even when $Q = \emptyset$, we produce again a reduction from $\textsc{Min Sat}(\{K\})$. As before, we start with a $\mathrm{CNF}(\{K\})$-formula $\varphi(\bar{x})$ and a satisfying truth assignment α; we then construct a $\mathrm{CNF}(S)$-formula $\chi(\bar{x}, \bar{z}, w_0, w_1)$ such that $\varphi(\bar{x}) \equiv \exists \bar{z} \chi(\bar{x}, \bar{z}, \mathbf{0}/w_0, \mathbf{1}/w_1)$. Consider now the $\mathrm{CNF}(S)$-formula $\tau(w_0, w_1)$ used in the proof of Step 3 of Theorem 3.2; this formula has the property that $\tau(\mathbf{0}/w_0, w_1)$ is satisfied by a truth assignment if and only if w_1 gets the value 1 under this assignment. We now consider the following instance of $(P;Q;Z)$-$\textsc{Min Sat}(S)$ with $Q = \emptyset$: the formula is defined to be $\chi(\bar{x}, \bar{z}, w_0, w_1) \wedge \tau(w_0, w_1)$ (thus w_1 is forced to take value 1 in every satisfying truth assignment that assigns 0 to w_0); P is defined to be the set consisting of the variables $\overline{x}$, w_0, and w_1; Q is the empty set; finally, Z and β are defined as in the previous reduction. It is now immediate that there is a truth assignment β' satisfying $\chi(\bar{x}, \bar{z}, w_0, w_1) \wedge \tau(w_0, w_1)$ and such that $\beta' <_{(P,Q,Z)} \beta$ if and only if $\varphi(\bar{x}) \equiv \exists \bar{z} \chi(\bar{x}, \bar{z}, \mathbf{0}/w_0, \mathbf{1}/w_1)$ has a satisfying truth assignment $\alpha' < \alpha$. This completes the NP-hardness proof ofthe second part of the theorem.

The tractability results follow easily, as in Theorem 3.2. ∎

We conclude by pointing out that the above Dichotomy Theorem 4.2 for $(P;Q;Z)$-$\textsc{Min Sat}(S)$ does *not* imply the Dichotomy Theorem 3.8 for $\textsc{Min Sat}(S)$ and $\textsc{Min Sat}_{\mathrm{C}}(S)$. Indeed, since $\textsc{Min Sat}(S)$ is a restricted case of $(P;Q;Z)$-$\textsc{Min Sat}(S)$, one cannot a priori rule out the existence of sets S of logical relations such that $(P;Q;Z)$-$\textsc{Min Sat}(S)$ is NP-complete, whereas

MIN SAT(S) is solvable in polynomial time. Actually, the familiar set $S = \{R_{1/3}\}$ has this property, thus manifesting that the dichotomy for MIN SAT(S) cannot be derived from the dichotomy for $(P; Q; Z)$-MIN SAT(S).

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5 Appendix: Proof of Lemma 3.3

For completeness, we give in this appendix a full proof of Lemma 3.3 due to Creignou and Hébrard [CH97]. To make the proof entirely self-contained, we first present the proof of a result due to Creignou and Hermann [CH96], which will be used in the sequel.

Proposition 5.1: [Creignou and Hermann, [CH96]] *If R is a 1-valid logical of arity k, then the following statements are equivalent:*

- *R is affine.*
- *For all $s, t \in R$, we have that $\bar{1} \oplus s \oplus t \in R$, where $\bar{1}$ is the all-ones k-tuple $(1, \ldots, 1)$.*

Proof: As mentioned in Section 2, Schaefer [Sch78] showed that a logical relation R is affine if and only if for all $t_1, t_2, t_3 \in R$, we have that $t_1 \oplus t_2 \oplus t_3 \in R$. It follows that if R is both 1-valid and affine, then for all $s, t \in R$, we have that $\bar{1} \oplus s \oplus t \in R$. For the other direction, assume that R is 1-valid and such that if $s, t \in R$, then $\bar{1} \oplus s \oplus t \in R$. Let $t_1, t_2, t_3 \in R$. It follows that $u = \bar{1} \oplus t_1 \oplus t_2 \in R$. By applying the closure property of R again, we get that $\bar{1} \oplus u \oplus t_3 \in R$. Since $\bar{1} \oplus \bar{1} = \bar{0}$ and $\oplus$ is associative, we have that $\bar{1} \oplus u \oplus t_3 = \bar{1} \oplus (\bar{1} \oplus t_1 \oplus t_2) \oplus t_3 = t_1 \oplus t_2 \oplus t_3$. ∎

Lemma 3.3: [Creignou and Hébrard [CH97]] *Let S be a 1-valid, non-Schaefer set of logical relations. Then at least one of the following two statements is true about the implication connective.*

1. *There exists a $\mathrm{CNF_C}(S)$-formula $\varepsilon(x, y)$ such that $(x \to y) \equiv \varepsilon(x, y)$.*
2. *There exists in $\mathrm{CNF_C}(S)$-formula $\eta(x, y, z)$ such that*

   (i) $(x \to y) \equiv (\exists z)\eta(x, y, z)$; (ii) $\eta(x, y, z)$ is satisfied by the truth assignment $(1, 1, 1)$; (iii) $\eta(x, y, z)$ is not satisfied by the truth assignment $(1, 1, 0)$.

   In other words, the formula $(\exists z)\eta(x, y, z)$ is logically equivalent to $(x \to y)$ and has the additional property that 1 is the only witness for the variable z under the truth assignment $(1, 1)$ to the variables (x, y).

Proof: Since S is a 1-valid, non-Schaefer set of logical relations, it must contain a 1-valid logical relation R that is not affine. Let k be the arity of R. From Proposition 5.1, it follows that there are two k-tuples $s, t \in R$ such that $\bar{1} \oplus s \oplus t \notin R$. Let $x_1, \ldots, x_k$ be propositional variables and let R' be a relation symbol of arity k that will be interpreted by R. For $(i, j) \in \{0, 1\}^2$, let V_{ij} be the set of all variables x_p, $1 \leq p \leq k$, such that the p-th coordinate of the tuple s is equal to i, and the p-th coordinate of the tuple t is equal to j. Let x, y, z, w be four new propositional variables and let $\varphi_1(x, y, z, w)$ be the $\mathrm{CNF}(S)$-formula $R'(x/V_{00}, y/V_{10}, z/V_{01}, w/V_{11})$ obtained from the formula $R'(x_1, \ldots, x_k)$ by substituting the variable x for all occurrences of the variables in V_{00}, and similarly for the variables y, z, and w. Also let $\varphi_2(x, y, z)$ be the $\mathrm{CNF_1}(S)$-formula $\varphi_1(x, y, z, \mathbf{1}/w)$. Now observe the following:

- the truth assignment $(1, 1, 1, 1)$ satisfies $\varphi_1(x, y, z, w)$, because $\bar{1} \in R$;
- the truth assignment $(0, 1, 0, 1)$ satisfies $\varphi_1(x, y, z, w)$, because $s \in R$;
- the truth assignment $(0, 0, 1, 1)$ satisfies the $\varphi_1(x, y, z, w)$, because $t \in R$;
- the truth assignment $(1, 0, 0, 1)$ does not satisfy $\varphi_1(x, y, z, w)$, because $\bar{1} \oplus s \oplus t \notin R$.

Therefore, $(1,1,1)$, $(0,1,0)$ and $(0,0,1)$ satisfy $\varphi_2(x,y,z)$, while $(1,0,0)$ does not.

For the remaining four truth assignments $(1,1,0),(0,1,1),(1,0,1)$ and $(0,0,0)$, we have no information as to whether or not they satisfy $\varphi_2(x,y,z)$. Consequently, we have sixteen possibilities to examine regarding the satisfiability of $\varphi_2(x,y,z)$ by these four truth assignments. To facilitate the case analysis, we introduce a notation that we explain by an example. Case [N(o)-Y(es)-N(o)-*] means that the following hold:

- $(1,1,0)$, the first truth assignment under consideration, does not satisfy $\varphi_2(x,y,z)$;
- $(0,1,1)$, the second truth assignment under consideration, satisfies $\varphi_2(x,y,z)$;
- $(1,0,1)$, the third truth assignment under consideration, does not satisfy $\varphi_2(x,y,z)$;
- $(0,0,0)$, the fourth truth assignment under consideration, may or may not satisfy φ_2.

We will show that for each of the sixteen possibilities there exists a $\mathrm{CNF}_\mathrm{C}(S)$-formula $\eta(x,y)$ that is logically equivalent to the implication $x \to y$, or there exists a $\mathrm{CNF}_\mathrm{C}(S)$-formula $\varepsilon(x,y)$ having the properties stipulated in Lemma 3.3, or there exists a $\mathrm{CNF}_C(S)$-formula that is logically equivalent to the disjunction $x \vee y$.

- Case [N-*-N-*]: Set $\eta(x,y,z) \equiv \varphi_2(x,y,z)$
- Case [*-*-Y-Y]: Set $\varepsilon(x,y) \equiv \varphi_2(x,0,y)$
- Case [*-Y-*-N]: Observe that $x \vee y \equiv \varphi_2(0,x,y)$
- Case [*-N-Y-*]: Set $\varepsilon(x,y) \equiv \varphi_2(y,x,1)$
- Case [Y-N-*-*]: Set $\varepsilon(x,y) \equiv \varphi_2(y,1,x)$
- Case [*-Y-N-*]: Set $\varepsilon(x,y) \equiv \varphi_2(x,y,1)$

It is easy to check that the above cases cover all sixteen possibilities. Note, however, that the proof of Lemma 3.3 has not been completed, because in Case [*-Y-*-N] we only succeeded to define $x \vee y$ using a $\mathrm{CNF}_C(S)$-formula. Since not every element of S is a dual Horn relation, S must contain a logical relation Q for which there are tuples $s,t \in Q$ such that $s \vee t \notin Q$ (here we use the closure properties of dual Horn relations, due to Dechter and Pearl [DP92], mentioned in Section 2). By arguments similar to the preceding ones, we can construct a $\mathrm{CNF}_C(S)$-formula $\psi_2(x,y,z)$ that is satisfied by $(1,1,1)$, $(0,1,0)$ and $(0,0,1)$, but it is not satisfied by $(0,1,1)$. Let $\psi_3(x,y,z)$ be the $\mathrm{CNF}_C(S)$-formula $\psi_2(x,y,z) \wedge (y \vee z)$. Observe that $\psi_3(x,y,z)$ is satisfied by $(1,1,1)$, $(0,1,0)$ and $(0,0,1)$, but it is not satisfied by $(0,1,1),(1,0,0),(0,0,0)$. We are now left with the triples $(1,1,0)$ and $(1,0,1)$ about which there is no information as to whether they satisfy $\psi_3(x,y,z)$ or not. We consider the following three exhaustive cases:

- If $(1,1,0)$ satisfies $\psi_3(x,y,z)$, then set
$$\varepsilon(x,y) \equiv \psi_3(y,1,x).$$
- If $(1,0,1)$ satisfies $\psi_3(x,y,z)$, then set
$$\varepsilon(x,y) \equiv \psi_3(y,x,1).$$

- If neither $(1, 1, 0)$ nor $(1, 0, 1)$ satisfies $\psi_3(x, y, z)$, then set

$$\eta(x, y, z) \equiv \psi_3(x, y, z).$$

This completes the proof of Lemma 3.3. ∎

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