On the Security of Modular Exponentiation

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by

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Abstract

Assuming the inractability of factoring, we show that the output of the exponentiation modulo a composite function $f_{N,g}(x) = g^x \bmod N$ (where $N = P \cdot Q$) is pseudorandom, even when its input is restricted to be half the size. This result is equivalent to the simultaneous hardness of the upper half of the bits of $f_{N,g}$, proven by Håstad, Schrift and Shamir. Yet, we supply a different proof that is significantly simpler than the original one. In addition, we suggest a pseudorandom generator which is more efficient than all previously known factoring based pseudorandom generators. Our work provides also an evidence for the difficulty of the Decisional Diffie-Hellman problem, when considered modulo a composite.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

One-way functions play an extremely important role in modern cryptography. Loosely speaking, these are functions which are easy to evaluate but hard to invert. A number theoretic function which is widely believed to be one-way, is the exponentiation function over a finite field. Its inverse, the discrete logarithm function, is the basis for numerous cryptographic applications. Most applications use a field of prime cardinality, though many of them can be adapted to work in other algebraic structures as well.

A concept tightly connected to one-way functions is the notion of *hard-core predicates*, introduced by Blum and Micali. A polynomial-time predicate b is called a hard-core of a function f, if all efficient algorithm, given $f(x)$, can guess $b(x)$ with success probability only negligibly better than half. Blum and Micali showed the importance of hard-core predicates in pseudorandom bit generation. Specifically, they showed that the modular exponentiation function over a field of prime cardinality, $f_{P,g}(x) = g^x \bmod P$, has a hard-core predicate, and used it in order to construct a pseudorandom bit generator. The study of hard-core predicates of $f_{P,g}$ has culminated in the work of Håstad and Näslund [HN], showing that all bits of $f_{P,g}$ are individually secure.

1.1.1 Hard core functions

The concept of a hard-core function (or the simultaneous security of bits) is a generalization of hard-core predicates. Intuitively, a group of bits associated to a one-way function f is said to be simultaneously secure, if no efficient algorithm can gain any information about the given group of bits in x, given only $f(x)$. Proving the simultaneous security of a group of bits in $f_{P,g}$ is a more desirable result, enabling the construction of more efficient pseudorandom generators as well as improving other applications. However, the best known result regarding the simultaneous security of bits in $f_{P,g}$ is due to Long and Wigderson [LW], Kalisky [Kal] and Peralta [P], who showed that $O(\log n)$ bits are simultaneously secure, where n is the size of the modulus P.

Better results were demonstrated when the modulus was taken to be a composite, thus allowing to relate hardness of bits to the factoring problem. Denote by $f_{N,g}$ the exponentiation modulo a composite function, defined as $f_{N,g}(x) = g^x \bmod N$, where N is an n-bit composite equal to the multiplication of two large primes and g is an element in the multiplicative group mod N. Håstad, Schrift and Shamir showed that under the factoring intractability assumption, all the bits in $f_{N,g}$ are individually hard, and that the upper $\lceil \frac{n}{2} \rceil$ bits and lower $\lceil \frac{n}{2} \rceil$ bits are simultaneously hard.

In the same setting (and under the same assumption that factoring is hard), we show that no efficient algorithm can tell apart $f_{N,g}(r)$ from $f_{N,g}(R)$, where r is a random $\lceil \frac{n}{2} \rceil$-bit string and R is a random string of full size. That is, one can work with an exponent x of half the size, and still obtain an element which "seems random" to all efficient algorithms. Note that all the cryptographic tools that use exponentiation in Z_N^* (and base their security on the discrete logarithm assumption) can greatly benefit from this fact, since the time consumed for exponentiation grows linearly with the size of the exponent (and is thus cut by a factor of two). Our result is in fact equivalent to the result of [HSS] on the simultaneous hardness of the upper $\lceil \frac{n}{2} \rceil$ bits of $f_{N,g}$. Nevertheless, we give an alternative proof for it while using some of their ideas and techniques. Our approach significantly simplifies the proof given in [HSS] and sheds a different light on it.

Our work has also two additional implications (to be further discussed below). The first one is the construction of a pseudorandom bit generator based on the computational indistinguishability of $f_{N,g}(r)$ from $f_{N,g}(R)$. Our generator is somewhat more efficient than all previously known factoring based pseudorandom generators. The second implication regards the *Decisional Diffie-Hellman Assumption*. We give an evidence that adds to our confidence in the above assumption, by showing a relation between the Decisional Diffie-Hellman problem when considered modulo a composite and the problem of factoring the modulus.

1.1.2 An efficient Pseudorandom generator

The notion of a pseudorandom bit generator, introduced by Blum and Micali [BM], plays a central role in cryptography. It enables the user to expand a short random seed into a longer sequence of bits, that can be used in any efficient application instead of a truly random bit sequence. Blum and Micali presented a pseudorandom bit generator based on the discrete log problem. Using the fact that the exponentiation function over a field of prime cardinality has a hard-core predicate, they suggested an iterative generator that yields one bit of output per each exponentiation. Furthermore, they conceived a general paradigm that constructs an iterative pseudorandom generator, given any length preserving one-way permutation f, and a hard-core predicate b for f.

The Blum-Blum-Shub pseudorandom generator [BBS](referred to as the "BBS generator"), is based on the above paradigm, taking f to be the modular squaring function, where the modulus N is a Blum integer.[1] Since, as shown by Rabin [Rab], the problem of factoring N can be reduced to the problem of extracting square roots in the multiplicative group mod N, the function f is a one-way function assuming the intractability of factoring Blum integers. Additionally, Blum, Blum and Shub showed that f induces a permutation over the set of quadratic residues in the multiplicative group mod N, and used the results of

[1] A Blum integer is equal to the multiplication of two primes of equal size, each congruent to 3 mod 4.

Alexi et.al. [ACGS] and Vazirani and Vazirani [VV], implying that the least significant bit constitutes a hard-core predicate for f. The BBS generator is by far more efficient than the Blum-Micali generator.[2] In particular, the BBS generator stretches an n-bit seed into a $2n$-bit pseudorandom string using $2n$ modular multiplications.

Another generator whose pseudorandomness is based on factoring, was suggested by Håstad, Schrift and Shamir [HSS] (we will refer to it as the "HSS generator"). The HSS generator relies on the simultaneous hardness of half of the bits in the exponentiation modulo a composite function $f_{N,g}$. Loosely speaking, the HSS generator takes an n-bit random seed x, (where n is the size of the modulus N) and outputs $f_{N,g}(x)$ followed by the lower half of the bits of x.[3] Observe that from an n-bit seed, the HSS generator obtains $1.5n$ bits of output, using n modular multiplications on the worst case, and $0.5n$ modular multiplications on the average case (we assume that the terms $g^{2^0}, \ldots, g^{2^n}$ are pre-computed together with the other parameters of the generator).

Even though our main result is equivalent to the simultaneous hardness of half of the bits in $f_{N,g}$, our result gives rise to a pseudorandom generator which is in a sense more natural than the HSS generator, as well as more efficient than it. Informally, we suggest a generator that takes a random seed x of size $\lceil n/2 \rceil$, and outputs $f_{N,g}(x)$. Observe that our generator doubles the length of its input. In particular, it obtains n bits of output from an $0.5n$-bit seed using $0.5n$ modular multiplications on the worst case, and $0.25n$ modular multiplications on the average case (once again, we assume that the terms $g^{2^0}, \ldots, g^{2^{\lceil n/2 \rceil}}$ are pre-computed).

The following table compares the three factoring based generators discussed above, each having the same security parameter n (the size of the modulus N). Note that the "cost" column refers to the average number of multiplications done in every application of the generator, and the "amortized cost" column refers to the average number of multiplications per output bit of the generator.[4]

	seed size	**output size**	**cost**	**amortized cost**
BBS construction	n	2n	2n	1
HSS construction	n	1.5n	0.5n	0.33
Our construction	0.5n	n	0.25n	0.25

An additional point is that our generator (as well as the HSS generator) has an efficient parallel implementation in time $O(\log n)$ using $\lceil n/2 \rceil$ processors $P_1, \ldots, P_{\lceil n/2 \rceil}$ (the input of

[2]The Blum-Micali generator obtains each bit of output at the cost of one modular exponentiation that is implemented by n modular multiplications, as opposed to one modular multiplication per output bit needed by the BBS generator.

[3]As a matter of fact, in order to achieve true pseudorandomness, universal hashing is applied. The formal construction will be presented in Chapter 4.

[4]Even though the correct way to compare the above generators is with respect to the same security parameter, one might consider a comparison with respect to the same seed length. In order to do that we must normalize the input/output sizes of our generator so that its seed length will be n. Thus, the output produced by our generator will be of length $2n$, the cost will be $0.5n$ and the amortized cost will be 0.25 multiplications per output bit. Note however, that the size of the security parameter in our construction will be twice its size in the BBS and the HSS constructions. Thus, our construction will be safer. On the other hand, we will need to work harder in order to produce the parameters N and g (but this is not too bad since these parameters are only obtained once).

each processor P_i is the i'th bit of the seed, s_i, and the output is the multiplication of the values $g^{2^{i-1} \cdot s_i}$ contributed by each processor). This is opposed to the BBS generator which is not known to have a fast parallel implementation (i.e. any faster than the straightforward sequential implementation).

1.1.3 The Decisional Diffie-Hellman Assumption

Let us try to formalize the exact complexity assumption used by the following protocol, called the Diffie-Hellman key exchange protocol [DH]: Alice and Bob fix a group G and a generator g. They respectively pick random a, b in $[1, |G|]$ and exchange g^a and g^b over a public channel. The secret key they now share is g^{ab}.

This protocol is totally breakable, if a passive eavesdropper that has seen the communication between Alice and Bob, can compute the secret key. An assumption which must hold, therefore, is the *Computational Diffie-Hellman Assumption* (CDH), stating that no efficient algorithm can compute g^{ab} given g, g^a and g^b. However, we need more than CDH in order to deem the protocol secure: A protocol is considered secure, if no efficient adversary can tell apart the secret key from a random value in G. This guarantees that no partial information about the secret key leaks to a computationally bounded eavesdropper. The assumption which formalizes this security requirement, is called the *Distributional Diffie-Hellman Assumption* (DDH), which states that no efficient algorithm can tell apart the distributions $\langle g, g^a, g^b, g^{ab} \rangle$ and $\langle g, g^a, g^b, g^R \rangle$, where R is uniformly distributed in $[1, |G|]$. An equivalent form is the *Decisional Diffie-Hellman Assumption*, stating that given $\langle g, g^a, g^b, y \rangle$, no efficient algorithm can tell whether $y = g^{ab}$ (see [NR] for the proof of equivalence).

The DDH assumption enables one to construct efficient cryptographic systems with strong security properties. One example for such application is an efficient public-key cryptosystem of Cramer and Shoup [CS], which was shown to be secure against adaptive chosen ciphertext attack.[5]

Although the Decisional Diffie-Hellman assumption appears to be a very strong assumption, the best known method for breaking it is by computing discrete log. Some of the evidence that adds to our confidence in the DDH assumption is surveyed by Boneh [B]. For instance, such evidence was given by Boneh and Venkatesan [BV], who showed that computing the $k \approx \sqrt{\log P}$ most significant bits of the Diffie-Hellman secret (over a cyclic group of prime order P) is as hard to compute as the entire secret.

Our work strengthens in a way the work of [BV] for the case of a composite modulus. However, we refer to the decision problem and not to the (seemingly harder) computational problem treated by [BV]. Specifically, we consider a hybrid between the two DDH distributions defined above, taken to be the distribution $\langle g, g^a, g^b, g^c \rangle$, where the exponent c constitutes of random bits in its upper half, and equals in its lower half to the lower half of ab (modulo the order of g). We show that under the assumption that factoring Blum integers is intractable, the above hybrid is computationally indistinguishable from $\langle g, g^a, g^b, g^R \rangle$, where R is random in $[1, |G|]$. At this point we confess that the original goal of this thesis was to show that the above hybrid is computationally indistinguishable from $\langle g, g^a, g^b, g^{ab} \rangle$ as

[5] Another example is the Naor and Reingold construction of a collection of efficient pseudorandom functions [NR], recently superseded by [NRR].

well, and thus show that the factoring assumption implies the DDH assumption modulo a composite. Unfortunately, we were only “half successful”.

Organization: The rest of this work is organized as follows: Basic definitions and notations are given in Section 1.2. In Chapter 2 the main theorem is proven (regarding the pseudorandomness of exponentiation with a short exponent), and is shown to be equivalent to the [HSS] result (additionally, both proofs are compared). Chapter 3 exhibits further results which are obtained using similar techniques. Among them, is the result regarding the DDH assumption modulo a composite. In Chapter 4 we present our construction of a pseudorandom generator versus the HSS construction.

1.2 Preliminaries

Statistical Difference: A basic notion from probability theory is the *statistical difference* between probability ensembles $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$. The statistical difference measures the distance between distributions and is defined to be

$$SD(X_n, Y_n) = \frac{1}{2} \cdot \sum_{\alpha} |\Pr[X_n = \alpha] - \Pr[Y_n = \alpha]|$$

Probability ensembles $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are called **statistically close** if their statistical difference is *negligible* in n (a function $\mu : \mathbb{N} \to [0,1]$ is called **negligible** if for every positive constant c and all sufficiently large n's, $\mu(n) < \frac{1}{n^c}$).

Computational Indistinguishability: A weaker notion of closeness between probability ensembles is the notion of indistinguishability by all efficient algorithms. When no efficient algorithm (that may be probabilistic) can tell apart the two ensembles, we call them computationally indistinguishable. Formally,

Definition 1 *We say that two ensembles $\{X_n\}_{n\in N}$ and $\{Y_n\}_{n\in N}$ are* **computationally indistinguishable**, *if for every probabilistic polynomial-time algorithm D, for every positive constant c and for all sufficiently large n's*

$$|\Pr[D(X_n, 1^n) = 1] - \Pr[D(Y_n, 1^n) = 1]| < \frac{1}{n^c}$$

A notation: Let A be a finite set, then $a \in_R A$ denotes that the element a is uniformly chosen from the set A (i.e. with probability $\frac{1}{|A|}$).

1.2.1 Pseudorandom Generators

Loosely speaking, a pseudorandom generator is a deterministic algorithm that stretches a random *seed* (i.e. input) into a longer bit sequence which is "pseudorandom". A pseudorandom bit sequence is defined as computationally indistinguishable from the uniform distribution (thus for all practical purposes we can use the output of the generator instead of a truly random string).

Definition 2 *A* **pseudorandom generator** *is a deterministic polynomial-time algorithm, G, satisfying the following two conditions:*

1. *There exists a function $l(n) : n \to n$ satisfying that $l(n) > n$ for all $n \in N$, such that $|G(s)| = l(|s|)$ for all $s \in \{0,1\}^*$.*

2. *The ensembles $\{G(U_n)\}_{n\in N}$ and $\{U_l(n)\}_{n\in N}$ are computationally indistinguishable.*

1.2.2 The Factoring Assumption

We denote by N_n the set of all n-bit integers $N = P \cdot Q$, where P and Q are two odd primes of equal size. The collection N_n can be sampled efficiently. Specifically, given input 1^n, it is possible to pick a random element in N_n in polynomial time (using a polynomial number of coin tosses).

The problem of factoring integers is widely believed to be intractable. Integers belonging to the set N_n are considered to be particularly hard to factor. Note that N_n is a non-negligible fraction of all n-bit integers. Currently, the best algorithm known can factor an integer picked randomly from N_n in (heuristic) running-time of $e^{1.92 n^{1/3} \log n^{2/3}}$.

Assumption 1 *[Factoring Assumption] Let A be a probabilistic polynomial-time algorithm. There is no constant $c > 0$ such that for all sufficiently large n's*

$$Pr\left[A(P \cdot Q) = P\right] > \frac{1}{n^c}$$

where $N = P \cdot Q$ is picked uniformly from N_n.

1.2.3 The group Z_N^*

Denote by Z_N^* the multiplicative group that consists of all the naturals which are smaller than N and are relatively prime to it. We represent the elements in Z_N^* by binary strings of size $n = \lceil \log N \rceil$.

Notations:

- Let x be an element in Z_N^*, and let $1 \leq j \leq i \leq n$. We denote by x_i the i'th bit in the binary representation of x, and by $x_{i,j}$ the substring of x including the bits from position j to position i.
- Denote by $ord_N(g)$ the order of an element g in Z_N^*, which is the minimal $k > 1$ for which $g^k = 1 \pmod{N}$.
- Denote by $\langle g \rangle$ the subgroup of Z_N^* generated by g. That is, $\langle g \rangle$ is the set of all elements of the form $g^x \bmod N$ for some $x < N$.
- Denote by P_n the set of pairs $\langle N, g \rangle$ where $N \in N_n$ and $g \in Z_N^*$. Note that P_n is efficiently samplable.

We now define the *exponentiation modulo a composite* function and its inverse the *discrete logarithm modulo a composite* function.

Definition 3 *Let $\langle N, g \rangle$ be a pair in P_n. The exponentiation modulo a composite function $f_{N,g} : \{0,1\}^* \rightarrow \langle g \rangle$ is defined to be $f_{N,g}(x) = g^x \bmod N$.*

Definition 4 *Let $\langle N, g \rangle$ be a pair in P_n. The discrete logarithm modulo a composite function $DL_{N,g}(Y) : \langle g \rangle \rightarrow [0, ord_N(g))$ is defined to be the unique natural $x < ord_N(g)$ for which $f_{N,g}(x) = Y$.*

Chapter 2

Exponentiation with a short exponent is pseudorandom

We introduce two probability ensembles, which we show to be computationally indistinguishable assuming the intractability of factoring.

Definition 5 *Let $\langle N, g\rangle$ be a uniformly distributed pair in P_n, let R be uniformly distributed in $[0, ord_N(g))$ and let r be uniformly distributed in $\{0,1\}^{\lceil \frac{n}{2} \rceil}$. We denote by $Full_n$ the distribution $\langle N, g, g^R \bmod N\rangle$ and by $Half_n$ the distribution $\langle N, g, g^r \bmod N\rangle$.*

Theorem 1 *The ensembles $\{Half_n\}_{n\in N}$ and $\{Full_n\}_{n\in N}$ are computationally indistinguishable.*

We use the hybrid technique in order to prove the indistinguishability of $Full_n$ and $Half_n$. For i's between $\lceil \frac{n}{2} \rceil$ and $n+\omega(\log n)$ we define a hybrid distribution H_n^i in the following way: H_n^i will consist of triplets of the form $\langle N, g, g^x \bmod N\rangle$, where $\langle N, g\rangle$ is uniformly distributed in P_n and x is uniformly distributed in $\{0,1\}^i$ (see Figure 2.1).

For a specific choice of a pair $\langle N, g\rangle$ in P_n we denote by $H_{N,g}^i$ the distribution $g^x \bmod N$ where x is uniformly distributed in $\{0,1\}^i$. (From now on we omit the expression "mod N" whenever it is clear from the context).

Clearly, $H_n^{\lceil n/2 \rceil} = Half_n$. Note that the distribution $H_n^{n+\omega(\log n)}$ is statistically close to $Full_n$, as asserted by the following claim:

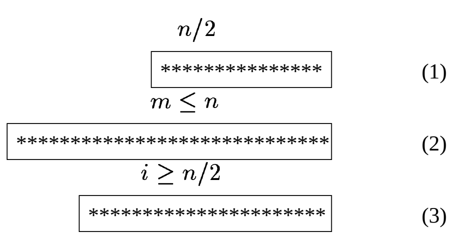

Figure 2.1: We denote random bits by '*' and the length of the binary expansion of $ord_N(g)$ by m. No. (1), (2), (3) show the exponents of $Half_n$, $Full_n$ and the hybrid H_n^i, respectively.

Claim 1.1 *The distributions $Full_n$ and $H_n^{n+\omega(\log n)}$ are statistically close.*

Proof: Let M denote $2^{n+\omega(\log n)}$. M can be written as $k \cdot ord_N(g) + r$ where k is an integer and $0 \leq r < ord_N(g)$. We now calculate the statistical difference between the distributions $Full_n$ and $H_n^{n+\omega(\log n)}$. Note that the first equality is implied from the fact that in $f_{N,g}(x)$ the exponent x is reduced modulo $ord_N(g)$.

$$\begin{aligned}
SD(Full_n, H_n^{n+\omega(\log n)}) &= \frac{1}{2}\left[r \cdot \left(\frac{k+1}{M} - \frac{1}{ord_N(g)}\right) + (ord_N(g) - r) \cdot \left(\frac{1}{ord_n(g)} - \frac{k}{M}\right)\right] \\
&= \frac{1}{2}\left[(ord_N(g) - 2r) \cdot \left(\frac{1}{ord_n(g)} - \frac{k}{M}\right) + \frac{r}{M}\right] \\
&= \frac{1}{2}\left[(ord_N(g) - 2r) \cdot \frac{r}{M \cdot ord_N(g)} + \frac{r}{M}\right] \\
&\leq \frac{r}{M}
\end{aligned}$$

Since $\frac{r}{M} < \frac{N}{M} \leq \frac{2^n}{2^{n+\omega(\log n)}}$, we have that $SD(Full_n, H_n^{n+\omega(\log n)})$ is negligible in n. □

Consequently, if there exists a probabilistic polynomial-time algorithm D, that distinguishes the ensemble $Half_n$ from $Full_n$, then D distinguishes (almost) as well $Half_n$ from $H_n^{n+\omega(\log n)}$. As the total number of hybrids is polynomial in n, a non-negligible gap between the extreme hybrids translates into a non-negligible gap between a pair of neighboring hybrids. Taking advantage of the structure of two neighboring hybrids, we use the distinguisher D in order to factor a composite in N_n, and thus contradict Assumption 1. In the following, let n be a sufficiently large natural and let i belong to the set $\{\lceil\frac{n}{2}\rceil,\ldots,n+\omega(\log n)\}$.

Lemma 2 (Main Lemma) *Suppose that the gap between the acceptance probability of D on the hybrids H_n^i and H_n^{i+1} is greater than $\frac{1}{n^c}$. Then, with probability at least $\frac{1}{8n^c}$ we can factor a composite N, uniformly distributed in N_n.*

2.1 Factoring vs Discrete Logarithm in Z_N^*

It turns out that there is a tight connection between factoring N and revealing the discrete logarithm of a certain element in Z_N^*. In order to factor a random integer $N = P \cdot Q$ in N_n, it is sufficient to find the discrete log of g^N for a randomly chosen $g \in Z_N^*$. This is due to the following trivial fact:

Fact 1 *Let $\langle N, g\rangle$ belong to P_n (say that $N = P \cdot Q$). Then, if $ord_N(g) > P + Q - 1$, the discrete logarithm $S = DL_{N,g}(g^N)$ is equal to $P + Q - 1$.*

Proof: Recall that the order of g divides the order of the group Z_N^* which is equal to $\varphi(N) = (P-1)(Q-1)$. Therefore, $g^N = g^{N-\varphi(N)} = g^{P+Q-1} \pmod N$. Consequently, if $ord_N(g) > P + Q - 1$ then $S = P + Q - 1$. □

The following proposition, established by Håstad et al. [HSS], claims that an element picked randomly in Z_N^* is very likely to be of high order:

Proposition 3 (Håstad et al.) *Let $\langle N, g\rangle$ be uniformly distributed in P_n, where N is equal to $P \cdot Q$. Then,*

$$\Pr\left[ord_N(g) < \frac{1}{n^k} \cdot (P-1)(Q-1)\right] \leq O\left(\frac{1}{n^{(k-4)/3}}\right)$$

The only use we make of the above proposition, is to show that with very high probability, $ord_N(g)$ cannot be too small. Specifically, Proposition 3 implies that with overwhelming probability $ord_N(g)$ is greater than $P+Q-1$. Therefore, as was first observed by Chor [Chor], we can solve the two equations $P+Q-1=S$ (according to Fact 1) and $P \cdot Q = N$ for the unknowns P and Q and thus factor N.

2.2 Proof of Main Lemma

The proof of Lemma 2 is basically a reduction. We show how to use the algorithm D that distinguishes H_n^i and H_n^{i+1} in order to calculate S and thus factor N.

2.2.1 Using D to discover the $(i+1)^{st}$ bit of the exponent

Let $W_n \subseteq P_n$ be the set of pairs $\langle N, g\rangle$ in P_n for which it holds that D distinguishes $H_{N,g}^i$ and $H_{N,g}^{i+1}$ with advantage at least $\frac{1}{2n^c}$. A standard averaging argument shows that the probability that a pair $\langle N, g\rangle$ chosen at random from P_n is in the set W_n is at least $\frac{1}{2n^c}$.
From now on we consider the case where $\langle N, g\rangle$ belongs to the set W_n, and therefore satisfies

$$\left|\Pr[D(N,g,g^x)=1|x \in_R \{0,1\}^i] - \Pr[D(N,g,g^x)=1|x \in_R \{0,1\}^{i+1}]\right| \geq \frac{1}{2n^c} \tag{2.1}$$

Observe that

$$\begin{aligned}\Pr[D(N,g,g^x)=1|x \in_R \{0,1\}^{i+1}] = \;& \tfrac{1}{2} \cdot \Pr[D(N,g,g^x)=1|x \in_R \{0,1\}^i] \;+ \\ & \tfrac{1}{2} \cdot \Pr[D(N,g,g^{2^i+x})=1|x \in_R \{0,1\}^i]\end{aligned} \tag{2.2}$$

From 2.1 and 2.2 we obtain the following:

$$\left|\Pr[D(N,g,g^x)=1|x \in_R \{0,1\}^i] - \Pr[D(N,g,g^{2^i+x})=1|x \in_R \{0,1\}^i]\right| \geq \frac{1}{n^c} \tag{2.3}$$

Denote by $\overline{H}_{N,g}^i$ the distribution g^{2^i+x} where x is drawn uniformly from $\{0,1\}^i$. Another way to state Inequality 2.3 is to say that the distinguisher D has advantage at least $\frac{1}{n^c}$ in distinguishing the distributions $H_{N,g}^i$ and $\overline{H}_{N,g}^i$. Let β and γ be the acceptance probabilities of D on input taken from $H_{N,g}^i$ and $\overline{H}_{N,g}^i$, respectively. That is, let

$$\beta \stackrel{def}{=} \Pr[D(N,g,g^x)=1|x \in_R \{0,1\}^i] \tag{2.4}$$

and

$$\gamma \stackrel{def}{=} \Pr[D(N,g,g^{2^i+x})=1|x \in_R \{0,1\}^i] \tag{2.5}$$

Without loss of generality assume that $\gamma > \beta$. Note that good approximations of β and γ can be easily obtained (in polynomial-time) by performing a-priori tests on D, using samples taken from $H^i_{N,g}$ and $\overline{H}^i_{N,g}$.

In the sequel we use the distinguisher D as an oracle, that enables us to "peek" into a 1-bit window on the $(i+1)^{st}$ location of an unknown exponent of length $(i+1)$. Specifically, we use D in order to derive the $(i+1)^{st}$ bit of an $(i+1)$-bit string x, given g^x.

2.2.2 Discovering S - a naive implementation

Suppose for a moment that we had a "perfect" oracle, that given input $Z = g^x$, where x is of length $(i+1)$, would supply us, with success probability 1, the $(i+1)^{st}$ bit of x. It would then enable us to extract x, using two simple operations:

Shifting to the left: By squaring Z we shift x by one position to the left.

Zeroing the j'th bit: By dividing Z by $g^{2^{j-1}}$ we zero the j'th position in x, in case it is known to be 1.

Therefore, we extract x from the most significant to the least significant bit by "moving" it under the $(i+1)^{st}$ window. Specifically, we query the oracle and determine the $(i+1)^{st}$ bit of x and zero it in case it equals 1. Next we shift x by one position to the left, query again the oracle to discover the next bit and so on.

As was explained earlier, we try to factor N by discovering $S = DL_{N,g}(g^N)$. An important property of S, is that with overwhelming probability its length is $\lceil n/2 \rceil + 1$, and is therefore smaller than $i+1$. We can thus manipulate $Y = g^N = g^S \pmod{N}$ and discover S.

However, as the oracle might give us erroneous answers and all we are guaranteed is that there is a $\gamma - \beta$ gap (which is greater than $\frac{1}{n^c}$) between the probability to get a correct 1-answer and the probability to get an erroneous 1-answer, our implementation needs to be more careful.

2.2.3 Discovering S - the actual implementation

We must randomize our queries to the oracle and learn the correct answer by comparing the proportion of 1-answers with β and γ. A straightforward way to learn the $(i+1)^{st}$ bit of x given Z, would be to query the oracle on polynomially many random multiples $Z \cdot g^{r_k}$ for known r_k's chosen uniformly from $\{0,1\}^i$, and based on the fraction of 1-answers to decide between 0 and 1. However this approach fails, since despite our knowledge of r_k, we cannot tell whether a carry from the addition of the i least significant bits of the known r_k and the unknown x effects the $(i+1)^{st}$ bit of their sum. Thus we cannot gain any information on the $(i+1)^{st}$ bit of x from the answer of the oracle on $Z \cdot g^{r_k}$.

We now give a rough description of a procedure that resolves this difficulty and computes S. The procedure consists of $\lceil n/2 \rceil + 1$ stages, where on the j'th stage we create a list L_j which is a subset of $\{0, \ldots, 2^j - 1\}$. We want two invariants to hold for the list L_j:

1. L_j contains an element e such that $S - e \cdot 2^{l(j)}$ belongs to the set $\{0,\ldots,2^{l(j)}-1\}$, where $l(j) \stackrel{def}{=} \lceil \frac{n}{2} \rceil + 1 - j$. (In other words, we want e to be equal to $S_{\lceil \frac{n}{2} \rceil + 1, l(j)}$).

2. The size of L_j is small, that is, it contains up to a polynomial number of values (where the polynomial is set a-priori).

On each stage, we keep the values of the list sorted. When we advance to the $(j+1)^{st}$ stage, we first let L_{j+1} be all the values v such that $v = 2u$ or $v = 2u+1$ where u is in L_j, thus doubling the size of the list L_j. Obviously, by this we maintain the first invariant specified above. In case the size of L_{j+1} exceeds the polynomial bound we fixed, we use repeatedly the *Trimming Rule* in order to throw false candidates out of the list L_{j+1} until we are within the maximal size allowed.

2.2.4 Keeping the size of L_j bounded

Suppose that we decide to trim L_j whenever the difference between the largest candidate in it, denoted by v^j_{max}, and the smallest candidate in it, denoted by v^j_{min}, exceeds a certain polynomial, say n^α (for some constant α). At least one of the values v^j_{max}, v^j_{min} is not the correct value $S_{\lceil \frac{n}{2} \rceil+1,l(j)}$. Therefore the trimming rule (to be defined in the sequel) may throw one of them out of the list. For this purpose, we are going to define a new secret S', for which $g^{S'}$ can be efficiently computed given $Y = g^S$, v^j_{max} and v^j_{min}. We will examine a certain position in it (which is a function of j), henceforth referred to as the *crucial position* (and shortly denoted *cp*). Essentially, S' will have the following properties:

1. If v^j_{min} is the correct candidate (i.e. $S_{\lceil \frac{n}{2} \rceil+1,l(j)} = v^j_{min}$) then the cp-bit in S' is 0, so are the $\lceil \alpha \log n \rceil$ bits to its right, and so are all the bits to its left.

2. If v^j_{max} is the correct candidate (i.e. $S_{\lceil \frac{n}{2} \rceil+1,l(j)} = v^j_{max}$) then the cp-bit in S' is 1, the $\lceil \alpha \log n \rceil$ bits to its right are all 0's, and so are all the bits to its left.

Consequently, in these two situations we will be able to perform the randomization we wanted. We first shift S' to the left until the *cp*-bit is placed in the $(i+1)^{st}$ location (by repeatedly squaring Y). We then multiply the result by g^r for some randomly chosen $r \in \{0,1\}^i$. The probability to have a carry into the $(i+1)^{st}$ location from the addition of r and the shifted S', is no more than $\frac{1}{n^\alpha}$ (a carry might occur only when $r_{i,i-\lceil \alpha \log n \rceil} = 11\ldots1$). Hence, by using a polynomial number of queries to the oracle (with independently chosen r's) we are able to deduce the value of the *cp*-bit by comparing the fraction of 1-answers with β and γ.

As the value of the cp-bit is revealed, we can discard one of the candidates v^j_{min} or v^j_{max} from the list: If $cp = 1$ we are guaranteed that v^j_{min} is not correct, and if $cp = 0$ we are guaranteed that v^j_{max} is not correct.

Note that in case neither v^j_{max} nor v^j_{min} are correct, we cannot ensure that the $\lceil \alpha \log n \rceil$ bits to the right of the *cp*-bit in S' will be zeros, so a carry may reach the $(i+1)^{st}$ position. Thus we can get the frequency of 1-answers altogether different from β and γ. Yet in that case, it is ok for the trimming rule to discard either one of the extreme values from the list.

We proceed with a formal presentation of the proof.

2.2.5 Definition of S' and cp

Recall that $l(j) = \lceil \frac{n}{2} \rceil + 1 - j$. We define the new secret S' (which is a function of j, S v^j_{min} and v^j_{max}) to be

$$S' = \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil \cdot (S - v^j_{min} \cdot 2^{l(j)})$$

where m is a natural number. We will see that in the choice of m there is a tradeoff between the running time and the probability of error: When m is large, the error probability is smaller. On the other hand, when m is small, the running-time is shorter (we will see that choosing m to be $\lceil \alpha \log n \rceil$ will be adequate). Note that $g^{S'}$ can be efficiently evaluated given $Y = g^S$, v^j_{min} and v^j_{max}. The Crucial Position in S' is defined to be

$$cp = \lceil \alpha \log n \rceil + m + l(j) + 1$$

Since we decided to trim L_j whenever the difference between the extreme values in it exceeds n^α, the trimming rule will be applied only for j's greater than $\lceil \alpha \log n \rceil$ (for smaller j's v^j_{max} and v^j_{min} will not differ by more than n^α). Therefore, the maximal value for cp will be $\lceil n/2 \rceil + m + 1$. Thus, for i's smaller than $\lceil n/2 \rceil + m$ it occurs that cp is greater than $i + 1$. For these i's we have to guess the $\lceil n/2 \rceil + m - i \leq m$ most significant bits of S (in order to keep the number of guesses polynomial, we restrict m to be logarithmic in n and prefered as small as possible).

2.2.6 The actual algorithms and their analysis

We first describe the procedure "find S" that on input $N \in N_n$ and i (the index of the hybrid for which the acceptance probability of D on $H^i_{N,g}$ and $\overline{H}^i_{N,g}$ differs by more than $\frac{1}{n^c}$), finds S. We proceed with an analysis of the procedure which leads us to the exact formulation of the trimming rule.

Procedure "Find S":

On input N and i execute the following steps:

1. Let $j_0 = max(\lceil n/2 \rceil + m - i, 0))$.
   Recall that $i \geq \lceil n/2 \rceil$, therefore $j_0 \in \{0, \ldots, m\}$.
2. If $j_0 > 0$ guess the j_0 most significant bits of S, and let $w \in \{0, \ldots, 2^{j_0} - 1\}$ denote the guess (if $j_0 = 0$ let $w \stackrel{\text{def}}{=} 0$).
   For each of these polynomial number of guesses do the following stages:
3. Let $L_{j_0} = \{w\}$.
4. For $j = j_0 + 1$ to $\lceil n/2 \rceil + 1$ do the following:
   (a) Let $L_j \stackrel{def}{=} \{2u,\ 2u + 1\ : u \in L_{j-1}\}$.
   Order the resulting list from the largest element v^j_{max} to the smallest element v^j_{min}.

(b) If $v^j_{max} - v^j_{min} > 2^{\lceil \alpha \log n \rceil}$ (we are guaranteed that $v^j_{max} - v^j_{min} \le 2 \cdot 2^{\lceil \alpha \log n \rceil}$ by the previous stage) use the trimming rule (to be specified) repeatedly until the difference between the largest element in the list and the smallest one is no more than $2^{\lceil \alpha \log n \rceil}$.

5. Check all values $v \in L_{\frac{n}{2}+1}$ and see whether g^v equals Y. If such a value is found, then it is S.

Two facts: We turn to make two observations which lead us to the formulation of the rule by which we trim L_j (assuming that $2^{\lceil \alpha \log n \rceil} < v^j_{max} - v^j_{min} \le 2^{\lceil \alpha \log n \rceil + 1}$):

Fact 2 *Suppose that v^j_{min} indeed equals $S_{\lceil n/2 \rceil + 1, l(j)}$. Then, the cp-bit in S' is 0, all the bits to its left are 0's, and the $\lceil \alpha \log n \rceil$ bits to its right are 0's as well.*

Fact 3 *Suppose that v^j_{max} indeed equals $S_{\lceil n/2 \rceil + 1, l(j)}$. Then, the cp-bit in S' is 1, all the bits to its left are 0's, and the $l \stackrel{def}{=} min(\lceil \alpha \log n \rceil, m-1) - 1$ bits to its right are 0's as well.*

Proof:(of Fact 2) Using $v^j_{max} - v^j_{min} > 2^{\lceil \alpha \log n \rceil}$, observe that

$$\begin{aligned} S' &= \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil \cdot \left(v^j_{min} \cdot 2^{l(j)} + S_{l(j),1} - v^j_{min} \cdot 2^{l(j)} \right) \\ &\le 2^m \cdot S_{l(j),1} \\ &\le 2^{m+l(j)} \\ &= 2^{cp - \lceil \alpha \log n \rceil - 1} \end{aligned}$$

□

Proof:(of Fact 3) Observe that

$$\begin{aligned} S' &= \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil \cdot \left(v^j_{max} \cdot 2^{l(j)} + S_{l(j),1} - v^j_{min} \cdot 2^{l(j)} \right) \\ &= \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil \cdot \left((v^j_{max} - v^j_{min}) \cdot 2^{l(j)} + S_{l(j),1} \right) \\ &= 2^{\lceil \alpha \log n \rceil + m + l(j)} + \delta \cdot (v^j_{max} - v^j_{min}) \cdot 2^{l(j)} + \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil \cdot S_{l(j),1} \end{aligned}$$

where $\delta = \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil - \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \in [0, 1)$.

Let $U_1 = \delta \cdot (v^j_{max} - v^j_{min}) \cdot 2^{l(j)}$ and let $U_2 = \left\lceil \frac{2^{\lceil \alpha \log n \rceil + m}}{v^j_{max} - v^j_{min}} \right\rceil \cdot S_{l(j),1}$.
We can write S' as

$$S' = 2^{cp-1} + U_1 + U_2$$

Recall that $2^{\lceil \alpha \log n \rceil} < v^j_{max} - v^j_{min} \le 2 \cdot 2^{\lceil \alpha \log n \rceil}$. Therefore, $U_1 \le 2 \cdot 2^{\lceil \alpha \log n \rceil + l(j)} = 2^{cp-m}$ and $U_2 \le 2^{m+l(j)} = 2^{cp - \lceil \alpha \log n \rceil - 1}$. Also, both $U_1, U_2 \ge 0$.
Consequently S' is of the following form:

- The cp-bit in S' is 1 and all the bits to its left are 0's.
- Let $l = min(\lceil \alpha \log n \rceil, m-1)-1$. Then the l bits to the right of the cp-bit in S' are 0's.

□

Fact 3 implies that we must choose m to be at least $\lceil \alpha \log n \rceil$, otherwise there wouldn't be enough 0's to the right of the cp-bit to enable the randomization. On the other hand, the larger m is, the more bits we have to guess in Step (1) of the procedure "Find S". We therefore set m to be $\lceil \alpha \log n \rceil$, and respectively define

$$S' = \left\lceil \frac{2^{2\lceil \alpha \log n \rceil}}{v^j_{max} - v^j_{min}} \right\rceil \cdot (S - v^j_{min} \cdot 2^{l(j)})$$

and

$$cp = 2\lceil \alpha \log n \rceil + l(j) + 1$$

We now formally state the trimming rule:

Trimming Rule:

1. Shift S' by $i + 1 - cp$ bits to the left (by computing $Y' = g^{S' \cdot 2^{i+1-cp}}$) therefore placing the crucial position in S' on location $i + 1$.
2. Pick $t(n) = n^{2c+4}$ random elements $x_1, \ldots, x_{t(n)} \in \{0,1\}^i$.
3. For each $1 \le k \le t(n)$ query the oracle on $Y' \cdot g^{x_i} \pmod N$ and denote by b_k its answer (i.e. $b_k = D(g^{S' \cdot 2^{i+1-cp} + x_i})$). Denote by M the mean $\frac{\sum_{k=1}^{t(n)} b_k}{t(n)}$.
4. If $M \le (\beta + \frac{\gamma - \beta}{2})$ discard the candidate value v^j_{max} from the list L_j. Otherwise (i.e. when $M > (\beta + \frac{\gamma-\beta}{2})$) discard the candidate value v^j_{min}.

Note that the trimming rule is applied only for j's which are greater than $j_0 + \lceil \alpha \log n \rceil$. Therefore, we have that $i+1$ is always greater or equal to cp, making Step (1) in the trimming rule well defined.

For explanations on the choice of the parameters see the Appendix. The bottom line is, however, that the probability of a mistake by the trimming rule (i.e. the probability that the correct value will be discarded from the list) is exponentially small.

Claim 3.1 *The Procedure "Find S" combined with the Trimming Rule above can factor integers picked randomly from N_n with probability greater than $\frac{1}{8n^c}$.*

Proof: As previously mentioned, for a pair $\langle N, g \rangle$ uniformly chosen from P_n (where N is equal to $P \cdot Q$), the following two facts hold:

1. With overwhelming probability $ord_N(g) > P + Q - 1$.
2. With probability greater than $\frac{1}{2n^c}$ the pair $\langle N, g \rangle$ belongs to the set W_n.

Therefore, given a random $N = P \cdot Q$ in N_n, we can pick g randomly in Z_N^* and with probability higher than $\frac{1}{4n^c}$ both of the above conditions hold. Hence, S will be equal to $P + Q - 1$ (according to Fact 1) and the algorithm D will have advantage of at least $\frac{1}{2n^c}$ in distinguishing the distributions $H^i_{N,g}$ and $\overline{H}^i_{N,g}$ (see Equation 2.3). Since the probability of error by the trimming rule is exponentially small, and since the trimming rule is used polynomially many times throughout the procedure "Find S", with probability greater than $\frac{1}{8n^c}$ the value S will be found. □

Note that the procedure "Find S" combined with the Trimming Rule yields at most $2^{j_0} \cdot 2^{\lceil \alpha \log n \rceil} \leq 2^{2\lceil \alpha \log n \rceil} = n^{O(1)}$ possible values for S, and is therefore polynomial time. Thus Claim 3.1 finishes the proof of Lemma 2.

We now go back to the proof of Theorem 1. Assume that the gap between the acceptance probability of D on the extreme hybrids $H_n^{\lceil n/2 \rceil}$ and $H_n^{n+\omega(\log n)}$ is greater than $\frac{1}{n^d}$. We construct an algorithm A that factors integers uniformly distributed in N_n. On input N, algorithm A picks a random i in $\{\lceil \frac{n}{2} \rceil, \ldots, n+\omega(\log n)\}$ and runs the procedure "Find S" on (N, i). By Lemma 2, the probability that "Find S" indeed factors N, is greater than one eight of the gap between the acceptance probabilities of D on H_n^i and H_n^{i+1}, for a random i as above. Denote the number of hybrids, $\lfloor n/2 \rfloor + \omega(\log n)$, by $m(n)$. Then, we have that for all sufficiently large n's

$$\begin{aligned}
\Pr[\text{A factors } N] &= \frac{1}{m(n)} \sum_{i=\lceil n/2 \rceil}^{n+\omega(\log n)} \Pr[\text{"Find S" on input } (N,i) \text{ factors } N] \\
&\geq \frac{1}{m(n)} \sum_{i=\lceil n/2 \rceil}^{n+\omega(\log n)} \frac{1}{8} \cdot \left| \Pr[D(H_n^{i+1}) = 1] - \Pr[D(H_n^i) = 1] \right| \\
&\geq \frac{1}{m(n)} \cdot \frac{1}{8} \cdot \left| \Pr[D(H_n^{n+\omega(\log n)}) = 1] - \Pr[D(H_n^{\lceil n/2 \rceil}) = 1] \right| \\
&\geq \frac{1}{n^{d+1}}
\end{aligned}$$

thus contradicting Assumption 1.

Note: In fact, Theorem 1 holds even when the distribution $Half_n$ is defined to include all triplets of the form $\langle N, g, g^x \rangle$ where $\langle N, g \rangle \in_R P_n$ and $x \in_R \{0,1\}^{\lceil n/2 \rceil - O(\log n)}$ (rather than $x \in_R \{0,1\}^{\lceil n/2 \rceil}$). The only change is in the proof of Lemma 2, where on Step (1) of the procedure "Find S", j_0 may belong to the set $\{0, \ldots, m+O(\log n)\}$. Thus, we have to guess more bits from S, however the total number of possible guesses remains polynomial in n.

2.3 Equivalence to the HSS result

Theorem 1 is actually equivalent to the result by [HSS] on the simultaneous hardness of the upper $\lceil n/2 \rceil$ bits in the exponentiation function $f_{N,g}$. In order to show that, we discuss first an alternative version of Theorem 1. Recall the hybrid $H_n^{n+\omega(\log n)}$ defined in the proof of Theorem 1, including triplets $\langle N, g, g^R \rangle$, where $\langle N, g \rangle$ is uniformly distributed in P_n and R

is uniformly distributed in $\{0,1\}^{n+\omega(\log n)}$. Let us denote it by $\widetilde{Full}_n$. The following is a corollary from Theorem 1 and Claim 1.1:

Corollary 4 *The probability ensembles $\{Half_n\}_{n\in N}$ and $\{\widetilde{Full}_n\}_{n\in N}$ are computationally indistinguishable.*

We show that Corollary 4 is equivalent to the result of [HSS]. But first, let us give the exact formulation of their result.

Definition 6 *Let $\langle N,g\rangle$ be uniformly distributed in P_n, let x be uniformly distributed in $\{0,1\}^n$ and let r be uniformly distributed in $\{0,1\}^{\lceil\frac{n}{2}\rceil}$. We define the following probability distributions:*

$$X_n \stackrel{def}{=} \langle N,g,f_{N,g}(x),x_{n,\lceil n/2\rceil}\rangle$$

and

$$Y_n \stackrel{def}{=} \langle N,g,f_{N,g}(x),r\rangle$$

Theorem 5 (Hastad et al.) *The probability ensembles $\{X_n\}_{n\in N}$ and $\{Y_n\}_{n\in N}$ are computationally indistinguishable.*[1]

2.3.1 The Equivalence

Proposition 6 *Theorem 5 holds if and only if Corollary 4 holds.*

Proof: We show how to transform a probabilistic polynomial-time algorithm D that distinguishes the ensemble $\{X_n\}$ from $\{Y_n\}$ into a probabilistic polynomial-time algorithm D' that distinguishes the ensemble $\{Half_n\}$ from $\{\widetilde{Full}_n\}$, and vice versa.

Transforming D into D': On input $\langle N,g,y\rangle$, pick z uniformly from $\{0,1\}^{\lceil n/2\rceil}$, run D on $\langle N,g,y\cdot g^{z\cdot 2^{\lceil n/2\rceil}},z\rangle$ and output D's answer. Observe that

1. If $\langle N,g,y\rangle$ is taken from $Half_n$, then $y=g^r$ where $r\in\{0,1\}^{\lceil n/2\rceil}$. Therefore, $\langle N,g,g^{z\cdot 2^{\lceil n/2\rceil}+r},z\rangle$ is distributed as X_n.
2. If $\langle N,g,y\rangle$ is taken from $\widetilde{Full}_n$, then $y=g^R$, where $R\in\{0,1\}^{n+\omega(\log n)}$. Let $\approx$ denote statistical closeness. Note that

$$\begin{aligned}(U_{n+\omega(\log n)}+z\cdot 2^{\lceil n/2\rceil}) \bmod ord_N(g) &\approx U_{n+\omega(\log n)} \bmod ord_N(g)\\ &\approx U_n \bmod ord_N(g)\end{aligned}$$

(the proof of each of the above transitions is similar to the proof of Claim 1.1). Therefore, $\langle N,g,g^{z\cdot 2^{\lceil n/2\rceil}+R},z\rangle$ is statistically close to Y_n.

[1] Actually, the simultaneous hardness of the upper $\lceil\frac{n}{2}\rceil$ in $f_{N,g}$ was defined differently by [HSS]. Their definition states that the two distributions $\langle\tilde{x}_{n,\lceil n/2\rceil},Z\rangle$ and $\langle r,Z\rangle$ are computationally indistinguishable, where $Z=g^x$ (for $x\in_R Z_N^*$), $\tilde{x}=DL_{N,g}(Z)$ and $r\in_R\{0,1\}^{\lceil n/2\rceil}$. However, this definition is problematic: At least the most significant bit in the first distribution, $\tilde{x}_n$, will be always 0, since $ord_N(g)$ is always smaller than $N/2$. Hence the above two distributions can be easily distinguished.

Thus, Theorem 5 is implied by Corollary 4.

Transforming D' into D: On input $\langle N, g, y, z\rangle$, run D' on $\langle N, g, y/g^{z\cdot 2^{\lceil n/2\rceil}}\rangle$ and output D''s answer. Observe that

1. If $\langle N, g, y, z\rangle$ is taken from X_n, then $y = f_{N,g}(x) = g^x$ and $z = x_{n,\lceil n/2\rceil}$. Therefore, $y/g^{z\cdot 2^{\lceil n/2\rceil}} = g^{x_{\lceil n/2\rceil,1}}$ and thus $\langle N, g, y/g^{z\cdot 2^{\lceil n/2\rceil}}\rangle$ is uniformly distributed in $Half_n$.
2. If $\langle N, g, y, z\rangle$ is taken from Y_n, then $y = f_{N,g}(x) = g^x$ and z is independent of x. Note that

$$\begin{aligned}(U_n - z\cdot 2^{\lceil n/2\rceil}) \bmod ord_N(g) &= U_n \bmod ord_N(g) - z\cdot 2^{\lceil n/2\rceil} \bmod ord_N(g)\\ &\approx U_{n+\omega(\log n)} \bmod ord_N(g) - z\cdot 2^{\lceil n/2\rceil} \bmod ord_N(g)\\ &= (U_{n+\omega(\log n)} - z\cdot 2^{\lceil n/2\rceil}) \bmod ord_N(g)\\ &\approx U_{n+\omega(\log n)} \bmod ord_N(g)\end{aligned}$$

   Therefore, $\langle N, g, y/g^{z\cdot 2^{\lceil n/2\rceil}}\rangle$ is statistically close to $\widetilde{Full}_n$.

Thus, Theorem 5 implies Corollary 4.

■

2.3.2 Discussion

Our proof of Theorem 1 simplifies to a great extent the proof given by [HSS] to Theorem 5. Basically, this is due to the following reasons:

1. Unlike in [HSS], we do not require that the order of g in Z_N^* will be very high (i.e. greater than $\frac{1}{n^k}\cdot(P-1)(Q-1)$). It suffices that the order of g will be greater than $P+Q-1$.
2. We do not need to consider separately the $O(\log n)$ most significant bits as done in [HSS] (where a very complex proof is given for these bits).
3. As a consequence from the different nature of the oracles, the randomization conducted by us (randomizing the bottom i bits) is different from the randomization done in [HSS] (randomizing the full range $[0, ord_N(g))$). Therefore many of the difficulties encountered in the work of [HSS] are not relevant in our proof. For example, we do not need to avoid a wrap around the order of g.

2.3.3 A refinement

Observe that Corollary 4 actually states that the hybrids $H_n^{\lceil n/2\rceil}$ and $H_n^{n+\omega(\log n)}$ are computationally indistinguishable. An equivalent phrasing would be to say that for all i's satisfying that $\lceil n/2\rceil \le i < n+\omega(\log n)$, the hybrids H_n^i and H_n^{i+1} are computationally indistinguishable. Theorem 5 can be formulated differently as well: As was shown by [HSS], the simultaneous hardness of the upper $\lceil n/2\rceil$ bits in $f_{N,g}$ is equivalent to saying that for all i's satisfying that $\lceil n/2\rceil \le i \le n$, the i'th bit in $f_{N,g}$ is *relatively hard to the left.*

Definition 7 *Let $\langle N, g\rangle \in_R P_n$. The i'th bit of $f_{N,g}$ is* **relatively hard to the left**[2]*, if for all probabilistic polynomial-time algorithms A, for all constants c and for all sufficiently large n's*

$$|\Pr[A(N, g, f_{N,g}(x), x_{n,i+1}) = 1 | x_i = 1] - \Pr[A(N, g, f_{N,g}(x), x_{n,i+1}) = 1 | x_i = 0]| < \frac{1}{n^c}$$

[Note that a more natural definition, saying that no efficient algorithm can predict the i'th bit of x given N, g, g^x and $x_{n,i+1}$ with probability of success significantly greater than $\frac{1}{2}$, works only for the $n - O(\log n)$ bottom bits of x. The reason is that the $O(\log n)$ upper bits of x might be biased (since N is smaller than 2^n, it can be that $\Pr[x_i = 1 | x \in_R [0, N)]$ significantly differs from $\frac{1}{2}$, for $i \in \{n - O(\log n), \ldots, n\}$). Thus these bits can be trivially predicted with success probability substantially greater than $\frac{1}{2}$. Definition 7, however, remains valid for these bits as well.]

Recall Proposition 6 above, asserting that Theorem 5 is equivalent to Corollary 4. We now demonstrate that Proposition 6 can be refined, by showing that the hybrids H_n^{i-1} and H_n^i are computationally indistinguishable if and only if the i'th bit of $f_{N,g}$ is relatively hard to the left.

We begin by specifing what is an *index function*, in order to properly define a probability ensemble indexed by $\mathbb{N}$, which has the hybrid H_n^i as its n'th random variable.

Definition 8 *An* **index function** $i : N \to N$ *is an integer function satisfying the following two conditions:*

1. *for all sufficiently large n's, $\lceil \frac{n}{2} \rceil < i(n) \leq n$.*
2. *i is polynomial-time computable.*[3]

Proposition 7 *Let i be an index function, and let $\langle N, g\rangle \in_R P_n$. The probability ensembles $\{H_n^{i(n)-1}\}_{n\in N}$ and $\{H_n^{i(n)}\}_{n\in N}$ are computationally indistinguishable if and only if the $i(n)$'th bit of $f_{N,g}$ is relatively hard to the left.*

Proof: For the sake of simplicity we treat i as a constant throughout the proof. We show how to transform a probabilistic polynomial-time algorithm A, that on input $\langle N, g, f_{N,g}(x), x_{n,i+1}\rangle$ tries to find x_i, and has a non-negligible gap between the probability of giving a correct 1-answer and the probability of giving an erroneous 1-answer, into a probabilistic polynomial-time algorithm D, that distinguishes with a non-negligible gap between the ensembles $\{H_n^{i-1}\}$ and $\{H_n^i\}$, and vice versa.

Transforming A into D: On input $\langle N, g, y\rangle$, pick z uniformly from $\{0,1\}^{n-i}$ and output $b \stackrel{def}{=} A(N, g, y \cdot g^{z\cdot 2^i}, z)$. Using $x = z \cdot 2^i + DL_{N,g}(y)$, observe that

$$\Pr[D(H_n^{i-1}) = 1] = \Pr[A(N, g, f_{N,g}(x), x_{n,i+1}) = 1 | x_i = 0]$$

[2]Analogously, one can define a bit being relatively hard to the right.

[3]In fact, since the input n for i can be given in unary, we can allow i to be computed in time exponential in n.

and

$$\begin{aligned}\Pr[D(H_n^i)=1] = & \Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1] \\ = & \frac{1}{2}\cdot\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=0] \;+ \\ & \frac{1}{2}\cdot\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=1]\end{aligned}$$

where in both cases $x \in_R \{0,1\}^n$. Thus,

$$|\Pr[D(H_n^{i-1})=1]-\Pr[D(H_n^i)=1]| = \\ \frac{1}{2}\cdot|\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=0]-\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=1]|$$

Therefore, the computational indistinguishability of $\{H_n^{i(n)-1}\}$ and $\{H_n^{i(n)}\}$ implies that the $i(n)$'th bit of $f_{N,g}$ is relatively hard to the left.

Transforming D into A: On input $\langle N,g,f_{N,g}(x),x_{n,i+1}\rangle$, let $y=f_{N,g}(x)/g^{x_{n,i+1}\cdot 2^i}$ and output $b \stackrel{def}{=} D(N,g,y)$. Observe that $y=g^{x_{i,1}}$ and thus

$$\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1]=\Pr[D(H_n^i)=1]$$

and

$$\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=0]=\Pr[D(H_n^{i-1})=1]$$

On the other hand,

$$\begin{aligned}\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1] = & \frac{1}{2}\cdot\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=1] \;+ \\ & \frac{1}{2}\cdot\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=0]\end{aligned}$$

and so,

$$\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=1] = \\ 2\cdot\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1]-\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=0]$$

Combining the above, we get

$$|\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=1]-\Pr[A(N,g,f_{N,g}(x),x_{n,i+1})=1|x_i=0]| = \\ 2\cdot(\Pr[D(H_n^i)=1]-\Pr[D(H_n^{i-1})=1])$$

Therefore, if the $i(n)$'th bit of $f_{N,g}$ is relatively hard to the left, then the ensembles $\{H_n^{i(n)-1}\}$ and $\{H_n^{i(n)}\}$ are computationally indistinguishable.

■

Chapter 3

Additional Results

3.1 Shifted exponents

When looking closely at the proof of indistinguishability of the ensembles $Full_n$ and $Half_n$, one sees that it relies on the fact that the ensemble $Half_n$ has a block of $\lceil n/2 \rceil$ consecutive random bits in the exponent of g. Recall that the problem of factoring N is reduced to computing S (where S is congruent to $N \bmod ord_N(g)$). Since S is of size $\lceil n/2 \rceil + 1$, we discover it by using the $\lceil n/2 \rceil$-bit block of random bits. At this point one may ask: Is the distribution obtained by shifting the block of $\lceil n/2 \rceil$ random bits by k positions to the left, where k is polynomial in n, still pseudorandom (i.e. computational indistinguishable from $Full_n$)? The answer is yes, provided that N is a *Blum-integer* and that g is a *quadratic residue* in Z_N^*.

We say that $N = P \cdot Q$ is a Blum integer, if N belongs to N_n, and both P and Q are congruent to $3 \bmod 4$. It is known that for such N's, squaring is a permutation in Z_N^*. We say that $x \in Z_N^*$ is a quadratic residue if $x = y^2$ for some $y \in Z_N^*$. We denote by $\widehat{P}_n$ the set of pairs $\langle N, g \rangle$ where N is a Blum integer of size n and g is a quadratic residue in Z_N^*. Note that one can efficiently pick a pair $\langle N, g \rangle$ uniformly distributed in $\widehat{P}_n$. When limiting N and g as above, and under the assumption that factoring Blum integers is hard, we can show that for every $k \geq 0$ which is polynomial in n, the ensembles $g^{r \cdot 2^k}$ and $Full_n$ are computationally indistinguishable (where r is a bit string of size $\lceil n/2 \rceil$). Note that the assumption that factoring Blum integers is hard, is implied by Assumption 1 (again, a non-negligible fraction of the Blum integers is a non-negligible fraction of all the integers in N_n). That is:

Definition 9 *Let $\langle N, g \rangle$ be a uniformly distributed pair in $\widehat{P}_n$, and let r be uniformly distributed in $\{0,1\}^{\lceil \frac{n}{2} \rceil}$. We denote by $Half_n^k$ the ensemble $\langle N, g, g^{r \cdot 2^k} \rangle$.*

Theorem 8 *Under the assumption that factoring Blum integers is hard, the ensembles $\{Half_n^k\}_{n \in N}$ and $\{Full_n\}_{n \in N}$ are computationally indistinguishable (as long as k is polynomial in n).*

We remark that the results of this section (including the above theorem) are based on the previous chapter, and particularly on the main Lemma.

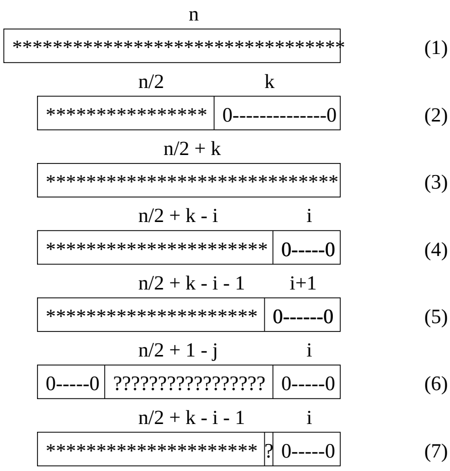

Figure 3.1: We denote random bits by '*', zeros by '0', and unknown bits of S by '?'. (1), (2), (3), (4), (5) show the exponents of $Full_n$, $Half_n^k$, $H_n^{\lceil n/2 \rceil + k}$, the hybrids $\widehat{H}_n^i$ and $\widehat{H}_n^{i+1}$, respectively. (6), (7) show the exponents of Y', on the j'th stage of the procedure that discovers S, before and after the randomization, respectively.

3.1.1 A right-shift technique

In order to prove the above assertion, it is required to perform right-shifts of the exponent S. Specifically, we use a technique of [HSS] for the following operation: Suppose the j bottom bits of S (i.e. $S_{j,1}$) are already known. We would like to truncate S off these bits and to shift it by one position to the right (i.e. we want to obtain $Z = \sqrt{g^{S - S_{j,1}}}$). In general, one cannot efficiently compute a square root modulo N without knowing the factorization of N. However, by choosing g in a clever way, such that its square root modulo N is known, we can derive Z efficiently by computing $(\sqrt{g})^{N - S_{j,1}}$). In particular, we can perform a polynomial number of shifts to the right. By choosing g to be $(g')^{2^t}$, where g' is a quadratic residue in Z_N^* and t is polynomial in n, we can shift S to the right by up-to t positions. Observe that although chosen in this manner, g is still a random quadratic residue in Z_N^*.

3.1.2 Proof of Theorem 8

Recall the hybrid $H_n^{\lceil n/2 \rceil + k} = \langle N, g, g^x \rangle$ as defined in the proof of Theorem 1, where x is uniformly distributed in $\{0,1\}^{\lceil n/2 \rceil + k}$. By Lemma 2 and by Claim 1.1, the hybrid $H_n^{\lceil n/2 \rceil + k}$ is computationally indistinguishable from $Full_n$. Therefore, by showing that the ensembles $Half_n^k$ and $H_n^{\lceil n/2 \rceil + k}$ are computationally indistinguishable, we will be done (see Figure 3.1, No. (1), (2) and (3)).

Defining new hybrids: We use again the hybrid argument, and define a hybrid $\widehat{H}_n^i$, in which the exponent of length $\lceil n/2 \rceil + k - i$ is shifted by i positions to the left. More formally, for every $0 \le i \le k$, we define

$$\widehat{H}_n^i = \langle N, g, g^{x \cdot 2^i} \rangle$$

where $x \in_R \{0,1\}^{\lceil n/2 \rceil + k - i}$ (see Figure 3.1, No. (4)). Note that $\widehat{H}_n^0 = H_n^{\lceil n/2 \rceil + k}$ and that $\widehat{H}_n^k = Half_n^k$.
Assume there exists a probabilistic polynomial-time algorithm D, that distinguishes the extreme hybrids with a non-negligible gap. Then D distinguishes a pair of neighboring hybrids $\widehat{H}_n^i$ and $\widehat{H}_n^{i+1}$ with a non-negligible gap.

An outline of the proof: Given a Blum integer N, we want to find $S = DL_{N,g}(g^N)$ as done in Theorem 1. Taking advantage of the structure of the two neighboring hybrids $\widehat{H}_n^i$ and $\widehat{H}_n^{i+1}$ (see Figure 3.1, No. (4) and (5)), we reveal S bit after bit from right to left, in $|S|$ stages, where on each stage the least significant bit of S is discovered and then truncated. Note that this is different than the proof of Theorem 1, where S is discovered from left to right (due to the different structure of the hybrids there, see Figure 2.1). In fact, this proof will be much simpler than the proof of Theorem 1, since we will not encounter difficulties in the randomization as we did there (where a carry from the addition of the random exponent affected the tested bit).[1]

The procedure that discovers S (while operating on $Y = g^S = g^N$) will be as follows: On the j'th stage, we assume the $(j-1)$'st least significant bits of S are already known. We zero them, and shift the exponent S, such that its j'th bit will be located at the $(i+1)$'st position. Call Y' the element obtained after these transformations (see Figure 3.1, No. (6)). Then, after several invocations of D on randomized instances of Y', we infer the value of the j'th bit of S. Two issues need to be further clarified:

1. **shifting S:** In order to shift S (after zeroing its $(j-1)$'st least significant bits) as described above, we need either to shift it to the left, if $i+1 > j$, or to shift it to the right, if $i+1 < j$. In the first case, the $(i+1)-j$ left shifts are done simply by raising $Y/g^{S_{j-1,1}}$ to the power of 2^{i+1-j}. In the second case, the $j-(i+1)$ right shifts are done by a-priori choosing g to be $(g')^{2^t}$ (for an appropriate choice of t, to be specified below), and by using the right-shift technique described in Subsection 3.1.1, which allows up-to t shifts to the right (implying that t must be at least $j-(i+1)$). Since on the j'th stage (for every $i+1 < j \leq \lceil \frac{n}{2} \rceil + 1$) we need to shift S by $j-(i+1)$ positions[2], the maximal number of shifts to the right that is needed throughout the procedure is $\lceil \frac{n}{2} \rceil + 1 - (i+1) = \lceil \frac{n}{2} \rceil - i$. Thus, setting $t = \lceil \frac{n}{2} \rceil$ suffices.

2. **randomizing Y′:** In order to randomize Y', we multiply it by a random element of the form $g^{x \cdot 2^{i+1}}$, where $x \in_R \{0,1\}^{\lceil n/2 \rceil + k - i - 1}$ (see Figure 3.1, No. (7)).
For large i's (and small j's) a problem of a wrap-around might arise; that is, the value of the shifted S plus $x \cdot 2^{i+1}$ may exceed $2^{\lceil n/2 \rceil + k}$. We solve it by guessing at the beginning of the procedure l bits of S (the l bottom bits), where $l = max(i+1-k+\lceil \alpha \log n \rceil, 0)$. Thus, our procedure will have $\lceil \frac{n}{2} \rceil + 1 - l$ stages, for $j = l+1, \ldots, \lceil \frac{n}{2} \rceil + 1$. In the j'th stage, after zeroing the $(j-1)$ bottom bits of S and shifting it as explained above, the binary length of S will be $\lceil \frac{n}{2} \rceil + 1 + (i+1-j) \leq \lceil \frac{n}{2} \rceil + 1 + i - l = \lceil \frac{n}{2} \rceil + k - \lceil \alpha \log n \rceil$.

[1]We have already benefited from Theorem 1: Due to its proof (i.e., Claim 1.1 and Lemma 2), we may focus on the hybrids $\widehat{H}_n^i$, which "bridge" $Half_n^k = \widehat{H}_n^k$ and $H_n^{\lceil n/2 \rceil + k} = \widehat{H}_n^0$.

[2]Note that if $i \geq \lceil \frac{n}{2} \rceil$ no right shifts are needed.

Therefore, the shifted S will become smaller than $2^{\lceil n/2\rceil+k-\lceil\alpha\log n\rceil}$, so we can add to it a random number of size approximately $2^{\lceil n/2\rceil+k}$ without risking in causing a wrap-around (in general). Note that if $i+1 \le k-\lceil\alpha\log n\rceil$ no bits from S need to be guessed.

A more formal description: Assuming that D distinguishes the hybrids $\widehat{H}_n^i$ and $\widehat{H}_n^{i+1}$ with a gap greater than $\frac{1}{n^c}$ implies that the ensembles $\langle N, g, g^{x\cdot 2^{i+1}}\rangle$ and $\langle N, g, g^{x\cdot 2^{i+1}+2^i}\rangle$ (where $\langle N, g\rangle \in_R \widehat{P}_n$ and $x \in_R \{0,1\}^{\lceil n/2\rceil+k-i-1}$) are distinguishable by D with a gap greater than $\frac{2}{n^c}$.

Given a Blum integer N, we pick a random quadratic residue g' in Z_N^* and let $g=(g')^{2^t}$, thus allowing up-to t shifts to the right. From a standard averaging argument, we get that with probability at least $\frac{1}{n^c}$, for these specific N and g

$$\left|\Pr[D(N,g,g^{x\cdot 2^{i+1}})=1]-\Pr[D(N,g,g^{x\cdot 2^{i+1}+2^i})=1]\right| \ge \frac{1}{n^c} \tag{3.1}$$

where the probability is taken only over the choice of $x \in_R \{0,1\}^{\lceil n/2\rceil+k-i-1}$. We assume for the rest of the proof that Equation 3.1 holds. Thus, the distinguisher D is used as an oracle, that on input $g^{z\cdot 2^i}$ (where $z \in_R \{0,1\}^{\lceil\frac{n}{2}\rceil+k-i}$) distinguishes with a gap greater than $\frac{1}{n^c}$ between the cases where the least significant bit of z is 0 or 1.

We begin by guessing the $l = max(i+1-k+\lceil\alpha\log n\rceil, 0)$ bottom bits of S. Since $i \le k$, we have that $l \le \lceil\alpha\log n\rceil+1$, thus the number of possible guesses is polynomial in n.

Recovering S:

We reveal S from right to left, while operating on $Y=g^N$, as follows:

1. Let $j_0 = l'$. For $j=j_0+1$ to $\lceil n/2\rceil+1$ find the j'th bit of S (assuming the $j-1$ least significant bits $S_{j-1,1}$ are known):
   (a) Zero the $j-1$ least significant bits of S and shift it such that its j'th bit will be located in the $(i+1)$'st position:
      - If $j \le i+1$ shift S to the left by computing $Y'=(Y/g^{S_{j-1,1}})^{2^{i+1-j}}$.
      - If $j > i+1$ shift S to the right by computing $Y'=(g'')^{N-S_{j-1,1}}$, where $g''={g'}^{2^{t-(j-i-1)}}$.

   (b) Derive the j'th bit of S by querying the oracle on n^{2c+4} randomized instances of Y', of the form $Y'\cdot g^{x\cdot 2^{i+1}}$, where $x \in_R \{0,1\}^{\lceil n/2\rceil+k-i-1}$.

2. After finding all bits of S, check whether $g^S = Y$.

Taking $\alpha = c+1$ and using a similar analysis to what is done in the proof of Theorem 1 (see Claim 3.1), it can be shown that for a uniformly distributed Blum integer N, the probability to factor N, given that we guessed correctly the l' bits of S, is $\Omega(\frac{1}{n^c})$. Therefore, the probability to factor N is at least $\frac{1}{poly(n)}\cdot\frac{1}{n^c}$.

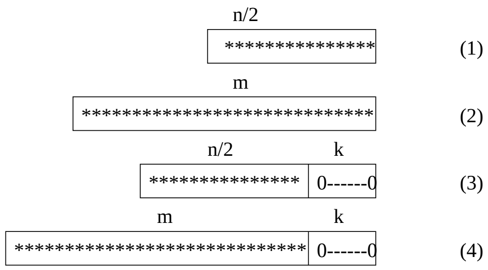

Figure 3.2: We denote random bits by '*', and zeros by '0'. (1), (2), (3), (4), show the exponents of $Half_n$, $Full_n$, $Half_n^k$ and $Full_n^k$, respectively.

3.1.3 Further Observations

Consider the following probability distribution, denoted $Full_n^k$, in which the exponent of g is a full size bit string, shifted by k positions to the left:

Definition 10 *Let $\langle N, g\rangle$ be a uniformly distributed pair in $\widehat{P}_n$, and let R be uniformly distributed in $[0, ord_N(g))$. We denote by $Full_n^k$ the ensemble $\langle N, g, g^{R\cdot 2^k}\rangle$ (see Figure 3.2).*

An immediate corollary from Theorem 1 is the following:

Corollary 9 *Under the assumption that factoring Blum integers is hard, the ensembles $\{Half_n^k\}_{n\in N}$ and $\{Full_n^k\}_{n\in N}$ are computationally indistinguishable (as long as k is polynomial in n).*

Proof: Consider the (polynomial-time computable) mapping $f : \langle N, g, y\rangle \mapsto \langle N, g, y^{2^k}\rangle$. Observe that f sends $Half_n$ to $Half_n^k$, and similarly sends $Full_n$ to $Full_n^k$. By Theorem 1 the ensembles $Half_n$ and $Full_n$ are computationally indistinguishable, therefore $f(Half_n) = Half_n^k$ and $f(Full_n) = Full_n^k$ are computationally indistinguishable. ■

At a first glance it looks as though the ensembles $\{Full_n^k\}_{n\in N}$ and $\{Full_n\}_{n\in N}$ are statistically close. Indeed, that would be the case if $ord_N(g)$ and 2^k would have been relatively prime (the above ensembles would then be identical, since multiplying by an element in $Z^*_{ord_N(g)}$ induces a permutation over $\{1, \ldots, ord_N(g)\}$). However, since it is very likely that $ord_N(g)$ is even, the above ensembles have a statistical difference of at least half. In light of the above, the following corollary may be of interest:

Corollary 10 *Under the assumption that factoring Blum integers is hard, the ensembles $\{Full_n^k\}_{n\in N}$ and $\{Full_n\}_{n\in N}$ are computationally indistinguishable (as long as k is polynomial in n).*

Corollary 10 is implied directly from Theorem 8 and Corollary 9. In fact, Corollary 10 together with Corollary 9 implies Theorem 8 as well.

3.2 Towards justifying the DDH Assumption

In order to strengthen our confidence in the DDH assumption, it is desirable to show a link between it and a known hard problem. We provide here an evidence, that when considering the DDH problem modulo a composite, there is some connection between DDH and factoring. Recall the two distributions $\langle N, g, g^a, g^b, g^{ab} \rangle$ and $\langle N, g, g^a, g^b, g^R \rangle$ (where N is a Blum integer and g is a quadratic residue in Z_N^*) that the DDH assumption claims to be computationally indistinguishable. We consider a hybrid between these distributions, defined as $\langle N, g, g^a, g^b, g^c \rangle$, where the exponent c constitutes of random bits in its upper half, and equals in its lower half to the lower half of ab modulo the order of g. We show that assuming the intractability of factoring Blum integers, the latter hybrid is computationally indistinguishable from $\langle N, g, g^a, g^b, g^R \rangle$. Details follow.

Definition 11 *Let $\langle N, g \rangle$ be uniformly distributed in $\widehat{P}_n$. Let $\xi \stackrel{def}{=} ord_N(g)$. Let m denote the size of $ord_N(g)$, that is, let $m \stackrel{def}{=} \lceil \log \xi \rceil$. Let $top^i_{N,g}$ denote the upper $m-i$ bits in $ord_N(g)$, that is, we let $top^i_{N,g} \stackrel{def}{=} \xi_{m,i+1}$ (i.e., $\xi = 2^i \cdot top^i_{N,g} + \xi_{i,1}$).[3] Let r be uniformly distributed in $[0, top^{\lceil n/2 \rceil}_{N,g})$, and let a, b, R be uniformly distributed in $[0, \xi)$. Let $[ab] \stackrel{def}{=} ab \bmod \xi$. We now define the following probability distributions:*

$$\begin{aligned} DDH_n &\stackrel{def}{=} \langle N, g, g^a, g^b, g^{ab} \rangle \\ Mid_n &\stackrel{def}{=} \langle N, g, g^a, g^b, g^{r \cdot 2^{\lceil \frac{n}{2} \rceil} + [ab]_{\lceil \frac{n}{2} \rceil, 1}} \rangle \\ R_n &\stackrel{def}{=} \langle N, g, g^a, g^b, g^R \rangle \end{aligned}$$

(see Figure 3.3, No. (1), (2), (3)).

Theorem 11 *Under the assumption that factoring Blum integers is hard, the ensembles $\{Mid_n\}_{n \in N}$ and $\{R_n\}_{n \in N}$ are computationally indistinguishable.*

We remark that the proof of the above theorem will use ideas from Section 3.1 (and will not be based at all on Chapter 2).

Proof (sketch): We use the hybrid argument and define for every $0 \leq i \leq \lceil n/2 \rceil$ the hybrid

$$H^i_n = \langle N, g, g^a, g^b, g^{x \cdot 2^i + [ab]_{i,1}} \rangle$$

where $x \in_R [0, top^i_{N,g})$ (see Figure 3.3, No. (4)). Clearly, $H^0_n = R_n$ and $H^{\lceil n/2 \rceil}_n = Mid_n$.

Again, assuming the existence of an efficient algorithm D that distinguishes the extreme hybrids with a non-negligible gap, implies that there is an i for which D distinguishes H^i_n and H^{i+1}_n with a non-negligible gap (say the gap is greater than $\frac{1}{n^c}$, for some constant c). We use D in order to factor Blum-integers.

[3] We allow ourselves to use freely the transition between strings and numbers.

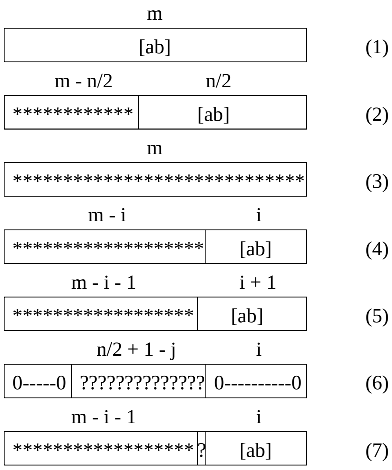

Figure 3.3: We denote random bits by '*', bits from $[ab]$ by '[ab]', and unknown bits of S by '?'. (1), (2), (3), (4), (5) show the exponents of DDH_n, Mid_n, R_n, the hybrids H_n^i and H_n^{i+1}, respectively. (6), (7) show the exponent of Y', on the j'th stage of the procedure that discovers S, before and after the randomization, respectively.

An outline of the proof: Assuming that D distinguishes the hybrids H_n^i and H_n^{i+1} (see Figure 3.3, No. (4), (5)), implies that D distinguishes as well (in fact, with a doubled gap) between

$$H_n^{i+1} = \langle N, g, g^a, g^b, g^{x \cdot 2^{i+1} + [ab]_{i+1,1}} \rangle$$

and

$$\overline{H}_n^i \stackrel{def}{=} \langle N, g, g^a, g^b, g^{x \cdot 2^{i+1} + \overline{[ab]}_{i+1} \cdot 2^i + [ab]_{i,1}} \rangle$$

where $\overline{[ab]}_{i+1}$ denotes $1 - [ab]_{i+1}$ and where $x \in_R [0, top_{N,g}^{i+1})$ (i.e. $|x| = m - i - 1$).

Note that $\xi = ord_N(g)$ is not known to us. Therefore, even if a and b are known, we cannot compute $[ab] = ab \bmod \xi$ and in particular $[ab]_{i,1}$ (but we can compute $g^{[ab]} = g^{ab}$). Let $e = x' \cdot 2^i + ab$, where $x' \in_R [0, top_{N,g}^i)$ (i.e. $|x'| = m - i$) and let $e' = e \bmod \xi = x' \cdot 2^i + [ab]$ (assume for now that $x' \cdot 2^i + [ab] < \xi$). Observe that e can be computed given a, b and x', and although e' is not known, $g^{e'} = g^e$ can be efficiently computed. The key point is that $e'_{i,1} = [ab]_{i,1}$ and that $e'_{i+1} = [ab]_{i+1} \oplus x'_1$. Therefore, we have that

- if $x'_1 = 0$ then $\langle N, g, g^a, g^b, g^e \rangle$ is distributed according to H_n^{i+1}.
- if $x'_1 = 1$ then $\langle N, g, g^a, g^b, g^e \rangle$ is distributed according to $\overline{H}_n^i$.

Given a Blum integer N, we find $S = DL_{N,g}(g^N)$ in a method which exploits the former observations. As in the proof of Theorem 8, we reveal S bit after bit from right to left in $|S|$ stages (while operating on $Y = g^N$). On the j'th stage, we assume the $(j-1)$'st least significant bits of S are already known. We zero them, and shift the exponent S, such that its j'th bit will be located at the $(i+1)$'st position (this is done as in the proof of Theorem 8). Call Y' the element obtained after these transformations (see Figure 3.3, No. (6)). The j'th bit of S is discovered by querying D on several randomized instances of $Y' \cdot g^{ab}$. A randomized instance will (roughly) be of the form $Y' \cdot g^{ab} \cdot g^{x \cdot 2^{i+1}}$, where $x \in_R [0, top_{N,g}^{i+1})$ (see Figure 3.3, No. (7)). Again, we have to deal with several difficulties:

1. First, without knowing ξ (the exact order of g), how can we choose a and b from $[0,\xi)$, and x (during the randomization) from $[0, top_{N,g}^{i+1})$? We therefore obtain an approximation of ξ, by guessing its size and its $\lceil \alpha \log n \rceil$ most significant bits.

2. Second, we want that ξ will be large enough in order to avoid a problem of a wrap around.[4] Specifically, we need that $m = \lceil \log \xi \rceil$ will be at least $n - \lceil \alpha \log n \rceil$. Using Proposition 3, this is assured with high probability.

3. Third, we should be careful not to cause a wrap around ξ when adding (during the randomization) the shifted S, [ab] and $x \cdot 2^{i+1}$ (where $x \in_R [0, top_{N,g}^{i+1})$). For this purpose, we do the following:

   - We guess the $\lceil \alpha \log n \rceil$ most significant bits of $[ab]$ (i.e. $[ab]_{m,m-\lceil \alpha \log n \rceil+1}$) and compute $Z = g^{ab-[ab]_{m,m-\lceil \alpha \log n \rceil+1}}$. Since we multiply Y' by g^{ab} in order to add $[ab]_{i+1,1}$ to the exponent and the rest of the bits of $[ab]$ are masked anyway (by $x \cdot 2^{i+1}$), we can multiply Y' by Z instead of g^{ab}. Note that this forces us to use the same a and b throughout the whole procedure.
   - We guess up-to $2\lceil \alpha \log n \rceil$ bits from S and then truncate them, thus reducing the size of the unknown part in S. The goal is to make the shifted S smaller than $2^{n-2\lceil \alpha \log n \rceil}$ which is (by comment 2 above) smaller than $2^{m-\lceil \alpha \log n \rceil}$. Thus, with high probability over the choice of $x \in_R [0, top_{N,g}^{i+1})$, when $x \cdot 2^{i+1} + [ab]_{m-\lceil \alpha \log n \rceil,1}$ is added to the shifted S, the sum will not cause a wrap around.

A more formal description: Given a Blum integer $N = P \cdot Q$, we pick a random quadratic residue g in Z_N^* in the same way as in the proof of Theorem 8. Let ξ denote $ord_N(g)$. By Proposition 3, we know that with high probability,

$$\xi > \frac{1}{n^\alpha} \cdot (P-1)(Q-1) \tag{3.2}$$

where $\alpha = 4c + 4$ (thus the probability that Equation 3.2 does not hold is $O(\frac{1}{n^{4c/3}})$). For the rest of the proof we assume that Equation 3.2 holds.
The procedure that discovers S is composed of two main stages, of which the first one includes initializations required prior to the actual computation of S, which is conducted in the second stage of the procedure.

<u>Stage 1: preprocessing</u>

1. We obtain an approximation of ξ, by guessing $m = \lceil \log \xi \rceil$, and the upper $\lceil \alpha \log n \rceil$ bits of ξ, that is, $\xi_{m,m-\lceil \alpha \log n \rceil+1}$.

   - We denote by $\tilde{\xi}$ the approximation of ξ, equal to $\xi_{m,m-\lceil \alpha \log n \rceil+1} \cdot 2^{m-\lceil \alpha \log n \rceil}$ (i.e., $\xi = \tilde{\xi} + \xi_{m-\lceil \alpha \log n \rceil,1}$).

[4]Note that this difficulty is not encountered in the proofs of Theorems 1 and 8. The reason is that in the distribution Mid_n (and the hybrids H_n^i), the distribution of the exponent of the last element depends on $ord_N(g)$, whereas in Theorem 8 and in the Main Lemma (in Chapter 2), the distribution of the exponents does not depend on $ord_N(g)$.

- For $i < m - \lceil \alpha \log n \rceil$ we have that

$$top^i_{N,g} = \xi_{m,i+1} = \xi_{m,m-\lceil \alpha \log n \rceil+1} \cdot 2^{m-i-\lceil \alpha \log n \rceil} + \xi_{m-\lceil \alpha \log n \rceil, i+1}$$

We denote by $\widetilde{top}^i_{N,g} = \xi_{m,m-\lceil \alpha \log n \rceil+1} \cdot 2^{m-i-\lceil \alpha \log n \rceil}$ the approximation of $top^i_{N,g}$ (i.e., $top^i_{N,g} = \widetilde{top}^i_{N,g} + \xi_{m-\lceil \alpha \log n \rceil, i+1}$). Note that for $i \geq m - \lceil \alpha \log n \rceil$, the value of $top^i_{N,g}$ is known precisely.

2. We pick random $a, b \in [0, \tilde{\xi})$.

3. We guess $[ab]_{m,m-\lceil \alpha \log n \rceil+1}$, and compute $Z = g^{ab-[ab]_{m,m-\lceil \alpha \log n \rceil+1}} = g^{[ab]_{m-\lceil \alpha \log n \rceil,1}}$.

4. We guess the l least significant bits of S, where $l \stackrel{def}{=} max(i - \lceil n/2 \rceil + 2\lceil \alpha \log n \rceil, 0)$. We choose l in that way in order that $\lceil \frac{n}{2} \rceil + 1 - l + i$ (the size of S after truncating its l least significant bits and shifting it by i positions to the left) would be smaller than $n - 2\lceil \alpha \log n \rceil$. Note that since $i \leq \lceil \frac{n}{2} \rceil$, we have that $l \leq 2\lceil \alpha \log n \rceil + 1$.

5. We approximate by sampling

$$\beta_{N,g,a,b} \stackrel{def}{=} \Pr[D(N, g, g^a, g^b, g^{x \cdot 2^{i+1} + [ab]_{i+1,1}}) = 1]$$

and

$$\gamma_{N,g,a,b} \stackrel{def}{=} \Pr[D(N, g, g^a, g^b, g^{x \cdot 2^{i+1} + \overline{[ab]}_{i+1} \cdot 2^i + [ab]_{i,1}}) = 1]$$

where the probabilities are taken over the choice of x uniformly from $[0, top^{i+1}_{N,g})$ (whereas N, g, a, and b are fixed). Note that when sampling, we use the approximation $\widetilde{top}^{i+1}_{N,g}$ in order to choose x's. Call $\tilde{\beta}$ and $\tilde{\gamma}$ the approximations of $\beta_{N,g,a,b}$ and $\gamma_{N,g,a,b}$ obtained in this way. Assuming the guesses from Step 1 were correct and using a standard averaging argument, we have that with probability at least $\frac{1}{n^c}$ (over the choice of N, g, a and b), the value $|\tilde{\gamma} - \tilde{\beta}|$ is greater than $\frac{1}{n^c}$ (w.l.o.g. say that $\tilde{\gamma} > \tilde{\beta}$). For the rest of the procedure, we assume that this is the case (i.e., $\tilde{\gamma} - \tilde{\beta} > \frac{1}{n^c}$).

Note that during Stage 1 no more than a polynomial number of guesses were made.

Stage 2: recovering S

1. Let $j_0 = l$. For $j = j_0 + 1$ to $\lceil \frac{n}{2} \rceil + 1$ find the j'th bit of S (assuming the $j-1$ least significant bits $S_{j-1,1}$ are known):

   (a) By operating on $Y = g^N$, zero the $j-1$ least significant bits of S and shift it such that its j'th bit will be located in the $(i+1)$'st position (this is done as in the proof of Theorem 8). Call Y' the element obtained after these operations.

   (b) Derive the j'th bit of S by querying D on $t(n) = n^{2c+4}$ *randomized* instances of $Y'' = Y' \cdot Z$ (recall that $Z = g^{[ab]_{m-\lceil \alpha \log n \rceil,1}}$ was computed during Step 3 of the first stage of the procedure).

i. Pick $x_1, \ldots, x_{t(n)} \in_R [0, \widetilde{top}^{i+1}_{N,g})$.

ii. For every $0 \leq q \leq t(n)$, let $b_q = D(N, g, g^a, g^b, Y'' \cdot g^{x_q \cdot 2^{i+1}})$. Let M denote the mean $\frac{\sum_{q=1}^{t(n)} b_q}{t(n)}$.

iii. If $M < \frac{\tilde{\gamma}-\tilde{\beta}}{2}$ infer that $S_j = 0$, otherwise (if $M \geq \frac{\tilde{\gamma}-\tilde{\beta}}{2}$), infer that $S_j = 1$.

2. After finding all bits of S, check whether $g^S = Y$.

It can be shown (analogously to Claim 3.1 in the proof of Theorem 1) that for a uniformly distributed Blum integer N, the probability to factor N, given that the guesses of the preprocessing stage were correct, is $\Omega(\frac{1}{n^c})$. Therefore, the probability to factor N is at least $\frac{1}{poly(n)} \cdot \frac{1}{n^c}$. ■

Chapter 4

Application to Pseudorandom Generators

An immediate application of Theorem 1 is an efficient factoring-based pseudorandom generator which nearly doubles the length of its input. The key tool used is a construction by Goldreich and Wigderson of a tiny family of functions which has good extraction properties [GW]. We discuss as well how the parameters of the generator (a composite $N \in N_n$ and an element $g \in Z_N^*$) can be chosen in a randomness-efficient way (which is polynomial-time). In particular, we present a method of choosing a random n-bit prime using only a linear number of random bits. This translates to a hitting problem which can be solved efficiently using methods described in the survey of Goldreich on samplers ([G2]).

4.1 Our construction vs. the HSS construction

Looking at Theorem 1, the first application that comes to mind is a pseudorandom generator that takes a seed r of length $\lceil n/2 \rceil$ and outputs $g^r \bmod N$ (for a fixed pair $\langle N, g \rangle$ in P_n). However, the output of the above so-called "pseudorandom generator" is not really pseudorandom. Even though it is computationally infeasible to distinguish between it and the distribution $g^R \bmod N$ (for a random R in $[0, ord_N(g))$), we are not guaranteed that it cannot be easily told apart from the uniform distribution on n-bit strings. The same applies for a "pseudorandom generator" implied directly by Theorem 5 (of [HSS]), which takes a seed x of length n, and outputs $g^x \bmod N$ followed by $x_{\lceil n/2 \rceil, 1}$ (again, for fixed $\langle N, g \rangle$ in P_n).

Denote by $Half_{N,g}$ the distribution $g^r \bmod N$, where r is uniformly distributed over $\lceil n/2 \rceil$-bit strings, and by $Full_{N,g}$ the distribution $g^R \bmod N$, where R is uniformly distributed over $[0, ord_N(g))$. Observe that the "amount of randomness" that $Full_{N,g}$ encapsulates in it is high, in the sense that it does not assign a too large probability mass to any value. More formally, we measure the "amount of randomness" in terms of *min-entropy.*

Definition 12 *Let X be a random variable. We say that X has min-entropy k, if for every x we have that* $\Pr(X = x) \leq 2^{-k}$.

The distribution $Full_{N,g}$ has min-entropy greater than κ, where

$$\kappa \stackrel{def}{=} \kappa(N, g) \stackrel{def}{=} \lfloor \log(ord_N(g)) \rfloor$$

The following fact is an immediate consequence of Proposition 3:

Fact 4 *Let $\langle N, g\rangle$ be a uniformly distributed in P_n, then $\kappa \leq n - \frac{1}{2}\log^2 n$ with negligible probability.*

Using hash functions which have good extracting properties, we are able to "smoothen" the distribution $Full_{N,g}$, and extract from it an almost uniform distribution over strings of length $n - \log^2 n$. To be more formal, we use a family of functions F having an extraction property, satisfying that for all but an ϵ fraction of the functions in F, a distribution over strings of length n having min-entropy $n - \frac{1}{2}\log^2 n$ is mapped to a distribution over strings of length $n - \log^2 n$ which is ϵ-close to uniform (we refer to ϵ, which is generally taken to be negligible in n, as the quality-parameter of the extraction property achieved by F). The price we pay for the use in extractors, hides in a lower expansion factor of the pseudorandom generators. Specifically, we need to use a part of the random seed in order to choose a random function in the family F we are using. Additionally, we lose a small quantity of pseudorandom bits when applying the extracting function.

Hastad et.al. used a universal family of hash functions [CW] in their construction of a pseudorandom generator. The quality parameter achieved by this family of functions is exponentially small in n (and therefore has the best possible quality). However, a universal family of hash functions has to be large: exponential in n. Thus the number of random bits needed to generate (and represent) a function in this family is polynomial in n, resulting in a considerably large loss in the expansion factor of their generator.

More recently, Goldreich and Wigderson [GW] presented an explicit construction of a family of functions, which exhibits a trade-off between the size of the family and the quality parameter ϵ of the extraction property it achieves. Specifically, they demonstrate a construction of a family of functions of size $poly(n/\epsilon)$ achieving the extraction property with quality ϵ (taking, for example, $\epsilon = n^{-\log n}$, yields a family of functions of very good quality - not exponentially small but still negligible in n, where each function in the family can be represented using $O(\log^2 n)$ bits).

4.1.1 The HSS construction

We present now the construction of the HSS pseudorandom generator. Even though the expansion factor of the HSS-generator can be increased using the function families of [GW], we bring the original construction which uses universal hashing.

Construction 1 ([HSS]): *Let $H_n^{\kappa-\log^2 n}$ be a universal family of hash functions which maps n-bit strings to $(\kappa - \log^2 n)$-bit strings, and suppose that every $h \in H_n^{\kappa-\log^2 n}$ is represented using $2n$ bits. The mapping $G_{N,g}^{HSS} : \{0,1\}^{3n} \rightarrow \{0,1\}^{3.5n-O(\log^2 n)}$ is defined as follows:*[1]
Let $x \in \{0,1\}^n$ and let $h \in H_n^{\kappa-\log^2 n}$. Then,

$$G_{N,g}^{HSS}(h,x) \stackrel{def}{=} \left(h, h(g^x), x_{\lceil n/2\rceil,1}\right)$$

[1] As a matter of fact, in the HSS construction, N is restricted to be the multiplication of two *safe primes*, see [HSS].

Note that applying the hash function causes a loss of $O(\log^2 n)$ bits in the length of the output. Therefore, the fact that $\lceil n/2 \rceil$ bits are simultaneously hard in $f_{N,g}$ (and not just $O(\log^2 n)$) is essential for the construction of $G^{HSS}_{N,g}$, since the addition of the $\lceil n/2 \rceil$ least significant bits to the output of the generator more than compensates for the loss of $O(\log^2 n)$ bits. Observe that the expansion factor obtained by the HSS-construction is approximately $\frac{7}{6}$ (whereas using the [GW] construction one can improve it to approximately $\frac{3}{2}$).

4.1.2 Our construction

We now present our construction of a pseudorandom generator achieving an expansion factor of nearly 2. But first we need to bring the exact formulation of the GW result (the GW construction itself is brought in Appendix C).

Theorem 12 (Extractors for High Min-Entropy [GW]): *Let $k < n$ and $m < n - k$ be integers, and $\epsilon > max\{2^{-(m-O(k))/O(1)}, 2^{-(n-m-O(k)/O(1))}\}$. (In particular, $m < n - O(k)$.) There exists a family of functions, each mapping $\{0,1\}^n$ to $\{0,1\}^m$, satisfying the following:*

- *each function is represented by a unique string of length $O(k + \log(\frac{1}{\epsilon}))$.*
- *there exists a logspace algorithm that, on input a description of a function f and a string x, returns $f(x)$.*
- *for every random variable $X \in \{0,1\}^n$ of min-entropy $n - k$, all but an ϵ-fraction of the functions f in the family satisfy*

$$SD(f(X), U_m) \leq \epsilon$$

In particular, taking $k = \frac{1}{2}\log^2 n$, $m = n - \log^2 n$ and $\epsilon = n^{-\log n}$, Theorem 12 implies the existence of a family of functions F, mapping $\{0,1\}^n$ to $\{0,1\}^m$, where each function $f \in F$ can be represented by a string of length $O(\log^2 n)$. We are now ready to exhibit our construction of a pseudorandom generator which uses the family F.

Construction 2 *We define the mapping $G_{N,g} : \{0,1\}^{\lceil \frac{n}{2} \rceil + O(\log^2 n)} \rightarrow \{0,1\}^n$ as follows: Let $x \in \{0,1\}^{\lceil \frac{n}{2} \rceil}$ and let $f \in F$. Then,*

$$G_{N,g}(f, x) \stackrel{def}{=} (f, f(g^x))$$

Theorem 13 *$G_{N,g}$ is a pseudorandom generator.*

Proof: Obviously $G_{N,g}$ is efficiently computable (since every $f \in F$ can be evaluated in polynomial time). Let F denote the random variable obtained by selecting uniformly a function f in the family F (although bearing the same name, it will be clear from the context whether we mean the random variable F or the function family F). Observe that

$$G_{N,g}(U_{\lceil \frac{n}{2} \rceil + O(\log^2 n)}) \equiv (F, F(Half_{N,g}))$$

$$U_n \equiv (F, U_m)$$

Consider now the hybrid $(F, F(Full_{N,g}))$. The theorem is directly implied from the following two claims:

Claim 13.1 *The ensembles*

$$\{(F, F(Half_{N,g}))\}_{n\in N}$$

and

$$\{(F, F(Full_{N,g}))\}_{n\in N}$$

are computationally indistinguishable.

Proof: The existence of an efficient distinguisher D between the above ensembles implies the existence of an efficient distinguisher D' between the ensembles $Half_n$ and $Full_n$: On input $\langle N, g, y\rangle$, the distinguisher D' picks an extractor f uniformly from F and outputs D's answer on input $(f, f(y))$. □

Claim 13.2 *The ensembles*

$$\{(F, F(Full_{N,g}))\}_{n\in N}$$

and

$$\{(F, U_m)\}_{n\in N}$$

are statistically close.

Proof: The third property of Theorem 12, ensures that for all but an ϵ-fraction of the functions f in F, the statistical difference between the ensembles $F(Full_{N,g})$ and U_m is bounded from above by ϵ. Thus, the statistical difference between $(F, F(Full_{N,g}))$ and (F, U_m) is no more than 2ϵ. Since ϵ was taken to be $n^{-\log n}$, we have that the ensembles $(F, F(Full_{N,g}))$ and (F, U_m) are statistically close. □

■

4.1.3 Increasing the expansion factor of the generator

The pseudorandom generator described above almost doubles the length of its input. However, such a small expansion factor has limited value in practice. Still, it is well known that even a pseudorandom generator G producing $n+1$ bits from an n-bit seed can be used in order to construct a pseudorandom generator G' having any arbitrary polynomial expansion factor (see e.g. [G, Sec. 3.3 Thm. 3.3.3]). Unfortunately, the cost of the latter transformation is rather high: Producing each bit in G''s output requires one evaluation of G. Nevertheless, since our generator $G_{N,g}$ has an expansion factor of nearly 2 to start with, we can do a little better than that: $G_{N,g}$ can be used to construct a generator $G'_{N,g}$ having an arbitrary polynomial expansion factor, such that for every $n/2 - O(\log^2 n)$ bits of output, one evaluation of $G_{N,g}$ is required. We remark that the issue of increasing the expansion factor of $G_{N,g}$ is relevant mostly due to the need to randomly pick the parameters N and g, which requires $O(n)$ additional random bits (as will be explained in the subsequent section). Our suggestion is to pick randomly N and g, set them once and for all, and construct a pseudorandom generator having a large expansion factor using this specific $G_{N,g}$. This way the cost of picking N and g becomes negligible (compared to our "profit" from the new generator).

We describe now how in general one uses a generator $G : \{0,1\}^n \to \{0,1\}^{n+l(n)}$ (for an integer function l) to construct a generator $G' : \{0,1\}^n \to \{0,1\}^{l(n)\cdot p(n)}$, for any arbitrary polynomial $p(\cdot)$.

Construction 3 *Let $l : N \to N$ be an integer function satisfying $l(n) > 0$ for every $n \in N$, let $p(\cdot)$ be a polynomial and let $G : \{0,1\}^n \to \{0,1\}^{n+l(n)}$ be a deterministic polynomial-time algorithm. Define $G'(s) = \tau_1 \ldots \tau_{p(n)}$, where $s_0 \stackrel{\text{def}}{=} s$, the string s_i is the n-bit long suffix of $G(s_{i-1})$ and τ_i is the $l(n)$-bit long prefix of $G(s_{i-1})$, for every $1 \leq i \leq p(n)$ (i.e., $\tau_i s_i = G(s_{i-1})$).*

Theorem 14 *If G is a pseudorandom generator then so is G'.*

Theorem 14 is a generalization of Theorem 3.3.3 proven in [G] (regarding a generator producing $n+1$ bits from an n bit seed). Observe that for every $l(n)$ output bits of G', one evaluation of G is required. Using our generator $G_{N,g}$ as the building block, we obtain a generator $G'_{N,g}$ that expands input of size $n/2 + O(\log^2 n)$ to output of size n^c using approximately $\frac{n^c}{2}$ modular multiplications.

4.2 An efficient choice of the parameters

In order to use in practice the generator $G_{N,g}$ we need to generate the parameters N and g from a primary seed in an "efficient" way, where by "efficient" we mean that both the running time and the amount of randomness used should be as small as possible. The major challenge is to generate efficiently two uniformly distributed primes P and Q, in order to obtain a random $N = P \cdot Q$ in N_n. A random element g in Z_N^* can be chosen using $O(n)$ random coins by picking a random number in $\{0,1\}^{n+\log^2 n}$ and reducing it modulo N (only with negligible probability the element obtained will not be relatively prime to N). We describe now a general method by which we can pick a random n-bit prime in polynomial time, using only a linear number of random coins.

4.2.1 Picking a random n-bit prime using $O(n)$ random bits

The trivial algorithm to choose a random n-bit prime is to repeat the following two stages until a prime x is output.

1. Choose a random integer x in $\{0,1\}^n$.

2. Test whether x is a prime. If it is, stop and output x.

Since the density of primes in $\{0,1\}^n$ is approximately $\frac{1}{n}$, the expected number of times that the above loop is performed is approximately n. Even assuming that we have a deterministic primality test, the above algorithm requires an expected $O(n^2)$ random bits. We now show how to perform poly(n) dependent iterations of the loop using only $O(n)$ random bits (rather than doing $O(n)$ independent iterations using $O(n^2)$ random bits). We will use, however, a probabilistic primality tester of Bach [Bach], which is a randomness-efficient version of the Miller-Rabin [M, R] primality tester.

Theorem 15 (randomness efficient primality tester [Bach]): *There exists a probabilistic polynomial-time algorithm that on input P uses $|P|$ random bits so that if P is a prime then the algorithm always accepts, and otherwise (i.e. P is composite) the algorithm accepts with probability at most $\frac{1}{\sqrt{P}}$.*

Corollary 16 *There exists a probabilistic polynomial-time algorithm that uses $2n$ random coins such that*

1. *with probability $\Theta(\frac{1}{n})$ outputs an n-bit prime. Furthermore, the probability to output a specific prime is 2^{-n}.*
2. *with probability $1 - \Theta(\frac{1}{n}) - \exp(-n)$ outputs a special failure sign, denoted $\perp$.*
3. *with probability at most $2^{-n/2}$ outputs a composite.*

4.2.2 A hitting problem

We refer to the algorithm guaranteed from Corollary 16 as a black-box. We associate every string $s \in \{0,1\}^{2n}$ with the output of the black-box given s as its random coins. Denote by W the set of strings in $\{0,1\}^{2n}$ which are associated with an n-bit prime. Corollary 16 implies that the density of W within $\{0,1\}^{2n}$ is $\frac{1}{n}$. The problem of uniformly picking an n-bit prime translates to a hitting problem, where we need to find a string $s \in W$ (which is subsequently used as random input for the black-box in order to yield a prime). An additional requirement is that the distribution of primes obtained in this way will be very close to uniform. Our goal now is to find an algorithm that hits W, whose randomness complexity is linear in n. The methods we use are described in the survey of Goldreich [G2] on samplers and will be adapted to (and analyzed in) our specific setting.

A pairwise-independent hitter Our first attempt uses a pairwise independent sequence of m uniformly distributed strings in $\{0,1\}^{2n}$. Such a sequence can be generated in the following way: We associate $\{0,1\}^{2n}$ with $F \stackrel{def}{=} GF(2^{2n})$, and select independently and uniformly $s, r \in F$. We let the i'th element in the sequence be $e_i = s + i \cdot r$ (with the arithmetic of F).[2] It can be easily seen that the generated sequence is indeed pairwise-independent.

Theorem 17 (A pairwise-independent hitter): *Let δ be an error parameter satisfying that $1/\delta = \text{poly}(n)$. There exists an efficient algorithm which uses $4n$ random coins for which the following holds:*

- *The probability to output a prime is at least $1 - \delta$.*
- *The probability to output a composite is at most $\exp(-n)$.*
  (With probability $1 - \delta - \exp(-n)$ a failure sign $\perp$ is output.)
- *The probability to output a specific prime is at least 2^{-n} and at most $\frac{n}{\delta} \cdot 2^{-n}$.*

[2] Note that the amount of pairwise independents strings one can generate in this way is limited to $2^{2n} - 1$.

Proof: We generate $m \stackrel{def}{=} \frac{n}{\delta}$ pairwise-independent samples $e_1, \ldots, e_m$ each uniformly distributed in $\{0,1\}^{2n}$, and run the black-box using each of the e_i's as random bits. Clearly, this procedure is efficient, since m is polynomial in n. Let

$$\zeta_i \stackrel{def}{=} \begin{cases} 1 & \text{if the black-box (using } e_i \text{ as random bits) outputs a prime} \\ 0 & \text{otherwise} \end{cases}$$

Corollary 16 implies that the expectation of ζ_i is $\frac{1}{n}$. Using Chebishev's Inequality we have

$$\begin{aligned} \Pr\left(\sum_{i=1}^{m} \zeta_i = 0\right) &\leq \Pr\left(\left|\frac{m}{n} - \sum_{i=1}^{m} \zeta_i\right| \geq \frac{m}{n}\right) \\ &\leq \frac{m \cdot \frac{1}{n}(1 - \frac{1}{n})}{(\frac{m}{n})^2} \\ &\leq \delta \end{aligned}$$

Regarding the probability to output a composite, using a union bound we get

$$\begin{aligned} \Pr[\text{a composite is output}] &= \Pr[\exists i \text{ s.t. } e_i \text{ yields a composite}] \\ &\leq \sum_{i=1}^{m} \cdot \Pr[e_i \text{ yields a composite}] \\ &\leq \frac{n}{\delta} \cdot \exp(-n) = \exp(-n) \end{aligned}$$

where the last inequality follows from the third item of Corollary 16 and from the fact that for every i the point e_i is uniformly distributed over $\{0,1\}^{2n}$.

As for the probability that a specific prime p is output, the first item of Corollary 16 implies that for every i, the probability that p is output using e_i as random coins is exactly 2^{-n} (since e_i is uniformly distributed). Thus,

$$\Pr[p \text{ is output}] \geq \Pr[e_1 \text{ yields } p] = 2^{-n}$$

On the other hand, using a union bound,

$$\Pr[p \text{ is output}] \leq \sum_{i=1}^{m} \cdot \Pr[e_i \text{ yields } p] = \frac{n}{\delta} \cdot 2^{-n}$$

■

If we were willing to settle with a polynomially small error δ the above algorithm would be sufficient for us. However, in order to achieve an overwhelming probability of success (i.e., $\delta = 2^{-n}$) we must take a somewhat more complex approach, which involves random walks on expander graphs (for definition and construction of expanders as well as the major theorem concerning random walks on expanders see Appendix B).

A combined hitter From the pairwise independent hitter emerges another hitting problem: Let W' be the set of strings in $\{0,1\}^{4n}$, that when supplied to the pairwise-independent hitter (with a constant error parameter δ) as a random seed, makes it hit W (i.e. yield a prime). From the first item of Theorem 17 we get that the density of W' within $\{0,1\}^{4n}$ is greater than $1-\delta$. Our new goal is to hit W' with an overwhelming probability of success.

In order to do that, we generate a random walk on an expander with vertex set $\{0,1\}^{4n}$, and use each of the vertices along the path as a seed for the pairwise-independent hitter. Taking advantage of the hitting property of expanders (see appendix B), we will have that a random walk of linear length (in n) will be sufficient in order to hit W'. Details follow.

Theorem 18 *There exists an efficient algorithm which uses $O(n)$ random coins such that the following holds:*

- *The probability that no prime is output is* $\exp(-n)$.
- *The probability that a composite is output is* $\exp(-n)$.
- *The probability that a specific prime is output is at least* 2^{-n} *and at most* $\text{poly}(n) \cdot 2^{-n}$.

Proof: We use an explicit construction of expander graphs with vertex set $\{0,1\}^{4n}$, degree d and second eigenvalue λ such that $\lambda/d < 0.1$. We generate a random walk of (edge) length n on this expander using $O(n)$ random coin flips ($4n$ bits are used to generate the initial vertex and $\log d$ bits are used to obtain each additional vertex on the path). We use each of the vertices $s_1, \ldots, s_n$ along the path as random coins for the pairwise-independent hitter which makes $m = 3n$ trials (i.e., for every $1 \leq i \leq n$ we generate a pairwise-independent sequence $e_1^i, \ldots, e_m^i$ from s_i and run the black-box using each one of the e_j^i's as random bits). Recall that W' was defined to be the set of coin tosses which make the pairwise-independent hitter output a prime. From Item 1 of Theorem 17 we have that $|W'|/2^{4n} \geq \frac{2}{3}$. Using Theorem ... the probability that all vertices of a random path reside in W'^c is bounded from above by $(0.34+0.1)^n < 2^{-n}$. Thus,

$$\Pr[\text{no prime is output}] < 2^{-n}$$

Let us now compute the probability to output a composite.

$$\begin{aligned}
\Pr\left[\text{a composite is output}\right] &= \Pr[\exists i \text{ s.t. } s_i \text{ yields a composite}] \\
&\leq \sum_{i=1}^{n} \cdot \Pr[s_i \text{ yields a composite}] \\
&\leq n \cdot \exp(-n)
\end{aligned}$$

where the last inequality follows from the second item of Theorem 17 and from the fact that, for every i, the seed s_i is uniformly distributed.

In order to bound the probability that a specific prime p is output, observe that for every i and j, the point e_j^i (i.e., the j'th point in the sequence of pairwise-independent strings generated from s_i) is uniformly distributed in $\{0,1\}^n$. Thus,

$$\Pr[p \text{ is output}] \geq \Pr[e_1^1 \text{ yields } p] = 2^{-n}$$

On the other hand, applying a union bound we get

$$\Pr[p \text{ is output}] \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \Pr[e_j^i \text{ yields } p] = n \cdot m \cdot 2^{-n} = 3n^2 \cdot 2^{-n}$$

■

4.2.3 Using almost uniformly distributed primes

Although the algorithm guaranteed from Theorem 18 does not yield uniformly distributed n-bit primes, the distribution of the primes it outputs is close to being uniform, in a sense that is quite sufficient for our needs: Denote by D_n the distribution of composites $N = P \cdot Q$ in N_n obtained by picking the primes P and Q using the algorithm of Theorem 18, and consider a slightly different factoring assumption, in which N is distributed according to D_n. Observe that the revised factoring assumption holds if and only if the original factoring assumption (with N uniformly distributed in N_n) holds: Let A be a probabilistic polynomial-time algorithm. Then, according to the third item of Theorem 18,

$$\frac{\sum_{N \in N_n} \Pr[\text{A factors } N]}{2^n} \leq \Pr[\text{A factors } N | N \sim D_n] \leq \frac{\sum_{N \in N_n} \Pr[\text{A factors } N]}{2^n/\text{poly}(n)} \quad (4.1)$$

where $N \sim D_n$ means that N is drawn according to the distribution D_n.
Note that the size of N_n is approximately $\frac{2^n}{n^2}$. Therefore,

$$\Pr[\text{A factors } N | N \in_R N_n] = \frac{n^2}{2^n} \sum_{N \in N_n} \Pr[\text{A factors } N] \quad (4.2)$$

From 4.1 and 4.2 we have that

$$\frac{\Pr[\text{A factors } N | N \in_R N_n]}{n^2} \leq \Pr[\text{A factors } N | N \sim D_n] \leq \frac{\Pr[\text{A factors } N | N \in_R N_n]}{n^2/\text{poly}(n)} \quad (4.3)$$

Thus, A does not violate the original factoring assumption if and only if it does not violate the revised factoring assumption.

Another important observation is that in all our theorems (and in particular, in Theorem 1), the values N and g are fixed throughout the whole proof. Thus, these theorems still hold when considering any distribution whatsoever of N (and g), provided that factoring is intractable for such a distribution.

Therefore, we have that under the standard factoring assumption (with N uniformly distributed in N_n), all our theorems hold even when the distribution of N is taken to be D_n, and the distribution of g is uniform over Z_N^*.

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Appendix A: Exact Analysis of Theorem 1

We show that the probability of error by the trimming rule is exponentially small. Suppose we want to trim the list L_j and that v^j_{min} is the correct value $S_{\lceil \frac{n}{2} \rceil +1, l(j)}$ (the analysis in the case where v^j_{max} is the correct value is analogous). The shifted S' is then a value smaller than $2^{i-\lceil \alpha \log n \rceil}$. Let us denote this value by δ. Recall that b_k is the answer of the oracle on the query $g^{shiftedS'+x_k}$ (see Step (3) of the trimming rule). We bound the expectation of b_k:

$$\begin{aligned} E(b_k) &= \Pr\left[D(g^{shiftedS'+x_k}) = 1\right] = \Pr\left[D(g^{x_k+\delta}) = 1\right] \\ &= \Pr\left[D(g^{x_k+\delta}) = 1 | 0 \le x_k \le 2^i - 1 - \delta\right] \cdot \Pr\left[0 \le x_k \le 2^i - 1 - \delta\right] \quad + \\ & \quad \Pr\left[D(g^{x_k+\delta}) = 1 | 2^i - 1 - \delta < x_k \le 2^i - 1\right] \cdot \Pr\left[2^i - 1 - \delta < x_k \le 2^i - 1\right] \end{aligned}$$

Let $y_k = x_k + \delta$. Then,

$$\begin{aligned} \Pr\left[D(g^{x_k+\delta}) = 1 | 0 \le x_k \le 2^i - 1 - \delta\right] &= \Pr\left[D(g^{y_k}) = 1 | \delta \le y_k \le 2^i - 1\right] \\ &\le \tfrac{2^i}{2^i - \delta} \cdot \Pr\left[D(g^{y_k}) = 1\right] = \tfrac{\beta \cdot 2^i}{2^i - \delta} \end{aligned}$$

and therefore

$$E(b_k) \le \tfrac{\beta \cdot 2^i}{2^i - \delta} \cdot \tfrac{2^i - \delta}{2^i} + 1 \cdot \tfrac{\delta}{2^i} = \beta + \tfrac{\delta}{2^i}$$

A standard application of Chernoff bound yields:

$$\begin{aligned} \Pr\left[\text{discard } v^j_{min}\right] &= \Pr\left[\textstyle\sum_{k=1}^{t} b_k > (\beta + \frac{\gamma-\beta}{2}) \cdot t\right] \\ &\le \Pr\left[\left|\textstyle\sum b_k - E(\sum b_k)\right| > (\beta + \frac{\gamma-\beta}{2}) \cdot t - E(\sum b_k)\right] \\ &= \Pr\left[\left|\textstyle\sum b_k - E(\sum b_k)\right| > \lambda \cdot E(\sum b_k)\right] \end{aligned}$$

for $\lambda = \frac{(\beta + \frac{\gamma-\beta}{2})t - E(\sum b_k)}{E(\sum b_k)}$.
Since

$$\begin{aligned} \tfrac{\lambda^2 E(\sum b_k)}{6} &= \tfrac{\left[(\beta + \frac{\gamma-\beta}{2})t - E(\sum b_k)\right]^2}{6 \cdot E(\sum b_k)} \ge \tfrac{\left[(\beta + \frac{\gamma-\beta}{2})t - (\beta + \frac{\delta}{2^i})t\right]^2}{6 \cdot (\beta + \frac{\delta}{2^i})t} \\ &= \tfrac{(\frac{\gamma-\beta}{2} - \frac{\delta}{2^i})^2}{6(\beta + \frac{\delta}{2^i})} \cdot t \ge \tfrac{(\frac{1}{2n^c} - \frac{1}{n^\alpha})^2}{6(\beta + \frac{1}{n^\alpha})} \cdot t \ge n \end{aligned}$$

for $\alpha = c + 1$ and for $t \ge n^{2c+4}$.

Therefore, the probability of discarding v^j_{min} from the list L_j is smaller than 2^{-n}. As mentioned above, a similar argument holds for the second case, where the correct candidate is v^j_{max}. Since for every $j_0 \le j \le n/2 + 1$ we use repeatedly the trimming rule for no more than n^α times, the overall probability of error is exponentially small.

Appendix B: Expanders and Random Walks

We now define expander graphs and families of expander graphs and describe an explicit construction of expanders due to Gabber and Galil [GG]. We also state the major theorem concerning random walks on expanders. Our exposition follows that of Goldreich in [G2].

B.1.1 Expanders

An (N, d, λ)-expander is a d-regular graph with N vertices so that the absolute value of all eigenvalues (except the biggest one) of its adjacency matrix is bounded by λ. A (d, λ)-family is an infinite sequence of graphs so that the n^{th} graph is a $(2^n, d, \lambda)$-expander. We are interested in explicit constructions of such families of graphs, which are *efficiently constructible*, by which we mean that there exists a polynomial-time algorithm that on input n (in binary), a vertex v and an index $i \in \{1, \ldots, d\}$, returns the i'th neighbor of v.

Gaber and Galil presented such a construction of a (d, λ)-family of expanders, for $d = 8$ and for some $\lambda < 8$ [GG]. Their expanders, however, are defined only for graph sizes which are perfect squares (i.e., only for even n's).

Construction 4 [Gaber-Galil] *Let $n = 2m$. The graph G_n is defined as follows: The vertex set includes all pairs in $Z_m \times Z_m$, and each node (x, y) is connected to the four nodes $(x+y, y)$, $(x + y + 1, y)$, $(x, x + y)$ and $(x, x + y + 1)$.*

In our applications we use (parameterized) expanders satisfying $\frac{\lambda}{d} < \alpha$ and $d = \text{poly}(1/\alpha)$, where α is an application-specific parameter. Such (parameterized) expanders are also efficiently constructible. For example, we may obtain them by taking paths of length $O(\log 1/\alpha)$ on an expander as in construction 4. Specifically, given a parameter $\alpha > 0$, we obtain an efficiently constructible (D, Λ)-family satisfying $\frac{\Lambda}{D} < \alpha$ and $D = \text{poly}(1/\alpha)$ as follows. We start with a constructible $(8, \lambda)$-family, set $k \stackrel{\text{def}}{=} \log_{8/\lambda}(1/\alpha) = O(\log 1/\alpha)$ and consider the paths of length k in each graph. This yields a constructible $(8^k, \lambda^k)$-family, and both $\frac{\lambda^k}{8^k} < \alpha$ and $8^k = \text{poly}(1/\alpha)$ indeed hold.

B.1.2 Random walks on Expanders

A fundamental discovery of Ajtai, Komlos, and Szemerédi [AKS] is that random walks on expander graphs provide a good approximation to repeated independent attempts to hit any arbitrary fixed subset of sufficient density (within the vertex set). The importance of this discovery stems from the fact that a random walk on an expander can be generated using much fewer random coins than required for generating independent samples in the vertex set. Precise formulations of the above discovery were given in [AKS, CoWi, GILVZ] culminating in Kahale's optimal analysis [Kah, Sec. 6].

Theorem 19 (Expander Random Walk Theorem [Kah, Cor. 6.1]): *Let $G = (V, E)$ be an expander graph of degree d and λ be an upper bound on the absolute value of all eigenvalues, save the biggest one, of the adjacency matrix of the graph. Let W be a subset of V and $\rho \stackrel{\text{def}}{=} |W|/|V|$. Then the fraction of random walks* (in G) *of* (edge) *length ℓ which stay within W is at most*

$$\rho \cdot \left(\rho + (1 - \rho) \cdot \frac{\lambda}{d}\right)^{\ell}$$

Appendix C: Tiny Families of Functions

We now bring the explicit construction of Goldreich and Wigderson of tiny families of functions designed for random variables with high min-entropy. Our exposition is taken from [GW].

C.1.3 The GW construction

We describe the GW construction of a family of functions, each mapping $\{0,1\}^n$ to $\{0,1\}^m$, such that all but an ϵ-fraction of them, map random-variables having min-entropy $n-k$ to a distribution whose distance from the uniform distribution is bounded by ϵ.

The construction uses an efficiently constructible expander graph, G, of degree d (power of two), second eigenvalue λ, and vertex set $\{0,1\}^m$, so that $\frac{\lambda}{d} \le \frac{\epsilon^2}{4 \cdot 2^{k/2}}$ (and $d = \mathrm{poly}(2^k/\epsilon)$). For every $i \in [d] \stackrel{\mathrm{def}}{=} \{1, 2..., d\}$ and $v \in \{0,1\}^m$, denote by $g_i(v)$ the vertex reached by moving along the i^{th} edge of the vertex v. The construction uses as well a universal hashing family, denoted H, that contains hash functions each mapping $(n-m)$-bit long strings to $[d]$.

Construction 5 *The family of functions, denoted F, is as follows: For each hashing function $h \in H$, we introduce a function $f \in F$ defined by*

$$f(x) \stackrel{def}{=} g_{h(\mathrm{lsb}(x))}(\mathrm{msb}(x))$$

where $\mathrm{lsb}(x)$ *returns the $n-m$ least significant bits of $x \in \{0,1\}^n$, and* $\mathrm{msb}(x)$ *returns the m most significant bits of x.*
Namely, $f(x)$ is the vertex reached from the vertex $v \stackrel{\mathrm{def}}{=} \mathrm{msb}(x)$ by following the i^{th} edge of v, where i is the image of the $n-m$ least significant bits of x under the function h.

As proven in [GW], Construction 5 above satisfies the requirements of Theorem 12 (stated in Chapter 4).

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