Electronic Colloquium on Computational Complexity, Report No. 4 (2001)

Recursive Analytic Functions of a Complex Variable

Tobias Gärtner
Günter Hotz
Fachbereich Informatik, Universität des Saarlandes
Postfach 15 11 50, 66041 Saarbrücken, Germany
http://www-hotz.cs.uni-sb.de

October 13, 2000

ISSN 1433-8092

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Chapter 1

Introduction

Two major concepts dominate the theory of computation: The concept using machines such as Turing machines on the one hand, and the concept of recursive functions, especially the μ-recursive and primitive recursive functions on the other hand. In the first approach, programs and functions computed by these programs are given by machine instructions. The computations are carried out by changing states of the machine, which can be seen as an iteration on the memory of the machine. In the second one, programs are specified by functions. Therefore we call the first approach the *iterative* approach, and the second one *functional* approach. Obviously there are connections to programming languages, since in imperative languages, programs are given by instructions, and in functional ones, they are given by functions. At first sight, these two approaches are completely different. But, as one knows, they give the same class of computable functions, the class of Turing-computable or μ-recursive functions.

These models deal with functions defined on the natural numbers. They can easily be extended to the integers or the rationals, but they do not adequately deal with real numbers. Therefore the concept of Turing machines has been extended in recent years.

So an extended machine model for computations on the reals has been developed by Blum, Shub and Smale [1], and there is a recursion theory and a complexity theory for these machines. They are called R-machines, where R is any ring, for example the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$. One can define complexity classes for these machines, formulate the $P \neq NP$ problem and much more. See [2], where this theory is presented.

So far, the machines are similar to Turing machines in the aspect that a computation is a finite sequence of states of the machine (or something equivalent). Going further, the next step is to allow infinite computations, to model limit processes that are quite common in analysis. Some approaches have been made, for example by Weihrauch [12], [13], [14]. The notion of *analytic machines* has been introduced by Hotz [7], [8]. Here, infinite computations are admissible, if the computation converges in some way to a limit. Chadzelek [4] has worked out the theory of analytic machines. A shorter version of this has been published by Chadzelek and Hotz [5].

To return to what has been said in the beginning, we see that the extension of a theory of computation to the real or complex numbers has only been made for the machine approach. The question arises what can be done to extend the *functional* approach of μ-recursive and primitive recursive functions. This is where we set in. We ask how we can define programs or computations on the real and complex numbers by functions. As in the therory of recursive functions over the natural numbers, the functions will be defined by *functional equations*, or *recursive equations*, as we will also call them in our context. Given functional equations, for example $f(x+1) = xf(x)$, for a function over the real and complex numbers, the question arises whether they have solutions in a certain class of functions, and whether they have *unique* solutions under certain circumstances. The last question is important, since we want to have unique semantics for our functional program.

Since it appears natural to us, we require the solutions of our equations to be analytic functions. This is reasonable, since there is an elaborate theory about these functions, and, above all, they are computable in a certain sense: It is known that one can represent analytic functions by power series. So, given the coefficients of the power series in some way, it is clear that these functions are computable by the analytic machines mentioned above. Moreover, one can view a power series as an infinite tape that is labelled by the coefficients of the series, see Hotz [7].

Our objective in this paper is to examine some classes of primitive recursive equations in view of the questions above. We examine a class of recursions that are linear, such as equations of the form $f(x+1) = g(x)f(x) + h(x)$ and similar ones. We call these *linear primitive recursions.* As it has been said, we focus on analytic functions, i. e. we want our solutions to be analytic functions. So the functions defining the recursions (in the example above the functions g and h) should be analytic functions. Therefore we also call the recursions *linear primitive analytic recursions.* For some cases, we show that analytic solutions exist, and we give conditions under which those solutions are unique, so that the semantics defined by the equations is unique. The solutions we give are all defined by infinite series, infinite products or limits of analytic functions. So they are computable by analytic machines.

Another point why extending the concept of recursion to $\mathbb{C}$ is interesting might be the following: It is hard to decide if different programs compute the same functions. So it would be of interest to find universal invariants for program transformations. If we are able to construct well defined continuations of the programs from $\mathbb{N}$ to analytical functions, the genus of the Riemann surface belonging to this function could serve as such an invariant. We can describe the construction of the Riemann surface from a given power series as a nondeterministic analytical machine [4], [5]. So the genus of the Riemann surface and the structure of singularities of the function may be considered as invariants of equivalent nondeterministic analytical machines in this sense.

1.1 Interpolation

We want to point out the difference between defining functions by recursions and extending primitive recursive functions from the natural numbers to the complex numbers. It is clear that an analytic function satisfying the same recursion as a primitive recursive one over the natural numbers is an extension of this function.

But it is more, since an extension of the function does not necessarily satisfy the functional equation given by the recursion. Moreover, using theorems of classic complex analysis, one can show that every recursive function over the naturals *has* an analytic extension: Consider the class of μ-recursive or the one of primitive-recursive functions. These are functions defined on the natural numbers. We can extend the domain of these functions to the complex numbers $\mathbb{C}$, such that these extensions are analytic functions. For any function f defined on the natural numbers, there is an analytic function F defined on $\mathbb{C}$ which has the property $F(n) = f(n)$ for all natural numbers n. Using some complex analysis, this is quite clear [6].
In fact, let $f : \mathbb{N} \to \mathbb{N}$ be any such function. By Weierstrass' factorization theorem, there is a holomorphic function $g : \mathbb{C} \to \mathbb{C}$ such that g has a zero of order 1 at each $n \in \mathbb{N}$. By Mittag-Leffler's theorem, there is a meromorphic function h that has the principal parts $\frac{f(n)}{z-n}$ at each natural number. This means that the Laurent series expansion of h at n has this principal part. Now put $F := gh$. Then $F : \mathbb{C} \to \mathbb{C}$ is an analytic function with $F|_{\mathbb{N}} = f$.
But this construction has a flaw: It interpolates the recursive function f, but in general, it is not the case that it is subject to the recursive equations by which f is defined. Therefore this is not the solution to our problem.

Chapter 2

Linear primitive recursion over $\mathbb{C}$

Let g, h be any functions which are defined on the complex numbers $\mathbb{C}$. We call the recursion

$$f(x+1) = g(x) \cdot f(x) + h(x) \tag{2.1}$$

linear primitive recursion. It shall be called *analytic* if g and h are analytic functions, and it will be called *polynomial* if $g, h \in \mathbb{C}[z]$ are polynomials.
Our main objectives are whether the recursion (2.1) has an analytic solution f, and if it is the case, how to distinguish a special one under the solutions in a natural way. First, we use the term 'natural' rather informally, we want to find uniqueness criteria which appear sound to us. If we have found such criteria, we call the solutions subject to these 'natural solutions'.
In the next section, we briefly discuss recursions which relate $f(z+\omega)$, $|\omega| = 1$ to $f(z)$, generalizing the concept of the usual recursion relating $f(z+1)$ to $f(z)$.

2.1 Recursions in arbitrary directions

The recursive equation

$$f(x+1) = g(x) \cdot f(x) + h(x) \tag{2.2}$$

comes from calculations on natural numbers, where the case $n+1$ is reduced to the case n. We want to extend functions and recursions to the complex numbers. In the complex plane, numbers can be interpreted as vectors, and numbers of modulus 1 can be interpreted as directions. We want to examine recursions not only in the direction of the positive real line, but in any direction. If $\omega \in \mathbb{C}$, $|\omega| = 1$ is such a direction, we want to solve the recursive equation

$$f(x+\omega) = g(x) \cdot f(x) + h(x). \tag{2.3}$$

Moreover, we want to know how the solutions of this equation are related to the solution of the standard recursion (2.1). We will show that, given a solution of (2.1), it is easy to solve (2.3). The idea is the following: Rotate the complex plane transforming ω into 1, then solve the recursion and finally undo the rotation.

Let g and h be analytic functions for which a solution of (2.3) is required. Then the functions defined by

$$\tilde{g}(z) := g(\omega z), \qquad \tilde{h}(z) := h(\omega z)$$

are analytic functions. Let $\tilde{f}$ be a solution of

$$\tilde{f}(z+1) = \tilde{g}(z) \cdot \tilde{f}(z) + \tilde{h}(z).$$

Then define

$$f(z) := \tilde{f}(\bar{\omega} z),$$

where $\bar{\omega}$ is the complex conjugate of ω. Now, since $|\omega| = 1$ and therefore $\omega\bar{\omega} = 1$, we have

$$\begin{aligned} f(z+\omega) &= \tilde{f}(\bar{\omega}z + \bar{\omega}\omega) \\ &= \tilde{f}(\bar{\omega}z + 1) \\ &= \tilde{g}(\bar{\omega}z) \cdot \tilde{f}(\bar{\omega}z) + \tilde{h}(\bar{\omega}z) \\ &= g(z) \cdot f(z) + h(z) \end{aligned}$$

Thus we have shown that we can solve (2.3), if we can solve (2.1) for slightly different functions $\tilde{g}$, $\tilde{h}$.
In the following, we will therefore focus on the standard recursion (2.1).

2.2 Reduction to special cases

We first consider two special cases. If $h = 0$, then our recursion has the form

$$f(x+1) = g(x) \cdot f(x) \tag{2.4}$$

For $g(x) = x$, a solution of this recursion is commonly known: It's the Γ-function. We will generalize this solution to the case that g is any polynomial in $\mathbb{C}[z]$.
The second special case is the case $g = 1$, i. e.

$$f(x+1) = f(x) + h(x) \tag{2.5}$$

The case $h(x) = x^k$ has been solved already by Bernoulli, a slight modification of the well-known Bernoulli Polynomials will satisfy this recursion.
Since the special cases (2.4) and (2.5) might be easier to solve, we want to reduce the general case (2.1) to these special cases.
Let g, h be any analytic functions for which we want to solve (2.1), and let us assume that we have a solution $f_1(x)$ of

$$f_1(x+1) = g(x) \cdot f_1(x)$$

and a solution $f_2(x)$ of

$$f_2(x+1) = f_2(x) + \frac{h(x)}{f_1(x+1)}.$$

These recursions are instances of the special cases (2.4) and (2.5). To solve (2.1), put $f := f_1 \cdot f_2$. Then we have

$$\begin{aligned} f(x+1) &= f_1(x+1) \cdot f_2(x+1) \\ &= f_1(x+1) \cdot \left(f_2(x) + \frac{h(x)}{f_1(x+1)} \right) \\ &= g(x) f_1(x) f_2(x) + h(x) \\ &= g(x) \cdot f(x) + h(x) \end{aligned}$$

Thus we have shown that it suffices to solve (2.4) and (2.5) in order to get a general solution for (2.1). For this reason, we will separately focus on these special cases, where (2.4) is called the *multiplicative case* and (2.5) the *additive case*.

2.3 The multiplicative case

2.3.1 A generalization of the Γ-function

The well-known Γ-function solves $\Gamma(x+1) = x\Gamma(x)$. We want to generalize its definition to obtain solutions of $\Gamma_p(x+1) = p(x)\Gamma_p(x)$ where p is a polynomial. Let $p(z) \in \mathbb{C}[z]$ be a polynomial of degree m and $f : \mathbb{N} \to \mathbb{C}$ defined by the primitive recursion

$$f(n+1) := p(n) \cdot f(n), \qquad f(1) := \alpha \tag{2.6}$$

If $p(\mathbb{N}) \subset \mathbb{N}$, this recursion defines a primitive-recursive function f on the natural numbers $\mathbb{N}$. If $p(\mathbb{N})$ has values not in $\mathbb{N}$, we have to generalize the notion of primitive-recursive functions sucht that these can have complex values. We look for a *natural* continuation of f to a function which is analytic on $\mathbb{C}$. Note that by an analytic function on a subset $A \subset \mathbb{C}$ of the complex numbers we denote functions which are holomorphic on A except for a discrete set of isolated singularities.

From (2.6) it follows that a nontrivial extension of f cannot be defined everywhere. The reason for this is the following: If, for example, the polynomial p has a zero at 0, then $f(1) = 0 \cdot f(0)$, such that the requirement $f(1) = 1$ cannot be fulfilled. So any nontrivial extension must have a singularity at $z = 0$. Regarding arbitrary polynomials p, we see that the extension of f can only be defined on $\mathbb{C}\backslash N$, where $N = \{z \in \mathbb{C} \mid \exists n \in \mathbb{N} : p(z+n) = 0\}$.

But how do we get a 'natural' extension of f? The definition should best be guided anyhow by the original definition, i. e. by the recursive equation. In order to do this, let s, $n \in \mathbb{N}$. From (2.6) it follows that

$$\begin{aligned} f(s) &= \alpha \cdot p(1) \cdot \ldots \cdot p(s-1) \\ &= \alpha \cdot p(1) \cdot \ldots \cdot p(s-1) \cdot \ldots \cdot p(n+s) \cdot \frac{1}{p(s) \cdot \ldots \cdot p(n+s)} \end{aligned}$$

This gives the following identity, which holds for all n, $s \in \mathbb{N}$:

$$f(s) = \alpha \cdot \frac{p(1)}{p(s)} \cdot \ldots \cdot \frac{p(n)}{p(s+n)} \cdot p(n)^s \cdot \left[\frac{p(n+1)}{p(n)} \cdot \ldots \cdot \frac{p(n+s)}{p(n)}\right] \tag{2.7}$$

Since we want to define the extension for complex numbers s instead of natural numbers, an idea is to take this definition and let s be any complex number. The problem is that we cannot do so in the term within the brackets, since there are s factors, which does not make sense for nonnatural numbers. But because the identity holds for all $n \in \mathbb{N}$, and the left side is independent of n, an idea is to let n tend to ∞. If the term in the brackets converges to 1, we could define the extension as such a limit. So now we show that this is indeed true:

If $z_1, z_2, \ldots, z_m$ are the zeros of p in $\mathbb{C}$, we have the factorization

$$p(z) = c \prod_{k=1}^{m} (z - z_k) \tag{2.8}$$

For large n, we have $p(n) \neq 0$. Now we have for $z \in \mathbb{C}$

$$\lim_{n\to\infty} \frac{p(n+z)}{p(n)} = \lim_{n\to\infty} \prod_{k=1}^{m} \frac{1 + \frac{z-z_k}{n}}{1 - \frac{z_k}{n}} = 1 \tag{2.9}$$

Thus we have shown that the term in the brackets converges to 1 for all s. This motivates the following: If we write $\Gamma_{p,\alpha}$ for the extension of f to $\mathbb{C}\backslash N$, we define

$$\Gamma_{p,\alpha}(z) := \lim_{n\to\infty} \alpha \frac{p(1)\cdot p(2)\cdot \ldots \cdot p(n)}{p(z)\cdot p(z+1)\cdot \ldots \cdot p(z+n)} \cdot p(n)^z$$

In case the limit exists, this definition leads for all z such that $n+z$ is no zero of p for all n to a 'termination of the infinite recursion'. For all $z \in \mathbb{N}$ the limit exists because of (2.7) and (2.9) stated above.
For all $n \in \mathbb{N}$ the function defined by

$$\Gamma_{p,\alpha,n}(z) := \alpha \frac{p(1)\cdot p(2)\cdot \ldots \cdot p(n)}{p(z)\cdot p(z+1)\cdot \ldots \cdot p(z+n)} \cdot p(n)^z$$

is analytic in $\mathbb{C}\backslash N$. A slight problem arises with the definition of $p(n)^z$. If $p(\mathbb{N}) \not\subset \mathbb{R}^+$ then the definition might require a branch of the logarithm for defining $p(n)^z = e^{z\log(p(n))}$. Branches of the logarithm can only be defined on simply connected regions. In the following, we assume that we have a simply connected region and a branch of the logarithm such that the definitions make sense. Furthermore, we assume that there do not arise problems with logarithm identities. It is clear that for real positive numbers all calculations are valid. But for complex numbers, some problems might arise with the logarithm, for example the functional equation $\log(wz) = \log(w) + \log(z)$ is not in general valid, but it requires a slight modification. In the following, we do not consider these problems, we simply assume that all needed prerequisites are given. This means no loss in generality, just a lot less to write down.
We show that $\Gamma_{p,\alpha,n}$ converges uniformly on compact subsets of $\mathbb{C}\backslash N$ to a function $\Gamma_{p,\alpha}$. Since the limit of a compact uniform convergent sequence of analytic functions is again analytic (cf. [11]), it follows that $\Gamma_{p,\alpha}$ is an analytic function.
Now, a simple manipulation of the definition of $\Gamma_{p,\alpha,n}$ gives

$$\begin{aligned}\Gamma_{p,\alpha,n}(z) &= \frac{\alpha}{p(z)} \cdot \prod_{i=1}^{n} \frac{p(i)}{p(z+i)} e^{z\cdot\log(p(n))} \\ &= e^{z\cdot(\log(p(n)) - \frac{m}{1} - \ldots - \frac{m}{n})} \cdot \frac{\alpha}{p(z)} \cdot \prod_{i=1}^{n} \frac{p(i)}{p(z+i)} \cdot e^{\frac{z\cdot m}{i}}\end{aligned}$$

Recall the factorization (2.8) $p(n) = c\prod_{k=1}^{m} n(1-\frac{z_k}{n})$. This gives

$$\begin{aligned}\frac{\log(p(n))}{m} &= \frac{\log(c)}{m} + \frac{1}{m}\sum_{k=1}^{m}\left(\log(n) + \log\left(1-\frac{z_k}{n}\right)\right) \\ &= \frac{\log(c)}{m} + \log(n) + \sum_{k=1}^{m} \frac{\log(1-\frac{z_k}{n})}{m}\end{aligned}$$

Note that for large n the real parts of n and of $1-\frac{z_k}{n}$ are positive and therefore we can apply the functional equation of the logarithm.
Combining the calculations above we obtain

$$\lim_{n\to\infty} e^{mz\cdot(\frac{\log(p(n))}{m} - 1 - \ldots - \frac{1}{n})} = e^{-\gamma\cdot m\cdot z + \log(c)\cdot z},$$

where

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log(n)\right)$$

is Euler's constant.
Consequently, it remains to be shown that the product

$$\prod_{i=1}^{n}\frac{p(i)}{p(z+i)}\cdot e^{\frac{z\cdot m}{i}}$$

converges uniformly on compact subsets of $\mathbb{C}\backslash N$. This is equivalent to the absolute and uniform convergence of $\sum_{i=1}^{\infty}\log\left(\frac{p(i)}{p(z+i)}\cdot e^{\frac{z\cdot m}{i}}\right)$ on compact sets [6].
In order to show this, we show that the summands of this series are bounded on compact sets by summands of a convergent series of positive numbers. To be more precise, we show that for any compact set $K\subset\mathbb{C}\backslash N$ there is a constant R_0, such that for all $z\in K$ $\left|\log\left(\frac{p(i)}{p(z+i)}\cdot e^{\frac{z\cdot m}{i}}\right)\right|\leq\frac{R_0}{i^2}$ holds for large i. By Weierstrass' criterion and the convergence of $\sum_i\frac{1}{i^2}$, this implies that the series converges normally (for a definition cf. [9]) and therefore uniformly on compact sets.
Recall the taylor series

$$\begin{aligned}\log(1+z) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}z^k\\ \log(1-z) &= -\sum_{k=1}^{\infty}\frac{1}{k}z^k\end{aligned}$$

for $|z|<1$. For sufficiently large i, consider the following:

$$\begin{aligned}\log\left(\frac{p(i)}{p(z+i)}\right) &= \log\left(\prod_{k=1}^{m}\frac{i-z_k}{i+z-z_k}\right)\\ &= \log\left(\prod_{k=1}^{m}\frac{1-\frac{z_k}{i}}{1+\frac{z-z_k}{i}}\right)\\ &= \sum_{k=1}^{m}\left(\log\left(1-\frac{z_k}{i}\right)-\log\left(1+\frac{z-z_k}{i}\right)\right)\\ &= \sum_{k=1}^{m}\left(-\sum_{l=1}^{\infty}\frac{1}{l}\left(\frac{z_k}{i}\right)^l+\sum_{l=1}^{\infty}\frac{(-1)^l}{l}\left(\frac{z-z_k}{i}\right)^l\right)\\ &= \sum_{k=1}^{m}\left(-\frac{z}{i}-\sum_{l=2}^{\infty}\frac{1}{l}\left(\frac{z_k}{i}\right)^l+\sum_{l=2}^{\infty}\frac{(-1)^l}{l}\left(\frac{z-z_k}{i}\right)^l\right)\\ &= -\frac{m\cdot z}{i}+\frac{1}{i^2}\sum_{k=1}^{m}\left(-\sum_{l=0}^{\infty}\frac{z_k^{l+2}}{(l+2)i^l}+\sum_{l=0}^{\infty}\frac{(-1)^{l+2}(z-z_k)^{l+2}}{(l+2)i^l}\right)\end{aligned}$$

Note that for sufficiently large i we can use the taylor expansion of the logarithm. Let $K\subset\mathbb{C}\backslash N$ be a fixed compact set. Then there is a $R>0$ such that

$|z| < R$ for all these z, and therefore $|z - z_k| < \tilde{K}$ for another constant $\tilde{K}$ and $k \in \{1, \ldots, m\}$. This shows that

$$\begin{aligned} \sum_{l=0}^{\infty} \left| \frac{(z-z_k)^{l+2}}{(l+2)i^l} \right| &\leq \tilde{K}^2 \sum_{l=0}^{\infty} \left| \frac{\tilde{K}^l}{(l+2)i^l} \right| \\ &= \tilde{K}^2 \sum_{l=0}^{\infty} \left| \left(\frac{\tilde{K}}{\sqrt{i}} \right)^l \frac{1}{(l+2)\sqrt{i}^l} \right| \\ &\leq C \end{aligned}$$

for a $C > 0$. The last sum converges for sufficiently large i. A similar calculation shows that there is a constant R_0 such that

$$\left| \sum_{k=1}^{m} \left(-\sum_{l=0}^{\infty} \frac{z_k^{l+2}}{(l+2)i^l} + \sum_{l=0}^{\infty} \frac{(-1)^{l+2}(z-z_k)^{l+2}}{(l+2)i^l} \right) \right| \leq R_0$$

We have therefore for $z \in K$

$$\left| \log \left(\frac{p(i)}{p(z+i)} \right) + \frac{m \cdot z}{i} \right| \leq \frac{R_0}{i^2}$$

and thus

$$\sum_{i=N_0}^{\infty} \left| \log \left(\frac{p(i)}{p(z+i)} e^{\frac{m \cdot z}{i}} \right) \right| \leq \sum_{i=N_0}^{\infty} \frac{R_0}{i^2} < \infty$$

if N_0 is large enough.
Since a series of functions $\sum f_k$ is normally convergent if for compact sets K $\sum ||f_j||_{K,\infty} < \infty$ by Weierstrass' criterion [9], we have thus shown that the sum converges uniformly to an analytic function on compact subsets of $\mathbb{C} \backslash N$, which implies that the product above does the same. Here for a continuous function f on a compact set K the sup norm $||f||_{K,\infty} := \sup_{z \in K} |f(z)|$ is defined in the usual way.
We have shown that

$$\Gamma_{p,\alpha} := \lim_{n \to \infty} \Gamma_{p,\alpha,n}$$

is an analytic function on the domain where the definition makes sense.
$\Gamma_{p,\alpha}$ satisfies the required recursion:

$$\begin{aligned} \Gamma_{p,\alpha,n}(z+1) &= \alpha \frac{p(1) \cdot p(2) \cdot \ldots \cdot p(n)}{p(z+1) \cdot p(z+2) \cdot \ldots \cdot p(z+n+1)} \cdot p(n)^{z+1} \\ &= \alpha \frac{p(1) \cdot p(2) \cdot \ldots \cdot p(n)}{p(z) \cdot p(z+1) \cdot \ldots \cdot p(z+n)} \cdot p(n)^z \cdot \frac{p(z) \cdot p(n)}{p(z+n+1)} \\ &= p(z) \cdot \Gamma_{p,\alpha,n}(z) \frac{p(n)}{p(z+n+1)} \to p(z) \cdot \Gamma_{p,\alpha}(z) \qquad \text{as} \quad n \to \infty \end{aligned}$$

Furthermore, $\Gamma_{p,\alpha}|_{\mathbb{N}} = f$, which shows that $\Gamma_{p,\alpha}$ is an extension of f on $\mathbb{C}$.
The definition of $\Gamma_{p,\alpha}$ generalizes the definition of the classical Γ-function, just take the polynomial $p(z) = z$ and receive the Γ-function.
We have shown

Theorem 2.1
Let $p \in \mathbb{C}[z]$ be a polynomial. Then the function $\Gamma_{p,\alpha}$ defined by

$$\Gamma_{p,\alpha}(z) := \lim_{n\to\infty} \alpha \frac{p(1)\cdot p(2)\cdot \ldots \cdot p(n)}{p(z)\cdot p(z+1)\cdot \ldots \cdot p(z+n)} \cdot p(n)^z \tag{2.10}$$

is an analytic function. It satisfies the recursion

$$f(z+1) = p(z)\cdot f(z), \qquad f(1) = \alpha.$$

We have the representation

$$\Gamma_p(z) = e^{-\gamma\cdot m\cdot z + z\log(c)} \cdot \frac{1}{p(z)} \prod_{i=1}^{\infty} \frac{p(i)}{p(z+i)} \cdot e^{\frac{z\cdot m}{i}} \tag{2.11}$$

where c is the leading coefficient of p.

This result is a generalization of the well-known Γ-function. In the following, we will just write Γ_p instead of $\Gamma_{p,\alpha}$, since it is clear that we can modify the value of the function at 1 by a multiplication.
The definition shows that Γ_p is computable by an analytic machine, since it is defined as a uniform limit of computable functions. (Provided the polynomial p is computable.)
Note that the definition $p(n)^z$ requires a branch of the logarithm. So it may be necessary to confine the definition of Γ_p to a simply connected region excluding the zeros of p and $p(z+n)$, $n \in \mathbb{N}$. This problem can be avoided if we define Γ_p on a suitable Riemann surface. In this work, we tacitly assume that the sets where the functions are defined have the required properties.
As it has been said in the introduction, the Riemann surface of Γ_p may be an invariant of the functions defined by such recursion, which may have some applications.

2.3.2 Uniqueness

In the last section, we have found a solution for the recursive equation (2.6). But, as it has been said, the question arises whether there are more solutions, and if there are, how we can identify a certain solution in a *natural* way to make the 'program' defined by the equation unique. This is important for possible applications, because the programs we define by equations should have unique semantics.
For the Γ-function, the uniqueness question has been dealt with before. In 1922, Bohr and Mollerup have shown that the Γ-function is the only continuous solution on the positive real line which is logarithmically convex (see [3]). This is a uniqueness criterion which seems natural to the authors, since the function n! has this property on the natural numbers.
It is easily seen that Γ_p is not the only solution for (2.6). For example, the functions

$$\widetilde{\Gamma_p}(z) := \Gamma_p(z)\cdot(\sin(2\pi kz)+1), \qquad k \in \mathbb{Z}$$

are also solutions of this recursion. We shall determine all functions which are solutions. If $f_1(z), f_2(z)$ are both solutions of (2.6), then

$$\frac{f_1(z+1)}{f_2(z+1)} = \frac{f_1(z)}{f_2(z)},$$

i. e. $\frac{f_1}{f_2}$ is a periodic function with period 1. If we demand the solutions to be analytic, we have thus that the quotient of two solutions is an analytic function with period 1. So our solution is unique modulo the periodic functions.
In the next paragraph, we use the notion of logarithmic convexity to make Γ_p the unique solution of our recursion which has this property.

2.3.3 Logarithmic convexity

First, recall some basic properties of convex functions. A function g is called *convex*, if for $0 \leq \lambda \leq 1$ and all a, b the following inequality holds:

$$g(\lambda a + (1-\lambda)b) \leq \lambda g(a) + (1-\lambda)g(b) \tag{2.12}$$

This means that the graph of g lies below each secant. It is easy to show that for differentiable functions this is equivalent to

$$g''(x) > 0.$$

Let p be a polynomial such that $p(x) > 0, \quad \forall x > x_0$ for some $x_0 \in \mathbb{R}$. We show that the function Γ_p above is logarithmically convex in an interval (a, ∞) for some $a > 0$. Recall that a function f is logarithmically convex, if $\log f$ is a convex function, which is equivalent to $(\log f)'' > 0$. By Γ_p we mean $\Gamma_{p,1}$ in order to simplify our calculations. Using (2.11)

$$\Gamma_p(x) = e^{-\gamma \cdot m \cdot x + \log(c)x} \cdot \frac{1}{p(x)} \prod_{i=1}^{\infty} \frac{p(i)}{p(x+i)} \cdot e^{\frac{x \cdot m}{i}},$$

we obtain

$$\log \Gamma_p(x) = -\gamma m x + x\log(c) - \log(p(x)) + \sum_{i=1}^{\infty} \left(\log(p(i)) - \log(p(x+i)) + \frac{xm}{i} \right)$$

and

$$(\log \Gamma_p)'(x) = -\gamma m + \log(c) - \frac{p'(x)}{p(x)} + \sum_{i=1}^{\infty} -\frac{p'(x+i)}{p(x+i)}$$

$$\begin{aligned} (\log \Gamma_{p,1})''(x) &= -\frac{p(x)p''(x) - p'(x)^2}{p(x)^2} + \sum_{i=1}^{\infty} -\frac{p(x+i)p''(x+i) - p'(x+i)^2}{p(x+i)^2} \\ &= \frac{p'(x)^2 - p(x)p''(x)}{p(x)^2} + \sum_{i=1}^{\infty} \frac{p'(x+i)^2 - p(x+i)p''(x+i)}{p(x+i)^2} > 0 \end{aligned}$$

if x is large enough. The last inequality holds, because $p'(x)^2 > p(x)p''(x)$ for large x. This is true because the degrees of p'^2 and pp'' are equal, but the leading coefficient of the first is greater. In fact, $\deg(p'^2) = 2(m-1) = \deg(p \cdot p'')$ and if $p(x) = a_m x^m + \ldots$, then $p'(x)^2 = m^2 a_m^2 x^{2(m-1)} + \ldots$, but $p(x)p''(x) = m(m-1)a_m^2 x^{2(m-1)} + \ldots$
We have shown

Proposition 2.2
Let $p \in \mathbb{R}[x]$ be a polynomial with real coefficients such that the leading coefficient is positive. Then there is an $x_0 > 0$ such that the function $\Gamma_{p,1}$ is logarithmically convex on the interval (x_0, ∞).

Furthermore, we show that $\Gamma_{p,1}$ is uniquely determined by this property, more precisely, we show the following

Theorem 2.3
Let p be a polynomial with $p(x) > 0$ if $x > 0$. Let f be an analytic function with the following properties:

- $f(x+1) = p(x) \cdot f(x) \quad (x > 0)$
- *f logarithmically convex on the positive real line*
- $f(1) = 1$

Then $f = \Gamma_p$.

Proof:
The proof is almost the same as the uniqueness proof for the Γ-function under the logarithmic convexity condition. Because of the functional equation it suffices to show the uniqueness for $0 < x < 1$. Since we have the convex combination $n + x = (1-x) \cdot n + x \cdot (n+1)$ (note that $0 < x < 1$), it follows with the logarithmic convexity of f (see (2.12)):

$$\log f(n+x) \leq (1-x) \cdot (\log f(n)) + x \cdot \log f(n+1) \qquad \text{and thus}$$

$$f(n+x) \leq f(n)^{1-x} \cdot f(n+1)^x = f(n) \cdot p(n)^x$$

$$f(n+x) \leq p(1) \cdot \ldots \cdot p(n-1) \cdot p(n)^x \tag{2.13}$$

In the same way $n + 1 = (1-x) \cdot (n+1+x) + x \cdot (n+x)$ implies

$$f(n+1) \leq f(n+1+x)^{1-x} \cdot f(n+x)^x = f(n+x) \cdot p(n+x)^{1-x}$$

$$f(n+1) \cdot p(n+x)^{x-1} \leq f(n+x)$$

$$p(1) \cdot \ldots \cdot p(n) \cdot p(n+x)^{x-1} \leq f(n+x) \tag{2.14}$$

Combining (2.13) and (2.14) leads to

$$p(1) \cdot \ldots \cdot p(n) \cdot p(n+x)^{x-1} \leq f(n+x) \leq p(1) \cdot \ldots \cdot p(n-1) \cdot p(n)^x$$

Since $f(n+x) = f(x) \cdot p(n+x-1) \cdot \ldots \cdot p(x)$, this is equivalent to

$$\underbrace{\frac{p(1) \cdot \ldots \cdot p(n)}{p(x) \cdot \ldots \cdot p(x+n)} \cdot p(n+x)^x}_{a_n} \leq f(x) \leq \underbrace{\frac{p(1) \cdot \ldots \cdot p(n-1)}{p(x) \cdot \ldots \cdot p(x+n-1)} \cdot p(n)^x}_{b_n}$$

Now, from

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{p(x+n)^x}{p(n)^x} \cdot \frac{p(n)}{p(x+n)} = 1$$

it follows that

$$f(x) = \lim_{n \to \infty} \frac{p(1) \cdot \ldots \cdot p(n-1)}{p(x) \cdot \ldots \cdot p(x+n-1)} \cdot p(n)^x = \lim_{n \to \infty} \frac{p(1) \cdot \ldots \cdot p(n)}{p(x) \cdot \ldots \cdot p(x+n)} \cdot p(n)^x.$$

This completes the proof. □

Remark 2.4
Obviously $\Gamma_{p,\alpha}$ is also uniquely determined by the logarithmic convexity for positive α. In our proof, we assume that Γ_p is logarithmically convex on the whole positive real line. But this is not necessarily the case, we have shown merely that $\Gamma_{p,1}$ is logarithmically convex on an Interval $(a,\infty) \subset \mathbb{R}^+$. But intuitively, this does not matter, since we simply can translate the polynomial p and correspondingly Γ_p. If Γ_p is logarithmically convex on (a,∞) for an $a > 0$, then translate the function to the left such that it is logarithmically convex on the positive real line. Then, by our theorem, the solution of the recursion of the translated polynomial is unique under the condition of logarithmic convexity, and since we demand a certain consistency of our solutions, the original Γ_p-function is also uniquely determined. The following proposition shows that translation of the defining polynomial results in the same translation of the corresponding Γ_p-function, multiplied with a certain factor.

Proposition 2.5
Let $N \in \mathbb{N}$, p be a polynomial and q be the polynomial defined by

$$q(x) := p(x+N),$$

and

$$\alpha := \Gamma_{p,1}(N+1).$$

Then

$$\Gamma_{p,1}(x+N) = \Gamma_{q,\alpha}(x),$$

i. e. if we translate p, Γ_p is translated in the same way.

Proof:

$$\begin{aligned}
\Gamma_{q,\alpha}(x) &= \lim_{n\to\infty} \alpha \frac{p(1+N)\cdot\ldots\cdot p(n+N)}{p(x+N)\cdot\ldots\cdot p(x+N+n)} \cdot p(n+N)^x \\
&= \lim_{n\to\infty} \frac{p(1)\cdot\ldots\cdot p(n)}{p(x+N)\cdot\ldots\cdot p(x+N+n)} \cdot p(n+1)\cdot\ldots\cdot p(n+N)\cdot p(n+N)^x \\
&= \lim_{n\to\infty} \frac{p(1)\cdot\ldots\cdot p(n)}{p(x+N)\cdot\ldots\cdot p(x+N+n)} \cdot p(n)^{x+N} \\
&\qquad \cdot \underbrace{\frac{p(n+N)^x}{p(n)^x} \cdot \frac{p(n+1)}{p(n)} \cdot\ldots\cdot \frac{p(n+N)}{p(n)}}_{\to 1} \\
&= \Gamma_{p,1}(x+N)
\end{aligned}$$

□

In the next section, we will examine some identities for the Γ_p-functions. We show that the mapping $p \mapsto \Gamma_p$ is multiplicative, and that Γ_c for a constant c is the exponential with base c, as one expects. This will lead to an interesting representation of Γ_p by the original Γ-function.

2.3.4 Representation theorem

Proposition 2.6
a) $\Gamma_{p,\alpha} = \alpha\Gamma_{p,1}$

b) For a constant $c > 0$, $\Gamma_{c,\alpha}(z) = \alpha c^{z-1}$

c) $\Gamma_{pq,\alpha\beta} = \Gamma_{p,\alpha}\Gamma_{q,\beta}$

d) If p *is the monomial* $p(z) = (z-a)$, *then* $\Gamma_{p,1}(z) = \frac{\Gamma(z-a)}{\Gamma(1-a)}$, *where* Γ *is the original* Γ*-function.*

Proof:

a) This is trivial. But this is the reason why we just write Γ_p and mean $\Gamma_{p,1}$.

b) The definition of $\Gamma_{p,\alpha}$ for the constant polynomial $p(z) = c$ gives

$$\Gamma_{c,\alpha}(z) = \lim_{n\to\infty} \alpha \frac{c^n}{c^{n+1}} \cdot c^z = \alpha c^{z-1}$$

c) Just use the definition of $\Gamma_{p,\alpha}$:

$$\begin{aligned}
\Gamma_{pq,\alpha\beta}(z) &= \lim_{n\to\infty} \alpha\beta \frac{pq(1)\cdot\ldots\cdot pq(n)}{pq(z)\cdot\ldots\cdot pq(z+n)} \cdot pq(n)^z \\
&= \left(\lim_{n\to\infty} \alpha \frac{p(1)\cdot\ldots\cdot p(n)}{p(z)\cdot\ldots\cdot p(z+n)} \cdot p(n)^z\right)\left(\lim_{n\to\infty} \beta \frac{q(1)\cdot\ldots\cdot q(n)}{q(z)\cdot\ldots\cdot q(z+n)} \cdot q(n)^z\right) \\
&= \Gamma_{p,\alpha}(z)\Gamma_{q,\beta}(z)
\end{aligned}$$

d) Since a standard definition of Γ is (cf. [6])

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdot\ldots\cdot(z+n)},$$

we obtain

$$\begin{aligned}
\frac{\Gamma(z-a)}{\Gamma(1-a)} &= \lim_{n\to\infty}\left(\frac{n! n^{z-a}}{(z-a)\cdot\ldots\cdot(z-a+n)} \frac{(1-a)\cdot\ldots\cdot(1-a+n)}{n! n^{1-a}}\right) \\
&= \lim_{n\to\infty}\left(\frac{p(1)\cdot\ldots\cdot p(n)}{p(z)\cdot\ldots\cdot p(z+n)}(1-a+n)n^{z-1}\right) \\
&= \lim_{n\to\infty}\left(\frac{p(1)\cdot\ldots\cdot p(n)}{p(z)\cdot\ldots\cdot p(z+n)}p(n)^z\right) \\
&= \Gamma_p(z),
\end{aligned}$$

where we used that

$$\lim_{n\to\infty}\left(\frac{(1-a+n)n^{z-1}}{(n-a)^z}\right) = \lim_{n\to\infty}\left(\left(\frac{n}{n-a}\right)^{z-1} \frac{1-a+n}{n-a}\right) = 1$$

This completes the proof. □

This lemma leads to the following representation theorem:

Theorem 2.7 (Representation theorem of Γ_p)
Let $p \in \mathbb{C}[z]$ be a polynomial. If

$$p(z) = c \prod_{j=1}^{n} (z - a_j)$$

is the factorization of p, where a_j are the zeros, then

$$\Gamma_{p,\alpha}(z) = \alpha c^z \prod_{j=1}^{n} \frac{\Gamma(z - a_j)}{\Gamma(1 - a_j)}$$

Proof:
This follws immediately from Proposition 2.6. □

2.3.5 Rational functions

In the last paragraph, we found solutions for (2.6) in the case that p is a polynomial. A simple consideration shows that we can extend these results to rational functions r. Since rational functions are quotients of polynomials, $r(x) = \frac{p(x)}{q(x)}$, it is an immediate calculation that $\frac{\Gamma_p}{\Gamma_q}$ solves the recursion for r:

$$\frac{\Gamma_p}{\Gamma_q}(x+1) = \frac{p(x)\Gamma_p(x)}{q(x)\Gamma_q(x)} = r(x)\frac{\Gamma_p}{\Gamma_q}(x)$$

Now we can generalize the results of the last sections to rational functions and summarize them in

Theorem 2.8 (Multiplicative recursion for rational functions)
Let $r(z) = \frac{p(z)}{q(z)}$ be a rational function. Then

$$\Gamma_{r,\alpha}(z) := \lim_{n\to\infty} \alpha \frac{r(1) \cdot r(2) \cdot \ldots \cdot r(n)}{r(z) \cdot r(z+1) \cdot \ldots \cdot r(z+n)} \cdot r(n)^z \tag{2.15}$$

defines an analytic function which solves the recursion

$$\Gamma_{r,\alpha}(z+1) = r(z)\Gamma_{r,\alpha}(z), \qquad \Gamma_{r,\alpha}(1) = \alpha$$

If

$$p(z) = c \prod_{j=1}^{n} (z - a_j)$$

and

$$q(z) = d \prod_{j=1}^{m} (z - b_j),$$

we have the representation

$$\Gamma_{r,\alpha}(z) = \alpha \left(\frac{c}{d}\right)^z \prod_{j=1}^{n} \frac{\Gamma(z - a_j)}{\Gamma(1 - a_j)} \prod_{j=1}^{m} \frac{\Gamma(1 - b_j)}{\Gamma(z - b_j)}.$$

Proof:
The compact uniform convergence of the limit follows from the proof of the compact uniform convergence of Γ_p, since we simply can write $r = \frac{p}{q}$ and examine the limits of the numerator and denominator separately.
The representation is an immediate consequence of the representation theorem for Γ_p and the observation $\Gamma_{\frac{1}{p}} = \frac{1}{\Gamma_p}$, which immediately follows from the definitions. □

2.3.6 Uniqueness of rational solutions

We have already shown that the Γ_p function is uniquely determined by continuity and logarithmic convexity. But this was shown only for real polynomials that have positive values on the positive real line. We want to give constraints which make all Γ_p-functions unique. The representation theorem and lemma 2.6 show that the mapping $p \mapsto \Gamma_p$ has the properties of a homomorphism. These properties motivate the following:

Theorem 2.9
Consider the recursion

$$f(z+1) = p(z) \cdot f(z), \quad f(1) = 1 \tag{2.16}$$

where p is a polynomial. The solutions of all these recursions are required to fulfill the following properties:

(i) The solution of $f(z+1) = c \cdot f(z)$, i. e. the case that $p(z) = c$ is constant, is the exponential function c^{z-1}.

(ii) The solution of the linear polynomial $p(z) = z - a$ is the shifted Γ-function (up to a factor), i. e. the solution of $f(z+1) = (z-a)f(z)$ is $\frac{\Gamma(z-a)}{\Gamma(1-a)}$.

(iii) The solutions are multiplicative, i. e. if f_p is the solution of $f_p(z+1) = p(z)f_p(z)$ and f_q is the solution of $f_q(z+1) = q(z)f_q(z)$, then the solution f_{pq} of $f_{pq}(z+1) = p(z)q(z)f_{pq}(z)$ is $f_p \cdot f_q$.

Then the solution of (2.16) is uniquely determined as Γ_p. If we add the requirement

(iv) The solution of $f(z+1) = \frac{1}{p(z)} f(z)$ is the multiplicative inverse of the solution of (2.16),

then the solution for all rational functions p is uniquely determined by Γ_p.

Proof:
The fact that the functions Γ_p fulfill these constraints follows from Lemma 2.6 and the representation theorem.
Conversely, let p be any polynomial, and let f be a solution of (2.16). Then p has the factorization

$$p(z) = c \prod_{k=1}^{m} (z - z_k). \tag{2.17}$$

By (i) and (ii), the solutions of $f(z+1) = cf(z)$ and $f(z+1) = (z - z_k)f(z)$ are c^{z-1} and $\frac{\Gamma(z-z_k)}{\Gamma(1-z_k)}$, respectively, it follows from (iii) and the representation

theorem that $f(z) = \Gamma_p(z)$.
For rational functions, the proof is the same. □

Remark 2.10
If we have the uniqueness requirement of logarithmic convexity for the solutions for positive real polynomials, then this requirement is consistent with the requirements of the theorem above.

2.3.7 A new proof of Legendre's formula

This section contains a mathematical application of our Γ_p theory. We present a new proof for the known duplication formula by Legendre

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z)\Gamma(z+\frac{1}{2}).$$

Proof:
The function $F : z \mapsto \Gamma(2z)$ is an analytic function which satisfies the recursion

$$F(z+1) = \Gamma(2z+2) = 2z(2z+1)\Gamma(2z) = p(z)F(z), \quad F(1) = 1$$

with the polynomial $p(z) = 2z(2z+1) = 2^2 z\left(z+\frac{1}{2}\right)$.
By our uniqueness theorem (since $\Gamma(2z)$ is logarithmically convex on the positive reals), it follows

$$F(x) = \Gamma_p(x)$$

on the reals, and since these are holomorphic it follows for all z that apply here.
By our representation theorem 2.7, we have

$$\begin{aligned}
\Gamma_p(z) &= \left(2^2\right)^{z-1} \cdot \Gamma(z) \frac{\Gamma\left(z+\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}\right)} \\
&= \left(2^2\right)^{z-1} \cdot \Gamma(z) \frac{\Gamma\left(z+\frac{1}{2}\right)}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \\
&= \frac{1}{\sqrt{\pi}} 2^{2z-1} \cdot \Gamma(z)\Gamma\left(z+\frac{1}{2}\right),
\end{aligned}$$

where we used the known fact $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. □

Remark 2.11
This proof sheds new light on this formula, since it shows that the two Γ-functions appearing on the right hand come from the fact that $\Gamma(2z)$ is simply a Γ_p function for a polynomial p of degree 2.

A similar proof, which requires more calculation, also applies to Gauss' multiplication formula

$$\Gamma(nz) = \left(\frac{1}{2\pi}\right)^{\frac{n-1}{2}} n^{\frac{2nz-1}{2}} \Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdot\ldots\cdot\Gamma\left(z+\frac{n-1}{n}\right).$$

2.4 The additive case

We are interested in analytic solutions of the recursion

$$f(x+1) = f(x) + h(x), \qquad f(0) = 0 \tag{2.18}$$

where h is an analytic function.
Let us first make some general considerations. Given two additive recursions

$$f_i(x+1) = f_i(x) + h_i(x), \quad i = 1, 2$$

then

$$(f_1 + f_2)(x+1) = (f_1 + f_2)(x) + h_1(x) + h_2(x)$$

If α is a constant, then

$$(\alpha f)(x+1) = (\alpha f)(x) + \alpha h(x)$$

This shows that the recursion is linear, i. e. if $\Delta(h)$ denotes the solution f of (2.18), then the relation Δ has the properties

$$\Delta(g+h) = \Delta(g) + \Delta(h), \qquad \Delta(\alpha g) = \alpha\Delta(g)$$

We have to be careful about the equality sign, since we do not know yet anything about uniqueness of solutions let alone their existence. But our goal is to define such an operator on the space of analytic functions or at least on a subspace of this space. In order to do this, we will introduce an operation on power series which we call the Bernoulli Transform.

2.4.1 The Bernoulli Transform

We want to define functions $s_k(z)$ which are continuations of the primitive-recursive functions defined by

$$s_k(n+1) = s_k(n) + n^k, \qquad s_k(0) = 0. \tag{2.19}$$

Since the recursion is definied by a polynomial, we might expect that the solution is again a polynomial. Bernoulli has already dealt with this problem, and the solutions are modifications of the known Bernoulli Polynomials. The following definition of $s_k(z)$ satisfies this recursion, and the $s_k(z)$ are indeed polynomials:

Definition 2.12
The B-polynomials $s_k(z)$ *are defined by the functions appearing in the following power series expansion [9]:*

$$F(w,z) = \frac{e^{wz} - e^z}{e^z - 1} = \sum_{k=1}^{\infty} \frac{s_k(w)}{k!} z^k + w - 1, \qquad s_0 = 0 \tag{2.20}$$

In an analogous way, the well-known Bernoulli Polynomials $B_k(z)$ *are defined [9]:*

$$H(w,z) = \frac{ze^{wz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(w)}{k!} z^k \tag{2.21}$$

The number $B_k = B_k(0)$ *is the* k*-th Bernoulli Number.*

By a simple calculation, one can show that the functions defined by the coefficients of these power series expansions are indeed polynomials. Here are some basic properties of the B-Polynomials and the Bernoulli Polynomials:

Proposition 2.13
a) *The Bernoulli Polynomials can be written explicitly as*

$$B_k(w) = \sum_{i=0}^{k} \binom{k}{i} B_i w^{k-i}, \tag{2.22}$$

where $B_i = B_i(0)$ are the well-known Bernoulli Numbers.

b) *The B-Polynomials and the Bernoulli Polynomials are connected by the following identity:*

$$s_k(w) = \frac{1}{k+1}(B_{k+1}(w) - B_{k+1}(1)). \tag{2.23}$$

c) *The Bernoulli Numbers can be expressed as*

$$B_{2m} = 2\frac{(-1)^{m-1}(2m)!}{(2\pi)^{2m}} \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = 2\frac{(-1)^{m-1}(2m)!}{(2\pi)^{2m}}\zeta(2m) \tag{2.24}$$

And it is $B_{2m+1} = 0$.
Here $\zeta(z)$ denotes Riemann's Zeta function.

These are well-known facts, for a proof see [9].
Since the B-polynomials solve (2.19) and because of the linearity one might try to solve

$$f(z+1) = f(z) + h(z), \quad f(0) = 0, \qquad h(z) = \sum_{k=0}^{\infty} a_k z^k \tag{2.25}$$

by putting

$$f(z) = \sum_{k=0}^{\infty} a_k s_k(z),$$

i. e. we simply replace the powers z^k in the taylor expansion of h by the k-th B-polynomial $s_k(z)$. This replacement we will call *Bernoulli Transform.* Note that this Bernoulli Transform is, in our sense, computable, since by Proposition 2.13 we can compute the s_k.

Definition 2.14
Let $h(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function. The formal sum

$$\Delta(h) = \sum_{k=0}^{\infty} a_k s_k(z) \tag{2.26}$$

is called the Bernoulli Transform *of h.*

Note that this is only a formal sum, it may not converge.
If h is a polynomial, however, this is obviously a solution of the recursion:

$$\begin{aligned}\Delta(h)(z+1) &= \sum_{k=0}^{n} a_k s_k(z+1) \\ &= \sum_{k=0}^{n} a_k (s_k(z) + z^k) \\ &= \sum_{k=0}^{n} a_k s_k(z) + \sum_{k=0}^{n} a_k z^k \\ &= \Delta(h)(z) + h(z).\end{aligned}$$

This solution is a *natural* one, since if we demand the solution to be polynomial, it is necessarily unique. In fact, since we want a continuation of the function f defined on the natural numbers, infinitely many values of the continuation are fixed. So there can be only one polynomial having these values at the natural numbers. For polynomials, the uniqueness question of the introduction can easily be answered, and so for polynomials h, we can define polynomial solutions of (2.25) as natural.
But if h is any analytic function, it is not clear when $\sum a_k s_k(z)$ converges, and we will see that it does not so in the general case.

2.4.2 Pointwise Convergence

In order to achieve pointwise convergence of the Bernoulli Transform on $\mathbb{C}$, consider the following.
The function defined by $F(w,z) = \frac{e^{wz}-e^z}{e^z-1}$ has for $w \notin \mathbb{Z}$ in $\{z \in \mathbb{C} \mid |z| \leq 2\pi\}$ its only isolated singularity in $2\pi i$. Therefore the convergence radius of the series $\sum_{k=1}^{\infty} \frac{s_k(w)}{k!} z^k + z - 1$ is 2π. By Cauchy-Hadamard's formula for the convergence radius of a power series,

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|},$$

where R is the convergence radius of the series $\sum_{n=0}^{\infty} a_n z^n$, it follows for all $w \in \mathbb{C}$, $w \notin \mathbb{Q}$:

$$\limsup_{k\to\infty} \left(\sqrt[k]{\frac{|s_k(w)|}{k!}} \right) = \frac{1}{2\pi} \tag{2.27}$$

A sufficient criterion for pointwise convergence of the Bernoulli Transform is then

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k| k!} \right) < 2\pi$$

Since under this condition we have

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k| \cdot |s_k(w)|} \right) = \limsup_{k\to\infty} \left(\sqrt[k]{|a_k| k! \frac{|s_k(w)|}{k!}} \right) < 1$$

and the series $\sum_{k=0}^{\infty} a_k s_k(z)$ converges absolutely for all $z \in \mathbb{C}$.
But if we want to assure that the limit function is analytic, we need uniform convergence. A criterion for this will be shown in the next section.

2.4.3 Uniform Convergence

Now we will show that

$$\limsup_{k\to\infty}\left(\sqrt[k]{|a_k|k!}\right) < 2\pi \tag{2.28}$$

is a sufficient criterion for uniform convergence on compact subsets of $\mathbb{C}$. In order to prove this, we will first prove an estimation for the B-Polynomials.

Proposition 2.15
There is a constant $M > 0$ which is independent of k and z, such that

$$|s_k(z)| \le M\frac{k!}{(2\pi)^k}e^{2\pi|z|} \tag{2.29}$$

Proof:
By Proposition 2.13, we have

$$B_{2k+1} = 0, \quad B_{2k} = 2\frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$$

and therefore for $k > 1$:

$$\begin{aligned} |B_k| &\le 2\frac{k!}{(2\pi)^k}\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &\le 2\frac{\pi^2}{6}\frac{k!}{(2\pi)^k}, \end{aligned}$$

which holds because

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},$$

which is a commonly known fact. With the representation of the Bernoulli Polynomials in Proposition 2.13 we obtain for a constant C

$$\begin{aligned} |B_k(z)| &\le \sum_{i=0}^{k}\binom{k}{i}|B_i||w|^{k-i} \\ &\le C\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\frac{i!}{(2\pi)^i}|w|^{k-i} \\ &= C\frac{k!}{(2\pi)^k}\sum_{i=0}^{k}\frac{(2\pi|w|)^{k-i}}{(k-i)!} \\ &\le C\frac{k!}{(2\pi)^k}e^{2\pi|w|} \end{aligned}$$

and therefore by Proposition 2.13 with another constant M

$$|s_k(z)| \le M\frac{k!}{(2\pi)^k}e^{2\pi|w|},$$

which completes the proof. □

Now we are able to show that (2.28) is sufficient for the uniform convergence of the Bernoulli Transform of f. If for the series

$$\sum_{k=0}^{\infty} a_k z^k$$

the criterion

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k|k!} \right) < 2\pi$$

holds, there is by the definition of the limes superior a k_0, such that for all $k \geq k_0$

$$\sqrt[k]{|a_k|k!} < a < 2\pi$$

for some a and thus

$$|a_k|k! \leq a^k.$$

Let $U \subset \mathbb{C}$ be compact and

$$A = ||e^{2\pi|z|}||_{U,\infty} \qquad (||f||_{U,\infty} := \sup_{z\in U} |f(z)|)$$

$$||a_k s_k(z)||_{U,\infty} \leq \left|\left| a_k M \frac{k!}{(2\pi)^k} e^{2\pi|z|} \right|\right|_{U,\infty} \leq MA \left(\frac{a}{2\pi}\right)^k$$

Because $a < 2\pi$, the series

$$\sum_{k=1}^{\infty} a_k s_k(z)$$

is on U normally convergent, it is then uniformly convergent on compact sets and converges therefore to an analytic function. Let us summarize this result in

Proposition 2.16
Let

$$h(z) = \sum_{k=0}^{\infty} a_k z^k$$

be an analytic function, such that

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k|k!} \right) < 2\pi$$

holds. Then the Bernoulli Transform

$$\Delta(h)(z) = \sum_{k=0}^{\infty} a_k s_k(z)$$

defines an analytic function on $\mathbb{C}$, which satisfies

$$\Delta(h)(z+1) = \Delta(h)(z) + h(z).$$

Proof:
Since the sum converges absolutely and uniformly on compact sets, the function $\Delta(h)$ is a well-defined analytic function, and

$$\begin{aligned}\Delta(h)(z+1) &= \sum_{k=0}^{\infty} a_k s_k(z+1)\\ &= \sum_{k=0}^{\infty} \left(a_k s_k(z) + a_k z^k\right)\\ &= \sum_{k=0}^{\infty} a_k s_k(z) + \sum_{k=0}^{\infty} a_k z^k\\ &= \Delta(h)(z) + h(z)\end{aligned}$$

□

2.4.4 Necessary criterion

If we examine what has been said about pointwise convergence, we easily obtain a necessary criterion for pointwise convergence of the Bernoulli Transform: If

$$\lim_{k\to\infty} \left(\sqrt[k]{|a_k|k!}\right) > 2\pi,$$

then there are points $w \in \mathbb{C}$ such that $\sum a_k s_k(w)$ does not converge absolutely. In fact this holds for *all* $w \notin \mathbb{Q}$: Under these circumstances, by (2.27) it is

$$\limsup_{k\to\infty} \left(\sqrt[k]{\frac{|s_k(w)|}{k!}}\right) = \frac{1}{2\pi}$$

and thus for some $a > 0$

$$\limsup_{k\to\infty} \left(\sqrt[k]{\frac{|a_k||s_k(w)|}{k!}}\right) = b > a > 1$$

By the definition of the limes superior, this implies that there is a subsequence $(|a_{n_k}||s_{n_k}(w)|)$ converging to b, which implies that almost all $|a_{n_k} s_{n_k}|$ are $> a$. But this means that in the series $\sum |a_k s_k(z)|$ infinitely many summands are > 1, which shows that this series cannot converge.

2.4.5 The Space $\mathcal{B}_0$

We will see that the set of all functions satisfying (2.28) is a vector space, which has some additional closure properties.

Definition 2.17
The space $\mathcal{B}_0$ is defined as

$$\mathcal{B}_0 := \left\{ \sum_{k=0}^{\infty} a_k z^k \,\middle|\, \limsup_{k\to\infty} \left(\sqrt[k]{|a_k|k!}\right) < 2\pi \right\} \tag{2.30}$$

We show that $\mathcal{B}_0$ is a linear space: If

$$\limsup_{k\to\infty}\left(\sqrt[k]{|a_k|k!}\right) < 2\pi,$$

then for all $\lambda \in \mathbb{C}$

$$\limsup_{k\to\infty}\left(\sqrt[k]{|\lambda a_k|k!}\right) < 2\pi$$

and if

$$\limsup_{k\to\infty}\left(\sqrt[k]{|a_k|k!}\right) < 2\pi, \qquad \limsup_{k\to\infty}\left(\sqrt[k]{|b_k|k!}\right) < 2\pi$$

then also

$$\begin{aligned}\limsup_{k\to\infty}\left(\sqrt[k]{|a_k+b_k|k!}\right) &\le \max\left(\limsup_{k\to\infty}\left(\sqrt[k]{|a_k|k!}\right), \limsup_{k\to\infty}\left(\sqrt[k]{|b_k|k!}\right)\right)\\ &< 2\pi.\end{aligned}$$

This shows that $\mathcal{B}_0$ is a vector space. The Bernoulli Transform Δ is a linear operator on this vector space:

$$\Delta : \mathcal{B}_0 \to \mathcal{O}(\mathbb{C}),\ f \mapsto \Delta f$$

Moreover, this space is closed under differentiation: If $h(z) = \sum_{k=0}^{\infty} a_k z^k$, then $h'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$. Examining the coefficients, we get

$$\limsup_{k\to\infty}\left(\sqrt[k]{(k+1)|a_{k+1}|k!}\right) < 2\pi$$

2.4.6 Uniqueness

As in the multiplicative case, the solutions of $f(x+1) = f(x) + h(x)$, $f(0) = 0$ are not unique, since we simply can add periodic functions. So we look for criteria under which the solutions are unique. These criteria should be as simple as possible. For example, the requirement that for polynomials the soulutions are again polynomials is simple, and it's plausible to make this requirement. This is what we informally call 'natural'. Our aim is to give plausible criteria for uniqueness, and the solutions that are subject to these conditions we then call natural. We will see that we can define solutions of the additive recursion uniquely for those h for which the Bernoulli Transform converges, if we require the two following uniqueness conditions:

(i) First, we demand that the solution f of $f(x+1) = f(x) + p(x)$, $f(0) = 0$, where p is a polynomial, is again a polynomial.

(ii) Δ should be a linear operator, i. e. if f_1, f_2 are solutions of $f_1(z+1) = f_1(z) + h_1(z)$, $f_2(z+1) = f_2(z) + h_2(z)$, then for λ, μ the solution of $f(z+1) = f(z) + \lambda h_1(z) + \mu h_2(z)$ should be $\lambda f_1(z) + \mu f_2(z)$. Note that this restriction has to be made, indeed it is clear that $\lambda f_1(z) + \mu f_2(z)$ solves the new recursion, but we could have taken a different solution. But this would be not consistent with linearity.

(iii) Δ should be a continuous linear operator, it should be continuous with respect to the topology of the ring $\mathcal{O}(\mathbb{C})$ of holomorphic functions, which is given by uniform convergence on compact subsets, i. e. if $(h_n)_{n=1}^{\infty}$ is a convergent sequence of functions with

$$h_n \to h$$

and $f_n = \Delta(h_n)$ is a solution of $f_n(x+1) = f_n(x) + h_n(x)$, then we demand that

$$f_n \to f,$$

where $f = \Delta(h)$ is solution of $f(x+1) = f(x) + h(x)$.

It is easily seen that these restrictions make our solutions unique:
If $f(0) = 0$ and $f(x+1) = f(x) + p(x)$ for a polynomial p, then f is uniquely determined on the natural numbers, and therefore there can only be one polynomial f which satisfies the recursion.
The general case $f(x+1) = f(x) + h(x)$ is unique because of the continuity, since for any h there is a sequence of polynomials $p_n(x)$ with $p_n(x) \to h(x)$. In fact, we simply can take the n-th Taylor approximation of h as p_n.
The next step would be to show that our solution satisfies these conditions: First, it is clear that for a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ the Bernoulli Transform $\Delta p(z) = \sum_{k=0}^{n} a_k s_k(z)$ is again a polynomial, since the $s_k(z)$ are and so are linear combinations of the $s_k(z)$.
But unfortunately, the linear operator Δ is not continuous on $\mathcal{B}_0$ with respect to the topology of uniform convergence on compact subsets of $\mathbb{C}$, as the following counterexample shows: Let

$$f_n(z) = \frac{z^n 10^n}{n!}.$$

Clearly, $f_n \to 0$ uniformly on compact $K \subset \mathbb{C}$. But

$$\Delta f_n(z) = \frac{s_n(z) 10^n}{n!}$$

does not converge to $\Delta(0) = 0$, since, by (2.27) we have for $w \notin \mathbb{Q}$:

$$\limsup_{k \to \infty} \left(\sqrt[k]{\frac{|s_k(w)|}{k!}} \right) = \frac{1}{2\pi}$$

and therefore by the definition of the limes superior, there is a subsequence

$$\left(\sqrt[k_j]{\frac{|s_{k_j}(w)|}{k_j!}} \right)_{j=1}^{\infty}$$

converging to $\frac{1}{2\pi}$. But this means, since $\frac{1}{10} < \frac{1}{2\pi}$, that

$$\frac{|s_{k_j}(z)|}{k_j!} \geq \left(\frac{1}{10} \right)^{k_j}$$

for j large enough and therefore

$$|\Delta f_n(z)| \geq 1$$

for infinitely many n. Thus, Δf_n does not converge to 0.
This problem, however, can be avoided if we define a suitable norm on $\mathcal{B}_0$.

2.4.7 The Bernoulli Space $\mathcal{B}$

We will define a norm for analytic functions which are subject to a certain condition.

Definition 2.18
For an analytic function $h(z) = \sum_{k=0}^{\infty} a_k z^k$, *define*

$$||h||_{\mathcal{B}} := \sum_{k=0}^{\infty} |a_k| \frac{k!}{(2\pi)^k}. \tag{2.31}$$

The Bernoulli Space $\mathcal{B}$ *is defined as*

$$\mathcal{B} := \{h \in \mathcal{O}(\mathbb{C}) \mid ||h||_{\mathcal{B}} < \infty\}. \tag{2.32}$$

We used the term 'space'. In fact, we have

Proposition 2.19
The space $(\mathcal{B}, ||.||_{\mathcal{B}})$ *is a normed linear space, and it is* $\mathcal{B}_0 \subset \mathcal{B}$.

Proof:
Let $f = \sum a_k z^k$, $g = \sum b_k z^k$ be in $\mathcal{B}$ and $\lambda \in \mathbb{C}$.

- $f = 0 \Leftrightarrow a_k = 0 \forall k \Leftrightarrow ||f||_{\mathcal{B}} = 0$
- $||f+g||_{\mathcal{B}} = \sum_{k=0}^{\infty} |a_k + b_k| \frac{k!}{(2\pi)^k} \leq \sum_{k=0}^{\infty} |a_k| \frac{k!}{(2\pi)^k} + \sum_{k=0}^{\infty} |b_k| \frac{k!}{(2\pi)^k}$
- $||\lambda f|| = \sum_{k=0}^{\infty} |\lambda a_k| \frac{k!}{(2\pi)^k} = |\lambda| \sum_{k=0}^{\infty} |a_k| \frac{k!}{(2\pi)^k}$

This shows that $\mathcal{B}$ is a normed linear space. Now, let $h(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}_0$, then, by (2.28) we have

$$\limsup_{k \to \infty} \left(\sqrt[k]{|a_k| \frac{k!}{(2\pi)^k}} \right) < 1,$$

and this implies that the series in $||h||_{\mathcal{B}}$ converges by a convergence criterion for infinite series. □

Now we can extend the linear operator Δ to $\mathcal{B}$. If $h(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}$, then we have by Proposition 2.15 and any compact $U \subset \mathbb{C}$ and a constant C

$$||a_k s_k(z)||_{U,\infty} \leq \left|\left| a_k M \frac{k!}{(2\pi)^k} e^{2\pi|z|} \right|\right|_{U,\infty} \leq C |a_k| \frac{k!}{(2\pi)^k}$$

But by Weierstrass' majorant criterion, this implies that the series $\sum_{k=0}^{\infty} a_k s_k(z)$ converges uniformly on compact sets to an analytic function. Therefore, we have that

$$\Delta : (\mathcal{B}, ||\cdot||_{\mathcal{B}}) \to \mathcal{O}(\mathbb{C})$$

is a linear operator. We now show that this operator is continuous, where on $\mathcal{O}(\mathbb{C})$ the standard-topology of uniform convergence on compact subsets is assumed and $\mathcal{B}$ is normed with $||\cdot||_{\mathcal{B}}$.

Proposition 2.20
The linear operator

$$\Delta : (\mathcal{B}, ||\cdot||_{\mathcal{B}}) \to \mathcal{O}(\mathbb{C})$$

is a continuous linear operator from the space $\mathcal{B}$ to the space of holomorphic functions.

Proof:
Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $\mathcal{B}$ with $f_n(z) = \sum_{k=0}^{\infty} a_{n,k} z^k$, such that $f_n \to f \in \mathcal{B}$ with respect to $||\cdot||_{\mathcal{B}}$, where f has the power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$ This means

$$\sum_{k=0}^{\infty} |a_{k,n} - a_k| \frac{k!}{(2\pi)^k} \to 0$$

We have to show that $\Delta f_n \to \Delta f$ uniformly on compact sets in $\mathbb{C}$. Let $K \subset \mathbb{C}$ be any such set, and denote $||\cdot||_{K,\infty}$ the sup-norm on K. Then, using (2.29),

$$\begin{aligned} ||\Delta f - \Delta f_n||_K &\leq \sum_{k=0}^{\infty} |a_k - a_{n,k}| \cdot ||s_k||_K \\ &\leq \sum_{k=0}^{\infty} |a_k - a_{n,k}| M \frac{k!}{(2\pi)^k} ||e^{2\pi|z|}||_K \\ &\leq C \sum_{k=0}^{\infty} |a_k - a_{n,k}| \frac{k!}{(2\pi)^k} \to 0 \end{aligned}$$

with the constant $C = M||e^{2\pi|z|}||_K < \infty$, since K is compact. But this means that $\Delta f_n \to \Delta f$ on compact sets $K \subset \mathbb{C}$. □

Now we are able to show that the solution of the additive recursion for $f \in \mathcal{B}$ is indeed unique, if we demand the solution of polynomials to be polynomials and if we demand the recursion to be continuous (with respect to the norm introduced above). We only have to show that the polynomials are dense in $\mathcal{B}$ with respect to the norm $||\cdot||_{\mathcal{B}}$, because if we establish this fact, then for any $f \in \mathcal{B}$ there is a sequence of polynomials $p_n \in \mathcal{B}$ with $||f - p_n||_{\mathcal{B}} \to 0$ and, because of continuity it is necessary that

$$\Delta(f) = \Delta(\lim_{n\to\infty}(p_n)) = \lim_{n\to\infty} \Delta(p_n),$$

where the limit on the right hand means uniform convergence on compact subsets and the one on the left hand means convergence with respect to $||\cdot||_{\mathcal{B}}$, which means that $\Delta : \mathcal{B} \to \mathcal{O}(\mathbb{C})$ is continuous.
But the fact that polynomials are dense in $\mathcal{B}$ is clear, since for $f(z) = \sum_{k=0}^{\infty} a_k z^k$, the sequence $(p_n)_{n=1}^{\infty}$ with

$$p_n(z) = \sum_{k=0}^{n} a_k z^k$$

converges to f in the norm $||\cdot||_{\mathcal{B}}$:

$$||f - p_n||_{\mathcal{B}} = \sum_{k=n+1}^{\infty} |a_k| \frac{k!}{(2\pi)^k} \to 0,$$

which holds because of the convergence of the series $\sum_{k=0}^{\infty} |a_k| \frac{k!}{(2\pi)^k}$.
Let us summarize our results

Theorem 2.21
Let $\mathcal{B}$ be the Bernoulli Space as defined above. Then, for any function $h \in \mathcal{B}$, the Bernoulli Transform Δh is a well-defined analytic function on $\mathbb{C}$ which satisfies the recursion

$$\Delta h(z+1) = \Delta h(z) + h(z), \qquad \Delta h(0) = 0.$$

If we demand that the solution of this recursion of a polynomial is again a polynomial and that the linear operator mapping a function to its solution is continuous with respect to the norm on $\mathcal{B}$, then the solution of this recursion is unique for all $h \in \mathcal{B}$.

Before closing this section, we will examine some analytic properties of the space $\mathcal{B}$. If one has a normed space, then the question arises whether this space is complete with respect to the norm. The space is complete, if any Cauchy sequence of elements of $\mathcal{B}$ has a limit that is again in $\mathcal{B}$. Such complete normed spaces are called Banach spaces. We now show that $\mathcal{B}$ is a Banach space.

Proposition 2.22
The Bernoulli Space $\mathcal{B}$ with respect to the norm $||.||_{\mathcal{B}}$ is a Banach space.

Proof:
Let $(f_n(z) = \sum a_{n,k} z^k)_{n=0}^{\infty}$ be any Cauchy sequence in $\mathcal{B}$. This means that for all $\epsilon > 0$, there is a n_0, such that for all $n, m \geq n_0$

$$\sum_{k=0}^{\infty} |a_{n,k} - a_{m,k}| \frac{k!}{(2\pi)^k} < \epsilon$$

It follows that on compact sets $K \subset \mathbb{C}$

$$\sup_{z \in K} |f_n(z) - f_m(z)| \leq \sup_{z \in K} \sum_{k=0}^{\infty} |a_{k,n} - a_{k,m}| |z|^k \to 0 \quad (m, n \to \infty)$$

which implies that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the topology of $\mathcal{O}(\mathbb{C})$, and because of the completeness of this space, there is a function $f \in \mathcal{O}(\mathbb{C})$ with $f_n \to f$ uniformly on compact sets. This implies, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, that $a_k = \lim_{n \to \infty} a_{n,k}$ because uniform convergence of holomorphic functions implies uniform convergence of all derivatives, and $a_k k! = f^{(k)}(0)$. Now, we show that $f_n \to f$ in the norm $|| \cdot ||_{\mathcal{B}}$, which in particular implies that $f \in \mathcal{B}$.
By assumption that $(f_n)_{n \in \mathbb{N}}$ is a $|| \cdot ||_{\mathcal{B}}$-Cauchy sequence, we have for all N

$$\sum_{k=0}^{N} |a_{n,k} - a_{m,k}| \frac{k!}{(2\pi)^k} \leq \sum_{k=0}^{\infty} |a_{n,k} - a_{m,k}| \frac{k!}{(2\pi)^k} < \epsilon$$

and, if we let m tend to ∞

$$\sum_{k=0}^{N} |a_{n,k} - a_k| \frac{k!}{(2\pi)^k} \leq \epsilon$$

Now let $N \to \infty$ and obtain

$$\sum_{k=0}^{\infty} |a_{n,k} - a_k| \frac{k!}{(2\pi)^k} \leq \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily, this shows that $||f_n - f||_{\mathcal{B}} \to 0$, which completes the proof. □

Next, we will consider some functions which are not in $\mathcal{B}$.

2.4.8 The recursion $f(x+1) = f(x) + \frac{1}{x}$

The recursion

$$f(x+1) = f(x) + \frac{1}{x} \tag{2.33}$$

cannot be solved by the Bernoulli Transform, since $\frac{1}{x}$ is not in $\mathcal{B}$. But the Γ-function leads to a solution of this recursion. If we derive $\Gamma(x+1) = x\Gamma(x)$, we obtain

$$\Gamma'(x+1) = \Gamma(x) + x\Gamma'(x)$$

and thus

$$\begin{aligned} \left(\frac{\Gamma'}{\Gamma}\right)(x+1) &= \frac{\Gamma(x) + x\Gamma'(x)}{x\Gamma(x)} \\ &= \left(\frac{\Gamma'}{\Gamma}\right)(x) + \frac{1}{x} \end{aligned}$$

Therefore the function $T_1(x) := \left(\frac{\Gamma'}{\Gamma}\right)$ solves (2.33). Consider the derivative on both sides of

$$T_1(x+1) = T_1(x) + \frac{1}{x}$$

which leads to

$$T_1^{(k)}(x+1) = T_1^{(k)}(x) + \frac{(-1)^k k!}{x^{k+1}}.$$

Thus, if we define

$$T_k(x) := \frac{T_1^{(k-1)}(x)}{(-1)^{k-1}(k-1)!},$$

we obtain

$$T_k(x+1) = T_k(x) + \frac{1}{x^k} \tag{2.34}$$

These considerations lead to a general result.

2.4.9 General rational solution

The results of the last paragraph and the Bernoulli Transform lead to a general solution of

$$f(x+1) = f(x) + R(x) \tag{2.35}$$

where $R(x)$ is a rational function.

Theorem 2.23
Let $R(x)$ be a rational function which has a partial fraction expansion

$$R(x) = p(x) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} \frac{a_{ik}}{(x-\alpha_i)^k}, \tag{2.36}$$

where $p(x)$ is a polynomial and $\alpha_1, \cdots, \alpha_n$ are the zeros of the denominator-polynomial of $R(x)$. Let

$$f(x) = \Delta p(x) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} a_{ik} T_k(x-\alpha_i) \tag{2.37}$$

where Δp is the Bernoulli Transform of the polynomial p and the T_k are defined as in the preceding paragraph. Then f is analytic except at isolated singularities and satisfies

$$f(x+1) = f(x) + R(x)$$

Proof:
By former results, it is clear that f is analytic. It satisfies the recursion, since

$$\begin{aligned}
f(x+1) &= \Delta p(x+1) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} (a_{ik}T_k(x-\alpha_i+1)) \\
&= \Delta p(x) + p(x) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} \left(a_{ik}T_k(x-\alpha_i) + \frac{a_{ik}}{(x-\alpha_i)^k}\right) \\
&= \Delta p(x) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} (a_{ik}T_k(x-\alpha_i)) + p(x) + \sum_{i=1}^{n}\sum_{k=1}^{n_i} \frac{a_{ik}}{(x-\alpha_i)^k} \\
&= f(x) + R(x)
\end{aligned}$$

□

Remark 2.24
Concerning uniqueness of our solution of (2.35), consider the following. Take the uniqueness conditions of 2.4.6. If we can give a requirement for uniqueness of solutions of (2.34), then, because of (i) and the linearity (ii), the combination of the criteria gives unique solutions of (2.35).

(iv) The recursion should commute with differentiation, i. e. if $f(z+1) = f(z) + h(z)$, has the unique solution f, then $f'(z+1) = f'(z) + h'(z)$ has the unique solution f'. It is clear that f' is indeed a solution of the second recursion.

(v) The solution of $f(z+1) = f(z) + \frac{1}{z}$ should be $T_1(z)$.

The second of these criteria is plausible if you accept that the Γ-function is the unique natural solution of $f(z+1) = zf(z)$, $f(1) = 1$, and that the recursions should be consistent with taking the logarithm, i. e. $\log f(z+1) = \log(z) + \log f(z)$ has the unique solution $\log\Gamma(z)$. Differentiating the last recursive equation gives the recursion for $T_1(z)$.

2.4.10 Periodic Functions

The following consideration will lead to the solution for periodic functions with period $\omega \notin \mathbb{Q}$. Let G be a region which is of the form $G = \{z \in \mathbb{C} | a < \mathrm{Im} z < b\}$ with $\mathbb{R} \subset G$, and let $h : G \to \mathbb{C}$ be a holomorphic periodic function on G with $h(z + \omega) = h(z)$ for a $\omega \in \mathbb{R}$, $\omega \notin \mathbb{Q}$. Then it is a known theorem (cf. [9]) that h is of the form

$$h(z) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i}{\omega} nz}, \tag{2.38}$$

where the sum converges normally to h. To solve the recursion $f(z + 1) = f(z) + h(z)$, define

$$F_n(z) = \frac{e^{\frac{2\pi i}{\omega} nz} - 1}{e^{\frac{2\pi i}{\omega} n} - 1}$$

and obtain

$$\begin{aligned} F_n(z+1) &= \frac{e^{\frac{2\pi i}{\omega} n(z+1)} - 1}{e^{\frac{2\pi i}{\omega} n} - 1} \\ &= \frac{e^{\frac{2\pi i}{\omega} nz} e^{\frac{2\pi i}{\omega} n} - e^{\frac{2\pi i}{\omega} nz} - 1 + e^{\frac{2\pi i}{\omega} nz}}{e^{\frac{2\pi i}{\omega} n} - 1} \\ &= \frac{(e^{\frac{2\pi i}{\omega} n} - 1) e^{\frac{2\pi i}{\omega} nz} + e^{\frac{2\pi i}{\omega} nz} - 1}{e^{\frac{2\pi i}{\omega} n} - 1} \\ &= F_n(z) + e^{\frac{2\pi i}{\omega} nz}. \end{aligned}$$

Since the series in (2.38) converges normally, the function definied by

$$f(z) := \sum_{n=-\infty}^{\infty} c_n F_n(z) \tag{2.39}$$

also converges normally. Now, it is clear that

$$\begin{aligned} f(z+1) &= \sum_{n=-\infty}^{\infty} c_n F_n(z+1) \\ &= \sum_{n=-\infty}^{\infty} c_n (F_n(z) + e^{\frac{2\pi i}{\omega} nz}) \\ &= f(z) + h(z) \end{aligned}$$

Thus we have proven

Theorem 2.25
Let $G = \{z \in \mathbb{C} | a < \mathrm{Im} z < b\}$ with $\mathbb{R} \subset G$, and let $h : G \to \mathbb{C}$ be a holomorphic periodic function on G with $h(z + \omega) = h(z)$ for an $\omega \in \mathbb{R}$, $\omega \notin \mathbb{Q}$. The fourier expansion of h is

$$h(z) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i}{\omega} nz}.$$

Then the function f defined by

$$f(z) := \sum_{n=-\infty}^{\infty} c_n F_n(z), \quad \text{where } F_n(z) = \frac{e^{\frac{2\pi i}{\omega} nz} - 1}{e^{\frac{2\pi i}{\omega} n} - 1} \tag{2.40}$$

solves the additive recursion $f(z+1) = f(z) + h(z)$.

Chapter 3

Some nonlinear recursions

So far, we only have dealt with linear recursions. Now we will consider some nonlinear cases, especially those which are reducible to the linear case. But since the main theme of this work are linear recursions, we will only sketch some basic ideas and will not go further into details.

3.1 Powers of functions reduced to linear cases

Consider

$$f(x+1) = q(x) \cdot f(x)^{p(x)} \tag{3.1}$$

Let f_1 be a solution of $f_1(x+1) = p(x)f_1(x) + q(x)$, and put $f(x) = e^{f_1(x)}$. This gives

$$\begin{aligned} f(x+1) &= e^{f_1(x+1)} \\ &= e^{p(x)f_1(x)+q(x)} \\ &= e^{q(x)}\left(e^{f_1(x)}\right)^{p(x)} \\ &= e^{q(x)} \cdot f(x)^{p(x)} \end{aligned}$$

Therefore we can solve (3.1) if we can generally solve (2.1). If we set $\tilde{q}(x) = \log(q(x))$ we get from the solution of $\tilde{f}(x+1) = p(x)\tilde{f}(x) + \tilde{q}(x)$ by taking the exponential function a solution of $f(x+1) = q(x) \cdot f(x)^{p(x)}$.
Of course we have to be careful about defining $f(x)^{p(x)}$, since over the complex numbers this is not generally possible. Furthermore, we have to consider that it can pose problems to define $\log(p(x))$. So these equations above are to be understood for those x, for which this is possible and the common power laws hold.

3.2 Nonlinear recursions solved by infinite products

Let f be an analytic function such that the product

$$F(x) := \prod_{k=0}^{\infty} \left(f(x+k)\right)^{n^{-k}}, \qquad n > 1 \tag{3.2}$$

converges absolutely and uniformly on compact sets and defines therefore an analytic function. For the definition of absolute convergence of products cf. [6] or [11]. Then F satisfies

$$\begin{aligned} F(x+1) &= \prod_{k=0}^{\infty}\left(f(x+k+1)\right)^{n^{-k}} \\ &= \prod_{k=1}^{\infty}\left(f(x+k)\right)^{n^{-k+1}} \\ &= \left(\prod_{k=1}^{\infty}\left(f(x+k)\right)^{n^{-k}}\right)^{n} \\ &= \left(\frac{1}{f(x)}\prod_{k=0}^{\infty}\left(f(x+k)\right)^{n^{-k}}\right)^{n} \\ &= \left(\frac{F(x)}{f(x)}\right)^{n} \end{aligned}$$

Now, the question arises for which functions (3.2) converges uniformly on compact subsets. A sufficient condition for the uniform convergence of a product $\prod_{k=0}^{\infty} g_k$ on compact sets is the absolute convergence of $\sum_{k=k_0}^{\infty}\log(g_k)$ for some k_0 which may be dependent on the compact set [6]. With this tool we can examine some interesting classes of functions.

1. Monomials of the form $f(x)=(x-a)^j$, $j\in\mathbb{N}$. We want to show that

   $$\sum_{k=k_0}^{\infty}\log\left(((x-a)^j)^{n^{-k}}\right)$$

   converges absolutely and uniformly on compact sets. Since

   $$\begin{aligned} \log\left(((x-a+k)^j)^{n^{-k}}\right) &= n^{-k}\cdot j\cdot\log(x-a+k) \\ &= n^{-k}\cdot j\cdot\log\left(k\left(1+\frac{x-a}{k}\right)\right) \\ &= n^{-k}\cdot j\cdot\left(\log(k)+\log\left(1+\frac{x-a}{k}\right)\right) \\ &=: h_k(x), \end{aligned}$$

   and since it is clear that the sum $\sum h_k(x)$ converges uniformly on compact sets, the product (3.2) converges for these functions f. Note that the functional equation $\log(ab)=\log(a)+\log(b)$ applies to k and $1+\frac{x-a}{k}$, since for some sufficiently large k, the real parts of these numbers are positive. (In general, this functional equation is not valid for complex numbers, cf. [9].)

2. Rationals of the form $\frac{1}{(x-a)^j}$, $j\in\mathbb{N}$. The same computation as above shows that the product (3.2) converges for $f(x)=\frac{1}{(x-a)^j}$. Just replace $-j$ for j in the computation for the monomials.

3. Rational functions $r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials. Since the product converges for the former two types of functions, it is clear that it does the same for rational functions, since we simply can factorize:

$$r(x) = \frac{\prod_{j=1}^{l}(x - a_j)^{\mu_j}}{\prod_{j=1}^{m}(x - b_j)^{\nu_j}}.$$

4. Generalization of the first two types: $(x - a)^t$, $t \in \mathbb{R}$. We have to be careful with the definition of the general power. If there is a branch of the logarithm on a suitable region, one defines $(x - a)^t = \exp(\log(x - a) \cdot t)$. To show the convergence of the product (3.2) for these functions we want to make the same computation as in 1. But, as it is with the functional equation of the logarithm, the equation $\log(\exp(z)) = z$ does not in general hold. Instead, we have $\log(\exp(z)) = z - 2\pi i n$, where n is dependent on the section $(2n - 1)\pi < \mathrm{Im} z < (2n + 1)\pi$. But, similar as above, the imaginary part of $t\log(1+\frac{x-1}{k})$ remains in the same section if t is constant and k is large enough. In fact, we can achieve that $-\pi < \mathrm{Im}\left(t\log(1 + \frac{x-1}{k})\right) < \pi$ for all $k > k_0$ for some k_0. Therefore, for large k,

$$\begin{aligned}\log\left(((x-a+k)^t)^{n^{-k}}\right) &= n^{-k} \cdot \log\left(\exp\left(t\left(\log(k) + \log\left(1 + \frac{x-a}{k}\right)\right)\right)\right) \\ &= n^{-k} \cdot \left(t\left(\log(k) + \log\left(1 + \frac{x-a}{k}\right)\right)\right)\end{aligned}$$

With the same argument as before we see the convergence of (3.2).

The last class of functions and products and quotients of these show that (3.2) can be applied to roots of rational functions.

Theorem 3.1
Let $r(x)$ be a rational function. Put

$$s(x) := \frac{1}{r(x)^{\frac{1}{n}}} \tag{3.3}$$

and define

$$F(x) := \prod_{k=0}^{\infty} s(x)^{n^{-k}}. \tag{3.4}$$

Then the product defines an analytic function which satisfies

$$F(x+1) = r(x)F(x)^n. \tag{3.5}$$

Proof:
The convergence of the product has already been shown. By the results above, it follows that

$$F(x+1) = \left(\frac{F(x)}{s(x)}\right)^n = r(x)F(x)^n$$

□

With this theorem we conclude our investigation of nonlinear recursions. We remark that there are a lot more functions for which (3.2) converges, and this has to be examined further. There are lots of interesting nonlinear cases, especially the recursion

$$f(z+1) = p(f(z)),$$

where p is a polynomial. For some cases we have found solutions, but it is an open question whether there exist solutions for all polynomials p and how such solutions can be defined.

Chapter 4

Further work

This work has dealt mainly with linear primitive recursions. We gave solutions for a large class of functions including the rational functions. We further gave uniqueness criteria for the multiplicative and additive recursions and showed that our solutions fulfill these criteria.
We did not solve the linear primitive recursion for *all* analytic functions, it is still open whether there are solutions for all functions. This has to be further investigated. The nonlinear cases have to be examined more closely. For applications, recursions in several variables is important, and the theory of primitive recursive analytic functions has to be extended in this direction.
We want to remark that continuing the concept of primitive recursion to $\mathbb{C}$ by functional equations is not the only way to define functional specifications of analytic continuations. In the following last section, we briefly give the idea of an approach to the functional specification of analytic continuations of recursive functions, which differs from the one given by recursive equations. We have not examined this approach very closely, we just want to give the idea. Future work will show whether this approach is sound.
In recursion theory over the natural numbers, one has some initial functions, as the functions given by $I_{n,k}(x_1, \ldots, x_n) = x_k$, $O_n(x_1, \ldots, x_n) = 0$, and the successor function $S(x) = x + 1$. The scheme of primitive recursion is given by

$$F(X,0) = g(X),\ F(X,Sy) = h(X,y,F(X,y)), \qquad X = (x_1, \ldots x_n),\ n \geq 0.$$

In addition, we have the scheme of μ recursion, we can define the function F by

$$F(X) = \mu y\{g(X,y) = 0\},$$

where the μ-operator refers to the smallest y such that $g(X,y) = 0$. The class of general recursive functions, or μ recursive functions is obtained by the initial functions and repeated substitution and recursion.
Now, Julia Robinson [10] has shown that one can obtain *all* general recursive functions by taking the initial functions, adjoining a certain function E, the function $(x,y) \mapsto x + y$, inversion $A^{-1}x = \mu y\{Ax = y\}$ and substitution. This means that the class of general recursive function is the smallest class of function closed under all these operations and which contains the initial functions, E, and $x + y$. In this approach, the μ operator is replaced by inversion. If the function A is not injective, the smallest inverse is taken. One can prove that it

is possible to replace the function A by a function B, such that the equation $Bx = y$ has a unique solution, cf. [10]. The function E referred to above is given by $Ex = x - [\sqrt{x}]^2$.
We want to apply these ideas to specify analytic continuations of general recursive functions. The idea behind this is that inversion of analytic functions should be computable with analytic machines. Locally one can give the inverse of an analytic function by an integral formula. Another possibility is to locally invert the power series of the function. So the inverse of an analytic function is, in a certain sense, locally computable. Generally, the inverse of an analytic function is not a function on $\mathbb{C}$ or a subset, but it is given by a Riemann surface. To give a functional specification of analytic recursive functions, define an analytic function E which interpolates the function E on the naturals. That such a function exists follows from the interpolation theorem (cf. 1.1). Now take the initial functions, the analytic function E and $x + y$. A general recursive function over the naturals can be obtained by repeated application of the schemes inversion and substition. The analytic continuation (which is then defined on a Riemann surface) is given by the same repeated application.

Acknowledgement

The authors want to thank Timo von Oertzen, who has contributed to this the paper with many helpful remarks and who has supported the authors in many respects.

Bibliography

[1] Lenore Blum, Mike Shub, Steve Smale. On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines. Bulletin (New Series) of the American Mathematical Society, 21(1):1-46, July 1989. ISCI technical report TR-88-012.

[2] Lenore Blum, Felipe Cucker, Mike Shub, Steve Smale. Complexity and Real Computation. Springer-Verlag, New York, 1997.

[3] Constantin Caratheodory. Funktionentheorie. Birkhäuser, Basel, 1950.

[4] Thomas Chadzelek. Analytische Maschinen. Dissertation. Saarbrücken, 1998.

[5] Thomas Chadzelek, Günter Hotz. Analytic machines. Theoretical Computer Science, 219:151-167, 1999.

[6] Wolfgang Fischer, Ingo Lieb. Funktionentheorie. Vieweg.

[7] Günter Hotz. über Berechenbarkeit fraktaler Strukturen. Abhandlungen der mathematisch-naturwissenschaftlichen Klasse 1, Akademie der Wissenschaften und der Literatur, Mainz, 1994.

[8] Günter Hotz, Björn Schieffer, Gero Vierke. Analytic machines. Technical Report TR95-025, Electronic Colloquium on Computational Complexity, 1995.

[9] Reinhold Remmert. Funktionentheorie 1. Springer, 1992.

[10] Julia Robinson. General Recursive Functions. Proceedings of the AMS 1950, 703-718.

[11] Walter Rudin. Real and Complex Analysis. International Edition, McGraw-Hill, 1987.

[12] Klaus Weihrauch. Computability. Number 9 in EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1987.

[13] Klaus Weihrauch. A foundation of computable analysis. In: Ker-I Ko and Klaus Weihrauch, editors, Computability and Complexity in Analysis, number 190-9/1995 in Informatikberichte, D-58084 Hagen, 1995.

[14] Klaus Weihrauch. Computable Analysis. Springer, Berlin, 2000.

[15] E. Zeidler (Hg.). Teubner-Taschenbuch der Mathematik. Teubner, Leipzig, 1996.

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