Electronic Colloquium on Computational Complexity, Comment 1 on Report No. 049 (2001)

# Triangle-Freeness is Also Hard for Read-$r$ Times Branching Programs

S. Jukna

The $K_t$-freeness function is a boolean function $f_{t,N}$ in $N = \binom{n}{2}$ variables, corresponding to the edges of a complete $n$-vertex graph $K_n$, such that, for every subgraph $G$ of $K_n$, $f_{t,N}(G) = 1$ if and only if $G$ contains no $K_t$ as a subgraph. In [1] we have shown that every $r$-n.b.p. (these are usual nondeterministic branching programs with the restriction that along each path each variable can be tested at most $r$ times) for $f_{4,N}$ requires size exponential in $N/r^2$. For a constant $r$, this gives the first *truly exponential* (in the number of variables) lower bound for a very natural graph property.

Although not explicitly stated in [1], essentially the same argument gives the following (not truly but still exponential) lower bound also for the triangle–freeness function.

**Theorem.** *Every $r$-n.b.p. computing the triangle-freeness function $f_{3,N}$ has size $2^{\Omega(\sqrt{N}/r^2)}$.*

The proof of this theorem is the same as (and even simpler than) that of Theorem 3.3 for $f_{4,N}$ in [1]: it is enough to use the following lemma instead of Lemma 2.3.

**Lemma.** *There exists an absolute constant $c > 0$ and a set $\Delta$ of edge-disjoint triangles in $K_{n+1}$ with the following property. For every set of at most $k \leq 2^{c\lambda^4 n}$ $\lambda$-balanced partial red/blue colorings of the edges of $K_{1,n}$, at least $\Omega(\lambda^4 n)$ triangles in $\Delta$ are mixed under each of them.*

The proof of this lemma is almost identical with the proof of Lemma 2.2(ii) from [1]. For completeness we sketch the proof.

Take an arbitrary partial red/blue coloring of the edges in $K_{1,n} = \{v_0\} \times V$, and say that a vertex in $V$ is red (resp. blue) if it is incident to a red (resp. blue) edge. These sets are disjoint. Moreover, if the coloring is $\lambda$-balanced, both sets have at least $\lambda n$ vertices. Applying the Joining Lemma (Lemma 4.1 from [1]) we obtain a set $E$ of $N = \Theta(n/\lambda^2)$ edges in $V$ such that for *every* $\lambda$-balanced partial coloring of $K_{1,n}$, at least $n$ of the edges in $E$ connect red vertices with blue ones. Moreover, at most $L = O(n/\lambda^4)$ of pairs of edges in $E$ share a common endpoint.

Fix now an arbitrary set $\chi_1, \ldots, \chi_k$ of $k \leq 2^{c\lambda^4 n}$ $\lambda$-balanced partial colorings of $K_{1,n}$. Let $E_i$ be the set of all edges in $E$ connecting the red and blue vertices of the $i$-th coloring $\chi_i$; hence, $|E_i| \geq n$ for all $i = 1, \ldots, k$. It remains to show that there exists a set $M$ of mutually disjoint edges in $E$ (a matching in $V$) such that each of the sets $E_i \cap M$ are large enough.

To get the desired matching $M \subseteq E$, consider a "collision graph" $\mathcal{F}$ whose vertices are edges of $E$, and where two edges are adjacent if and only if they share a common endpoint. Hence, we can apply the general version of the Collision Lemma (Lemma 4.3 from [1]) with $r = 2$ and $\mu = \Omega(\lambda^2)$. Since in our case $p = p(\mathcal{F}) = \Omega(\mu N/L) = \Omega(\lambda^4)$ and $\mu p N = \Omega(\lambda^4 n)$, the lemma gives us an independent set $M \subseteq E$ in $\mathcal{F}$ (a matching in $V$) such that $|E_i \cap M| = \Omega(\lambda^4 n)$ for each $i = 1, \ldots, k$. Since $M$ is a matching, no two triangles in $\Delta = \{(v_0, e) : e \in M\}$ share a common edge. Since for each $i$ all the triangles in $\Delta_i = \{(v_0, e) : e \in E_i \cap M\}$ are mixed under $\chi_i$, we are done.

## Reference

[1] S. Jukna and G. Schnitger, Triangle-freeness is hard to detect, *Combinatorics, Probability and Computing* (to appear). A preliminary version appeared as ECCC Report Nr. 49 (2001).



ISSN 1433-8092