COMMENT ON TR01-29: A NOTE ON THE SUBGROUP MEMBERSHIP PROBLEM
FOR PSL(2, p)

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Abstract. We use a classification theorem of Dickson, to show that the subgroup membership problem for
PSL(2, p) is in NP \cap coNP.

1. Dickson’s Theorem

In his study of Linear groups [DL58], Dickson completely classified all the subgroups of PSL(2, p) (the
quotient of the group of 2 \times 2 matrices with entries in GF[p] of determinant 1, by its center). We can use
this theorem to show that the subgroup membership problem for PSL(2, p) is in NP \cap coNP. Specifically if we
can show that the factor groups of composition series of these groups are not isomorphic to the exceptional
family of simple groups then the membership problem for PSL(2, p) can be decided in NP \cap coNP (see
[CD01]). See Huppert’s book [Hup67] (Hauptsatz 8.27, p. 213) for the version of the theorem that we will
use.

Theorem 1.1 (Dickson). The subgroups of PSL(2, p) are isomorphic to one of the following families of
groups:
1. Elementary abelian p-groups;
2. Cyclic groups of order z, where z is a divisor of \frac{p^f + 1}{k} and k = (p^f - 1, 2);
3. Dihedral group of order 2z, where z is as defined in (2);
4. Alternating group A_4 this can occur only for p > 2 or when p = 2 and f \equiv 0 \mod 2;
5. Symmetric group S_4 this can occur only if p^{2f} \equiv 1 \mod 16.
6. Alternating group A_5 for p = 5 or p^{2f} \equiv 1 \mod 5;
7. A semidirect product of an elementary abelian group of order p^m with a cyclic group of order t, where t
is a divisor of (p^m - 1, p^f - 1);
8. The group PSL(2, p^m) for m a divisor of f, or the group PGL(2, p^m) for 2m a divisor of f.

Now we need to check that none of these families can have subgroups for which the exceptional groups
occur as factor groups of composition series. This is easily done. Groups (1) and (2) are abelian so we can
eliminate them. The dihedral groups, A_4, S_4 and the group family (7) are all solvable so again they can be
excluded. A_5 is simple. Finally PGL(2, p^m) has PSL(2, p^m) sitting inside it as a maximal normal subgroup
of index 2 if p is odd and if p = 2, then PSL(2, p^f) = PGL(2, p^f). Thus the exceptional family of groups
namely the Suzuki groups, the projective special unitary groups and the Ree groups do not occur as factor
groups of composition series of any subgroup of PSL(2, p^f). Thus we have as a corollary (see [CD01]):

Corollary 1.2. The subgroup membership problem for PSL(2, p^f) is in NP \cap coNP.

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References
1958.
[Hup67] Huppert, Bertram; Endliche Gruppen I, Die Grundlehren der Mathematischen Wissenschaften, Band 134 Springer-

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