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Relative to P promise-BPP equals APP

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Abstract

We show that for determinictic polynomial time computation, oracle access to **APP**, the class of real functions approximable by probabilistic Turing machines, is the same as having oracle access to promise-**BPP**. First we construct a mapping that maps every function in **APP** to a promise problem in **prBPP**, and that maps complete functions to complete promise problems. Then we show an analogue result in the opposite direction, by constructing a mapping from **prBPP** into **APP**, that maps every promise problem to a function in **APP**, and mapping complete promise problems to complete functions. Second we prove that $\mathbf{P}^{\mathbf{APP}} = \mathbf{P}^{\mathbf{prBPP}}$. Finally we use our results to simplify proofs of important results on **APP**, such as the **APP**-completeness of the function f_{CAPP} that approximates the acceptance probability of a Boolean circuit, or the possibility (similarily to the case of **BPP**) to reduce the error probability for **APP** functions, or the conditionnal derandomization result **APP** = **AP** iff **prBPP** is easy.

1 Introduction

The complexity class **BPP** is sometimes considered to be the class of all feasible computation. Nevertheless, it has been conjectured that **BPP** does not have any complete sets. One reason for this is the existence of a relativized world where **BPP** (and other semantic classes) do not have complete sets (see [Sip82] and [HH86]). This is because **BPP** is a semantic class (on every input, a **BPP** machine must have either at least 3/4 or at most 1/4 accepting paths). Thus the canonical complete language $L = \{(M, x, 1^t)|\ M$ is a **BPP** machine and M accepts x in at most t steps$\}$ is not **BPP**-complete, because the predicate $-$ *M is a* **BPP** *machine* $-$ is undecidable, thus L is not in **BPP**.

One way around this difficulty is to consider promise problems i.e. problems that need to be solved only on instances where a certain promise holds. Thus the canonical complete language L together with the promise that M is indeed a **BPP** machine, is promise-**BPP** (denoted **prBPP**) complete. Indeed once you know that M is a **BPP** machine, a probabilistic algorithm can simulate machine M on input x, thus deciding, with high probability, wether M accepts x or not; this puts L in **prBPP**.

Another approach was introduced in [KRC00]. They introduced a natural generalization of **BPP**, namely the class **APP** of real-valued functions $f : \{0,1\}^* \rightarrow [0,1]$ that can be approximated within any $\epsilon > 0$, by a probabilistic Turing machine running in time polynomial in the

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input size and the precision $1/\epsilon$. They showed that **BPP** is exactly the subset of all Boolean functions in **APP**. Moreover they proved that computing the acceptance probability of a given Boolean circuit is an **APP**-complete problem.

This paper shows that relative to **P**, the two complexity classes **APP** and **prBPP** are equal, i.e. $\mathbf{P}^{\mathbf{APP}} = \mathbf{P}^{\mathbf{prBPP}}$. Our main tool is the graphe of a function. Recall that for a real valued function $f : \{0,1\}^* \to [0,1]$, its graphe is defined as beeing the set of triples $(1^k, x, y)$ such that $f(x) = y$ within distance $1/k$. Our first result states that computing the graphe of the **APP**-complete function f_{CAPP} (where f_{CAPP} on input a Boolean circuit outputs its probability of acceptance), together with the promise that all querries $f(x) \stackrel{?}{=} y$ made to graphe(f_{CAPP}) have the property that the distance between $f(x)$ and y is either "very small" or "rather large", is **prBPP** complete. Then we prove that computing the graphe of any function in **APP** toghether with the same promise, is in **prBPP**. This yields a mapping from **APP** to **prBPP**, mapping each function in **APP** to a promise problem in **prBPP**, and mapping complete functions to complete promise problems.

For the other direction we first prove that, for any real-valued function $f : \{0,1\}^* \to [0,1]$ such that the problem of computing its graphe (toghether with same the promise as above) is in **prBPP**, f is in **APP**. Second we construct a mapping from **prBPP** to **APP**, that maps every promise problem to a real-valued function, and mapping complete promise problem to complete functions. We then prove that $\mathbf{P}^{\mathbf{APP}} = \mathbf{P}^{\mathbf{prBPP}}$.

Finally we use our results to simplify proofs of important results about **APP**. Namely it is shown in [KRC00] that similarly to the case of **BPP**, the error probability for **APP** functions can be reduced exponentially. Their proof is rather technical and relies on a rather involved argument of repeated trials; on the other hand the idea of our proof is very simple: let f be any function in **APP**. We first map f to its corrsponding promise problem (Q, R) in **prBPP**. Then using the fact that error probability is possible in **prBPP**, we reduce the error probability of the Turing machine that solves (Q, R). Finally by mapping (Q, R) to its corresponding function in **APP**, we obtain a function f' which is the same as f, but such that the Turing machine that computes f' has exponentially small error probability.

Second it is proved in [KRC00] that the function f_{CAPP} (where f_{CAPP} on input a Boolean circuit C outputs its probability of acceptance) is **APP**-complete under approximate polynomial time many-one reduction (the analogue of many-one reduction for **APP**). We cannot prove this directly from our results. Still we can prove a slightly weaker result, namely the completeness of f_{CAPP} under polynomial time Turing approximate reduction (the analogue of polynomial Turing reduction for **APP**).

Finally it is proved in [For01] that **APP** = **AP** iff **prBPP** is easy. We prove this by using our two mappings between **APP** and **prBPP**.

2 Preliminaries

Since we are working with real-valued functions, we need the following definition of approximate equality. Let $a, b \in [0,1]$ be two real numbers. We say that a and b are $\frac{1}{k}$- equal (denoted $\stackrel{\frac{1}{k}}{=}$) if $|a - b| \leq \frac{1}{k}$.

[KRC00] introduced the class **APP** of real valued function. Here is their definition.

Definition 1 *A family $f = \{f_n\}_{n\geq 0} : \{0,1\}^* \to [0,1]$ of real-valued functions is in* **APP**, *if there exists a probabilistic, polynomial-time Turing machine M such that, for all $k, n \in \mathbb{N}$, we have*

$$\Pr_w[M_w(1^k, x) \stackrel{\frac{1}{k}}{=} f_n(x)] \geq \frac{3}{4}.$$

Consider the following family of functions $f_{\text{CAPP}} : \{0,1\}^* \to [0,1]$, which takes on input a Boolean circuit C, and outputs its acceptance probability, i.e. $f_{\text{CAPP}}(C) = \Pr_w[C(w) = 1]$. It was proved in [KRC00] that the function f_{CAPP} is **APP**-complete under polynomial many-one approximate reduction. For two functions f and g in **APP**, f is polynomially many-one approximately reducible to g, denoted $f \lesssim^{\text{p}}_{\text{mo}} g$, if there is a polynomial family of reductions $r_{n,k} : \{1\}^k \times \{0,1\}^n \to \{0,1\}^{p(n)}$, for some polynomial p, such that, for all k, $n \in \mathbb{N}$,

$$f_n(x) \stackrel{\frac{1}{k}}{=} g_m(r_{n,k}(1^k, x)).$$

A promise problem is a formulation of a partial decision problem that has the structure

$$\text{Input } x \quad \text{Promise } Q(x) \quad \text{Property } R(x)$$

where Q and R are predicates. Formally, a promise problem is a pair of predicates (Q, R). A Turing machine solves (Q, R) if

$$\forall x [Q(x) \to [M(x) \text{ halts } \wedge [M \text{ accepts } x \leftrightarrow R(x)]]].$$

A solution of (Q, R), is a language A decided by a machine M (i.e. $A = L(M)$) such that M solves (Q, R).

prBPP is the class of all promise problems (Q, R), that have a solution in **BPP** (on instances where the promise is satisfied).

In order to define complete problems for **prBPP** we need the following definitions of reductions.

Definition 2 *A promise problem (Q, R) is uniformly Turing reducible in polynomial time to a promise problem (S, T), denoted $(Q, R) \leq^{\text{PP}}_{\text{UT}} (S, T)$, if there is a deterministic, polynomial time oracle Turing machine M such that, for every solution A of (S, T), M^A solves (Q, R).*

If machine M depends on the solution A, we simply call it Turing reducibility. Grollmann and Selman [GS88] showed that the two definitions are equivalent. Finally we say that a promise problem (Q, R) is uniformly many-one reducible in polynomial time to a promise problem (S, T), denoted $(Q, R) \leq^{\text{PP}}_{\text{mo}} (S, T)$, if there exists a partial polynomial time computable function $red : \{x \in \{0,1\}^* | \; Q(x)\} \to \{0,1\}^*$ in **FP**, such that for every solution A of (S, T), the set B defined by:

$$B(x) = \begin{cases} A(red(x)) & \text{if } Q(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

is a solution of (Q, R).

Unlike **BPP**, the canonical complete language yields a complete promise problem for **prBPP**. Consider the following promise problem $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$.
$\mathcal{Q}_{\text{prBPP}}(M, x, 1^t) = 1$ iff M is a probabilistic Turing machine that decides x **BPP**-wise, i.e. $\Pr_w[M_w(x) = 1] \geq \frac{3}{4}$ or $\leq \frac{1}{4}$.
$\mathcal{L}_{\text{prBPP}}(M, x, 1^t) = 1$ if M accepts x **BPP**-wise in at most t steps, i.e. $\Pr_w[M_w(x) = 1] \geq \frac{3}{4}$.
$\mathcal{L}_{\text{prBPP}}(M, x, 1^t) = 0$ if M rejects x **BPP**-wise in at most t steps, i.e. $\Pr_w[M_w(x) = 1] \leq \frac{1}{4}$.

The following result states that this promise problem is complete for **prBPP** under uniform polynomial time many-one reduction, and hence under uniform Turing polynomial time reduction.

Theorem 1 *The promise problem* $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$ *is* **prBPP***-complete under* $\leq_{\text{mo}}^{\text{PP}}$ *reduction.*

Proof

i) $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}}) \in$ **prBPP**.

Indeed when $\mathcal{Q}_{\text{prBPP}}(M, x, 1^t)$ holds, we know that machine M has a **BPP** behaviour on input x. Therefore a simulation of M on input x yields a **BPP** solution for $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$.

ii) $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$ is **prBPP**-hard under $\leq_{\text{mo}}^{\text{PP}}$ reduction.

Let (S, T) be any promise problem in **prBPP**. Let M be a probabilistic polynomial time Turing machine solving (S, T) and let p be its polynomial time bound. Consider the following deterministic polynomial time partial function

$$\begin{cases} red : \{x \in \{0,1\}^* \mid S(x)\} \to \{0,1\} \in \mathbf{FP} \\ x \mapsto (M, x, 1^{p(|x|)}) \end{cases}$$

We claim that *red* is a many-one reduction from (S, T) to $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$. Indeed let A be a solution of $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$. It is clear that first if $S(x)$ holds then $\mathcal{Q}_{\text{prBPP}}(M, x, 1^{p(x)})$ holds. Second the set B defined by

$$B(x) = \begin{cases} A(red(x)) & \text{if } \mathcal{Q}_{\text{prBPP}}(x) \\ 0 & \text{otherwise} \end{cases}$$

is a solution of (S, T).

□

3 A mapping from APP to prBPP

Our main tool to build a correspondance between **APP** and **prBPP**, is the graphe of a function.

Definition 3 *Let* $f = \{f_n\}_{n \geq 0} : \{0,1\}^* \to [0,1]$ *be a real valued function. We define its graphe by:*

$$gr(f) = \{(1^k, x, y) \in \{1\}^* \times \{0,1\}^* \times \{0,1\}^* \mid \quad y \stackrel{\frac{1}{k}}{=} f(x)\}.$$

Let $f : \{0,1\}^* \to [0,1]$ be a real-valued function. Consider the following promise problem $(\mathcal{P}_{\text{APP}}, gr(f))$, where

$$\mathcal{P}_{\text{APP}}(1^k, x, y) = \begin{cases} 1 & \text{if } d(f(x), y) \leq \frac{1}{2k} \text{ or } > \frac{3}{2k} \\ 0 & \text{otherwise} \end{cases}$$

The following result states that computing the graphe of the **APP**-complete function f_{CAPP} is a **prBPP**-complete problem.

Theorem 2 *Let $f_{\text{CAPP}} : \{0,1\}^* \to [0,1]$ be the* **APP***-complete function. Then $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$ is* **prBPP***-complete under $\leq_{\text{UT}}^{\text{PP}}$ reduction.*

Proof

i) $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}})) \in$ **prBPP**

Let M be the probabilistic transducer witnessing the fact $f_{\text{CAPP}} \in$ **APP**. Consider the following probabilistic polynomial time Turing machine N. Input $(1^k, x, y)$

- Simulate $M(1^{2k}, x)$ denote the output by $\tilde{y}$.
- Accept iff $d(y, \tilde{y}) \leq \frac{1}{k}$.

It is clear that first N has a **BPP**-like behaviour inside the promise. Second it is clear that N decides $gr(f_{\text{CAPP}}))$ correctly inside the promise; indeed by observing Figure 1 we see that wherever y and $\tilde{y}$ are in the interval $[f_{\text{CAPP}}(x) - \frac{1}{2k}, f_{\text{CAPP}}(x) + \frac{1}{2k}]$, N always accepts $(1^k, x, y)$ inside the interval $[f_{\text{CAPP}}(x) - \frac{1}{2k}, f_{\text{CAPP}}(x) + \frac{1}{2k}]$, and always rejects $(1^k, x, y)$ outside the interval $[f_{\text{CAPP}}(x) - \frac{3}{2k}, f_{\text{CAPP}}(x) + \frac{3}{2k}]$.

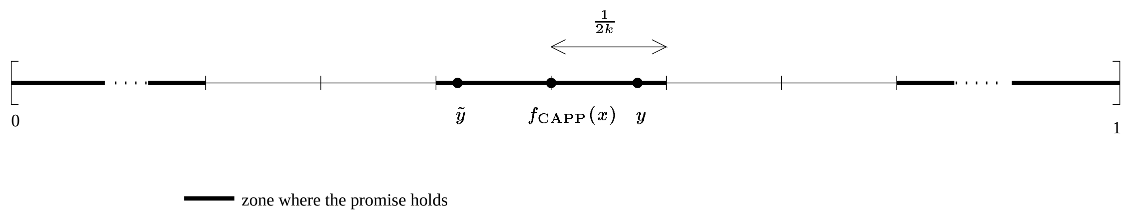

Figure 1: The intervall $[0, 1]$

ii) $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$ is **prBPP**-hard.

We prove that $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}}) \leq_{\text{UT}}^{\text{PP}} (\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$ which proves part ii). We construct a polynomial time deterministic Turing machine M, such that for every solution A of $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$, M^A solves $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$. So let $(N, x, 1^t)$ be an input for the promise problem $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$, where N is a probabilistic Turing machine, such that the promise $\mathcal{Q}_{\text{prBPP}}(N, x, 1^t)$ holds. Since the promise holds, N accepts or rejects input x **BPP**-wise. Therefore in order to decide $\mathcal{L}_{\text{prBPP}}$ inside the promise, M^A only needs to compute whether N accepts x in at most t steps.

Let A be a solution of $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$, and let $k = 10$. Consider the Boolean circuit $C_{N,x}$ which computes $N(x)$, i.e. $C(w) = N_w(x)$. Here is a description of M^A on input $(N, x, 1^t)$.

- Divide both $[0, \frac{1}{4} + \frac{1}{2k}]$ and $[\frac{3}{4} - \frac{1}{2k}, 1]$ in subintervals of size at most $\frac{1}{k}$. Let $y_0, \dots, y_t$ and $z_0, \dots, z_t$ be the endpoints of those subintervals.
- Ask the oracle whether $(1^k, C_{N,x}, y_i) \overset{?}{\in} A$ and whether $(1^k, C_{N,x}, z_i) \overset{?}{\in} A$ for $i = 1, 2, \dots, t$. Accept iff there is a z_i such that $z_i \in A$.

We show that M^A solves $\mathcal{L}_{\text{prBPP}}$ correctly.

Since N decides input x **BPP**-wise (because the promise holds), we know that either $f_{\text{CAPP}}(C_{N,x}) \leq \frac{1}{4}$ or $\geq \frac{3}{4}$. Suppose wlog that $f_{\text{CAPP}}(C_{N,x}) \geq \frac{3}{4}$. Therefore $f_{\text{CAPP}}(C_{N,x}) \in [z_{i_0}, z_{i_0+1}]$, for a certain i_0 where $0 \leq i_0 \leq t$. First we show that there is a z_i such that $(1^k, C_{N,x}, z_i) \in A$, and second $y_i \notin A$ for every $i = 0, 1, \dots, t$. The first statment is clear because since $f_{\text{CAPP}}(C_{N,x}) \in [z_{i_0}, z_{i_0+1}]$, we can suppose wlog that $d(f_{\text{CAPP}}(C_{N,x}), z_{i_0}) \leq \frac{1}{2k}$. Therefore A is correct for z_{i_0}, which implies $(1^k, C_{N,x}, z_{i_0}) \in A$. The second holds because for every $i = 0, 1, \dots, t$ we have $d(f_{\text{CAPP}}(C_{N,x}), y_i) > \frac{3}{2k}$, which implies the correctness of A for every y_i for $i = 0, 1, \dots, t$.

□

The proof of Theorem 2 can be applied to any function $f \in$ **APP**.

Theorem 3 *Let $f : \{0,1\}^* \to [0,1]$ be a real valued function in* **APP**. *Then the promise problem $(\mathcal{P}_{\text{APP}}, gr(f)) \in$* **prBPP**.

Proof

Similar to part i) of Theorem 2.

□

4 From prBPP to APP

In a sense Theorem 2 gives a mapping Ψ from **APP** to **prBPP** associating to each real-valued function in **APP** a promise problem in **prBPP**, and mapping complete function onto complete promise problems (see Theorem 6). The following result gives an inverse for Ψ.

Theorem 4 *Let $f : \{0,1\}^* \to [0,1]$ be a real valued function, such that $(\mathcal{P}_{\text{APP}}, gr(f)) \in$* **prBPP**. *Then f is in* **APP**.

Proof

By hypothesis, there is a solution A which decides $gr(f)$ correctly inside the promise, moreover $A \in$ **BPP** inside the promise, i.e. whenever $d(x, f(x)) \leq \frac{1}{2k}$ or $> \frac{3}{2k}$. We construct the following probabilistic polynomial time transducer M for f. Input: $(1^{k'}, x)$.

- Divide the interval $[0, 1]$ into $\frac{3k'}{2}$ subintervals of size at most $\frac{2}{3k'}$. Denote by $y_0, \dots, y_t$ the endpoints.

- Output the first y_i such that $(1^{\frac{3}{2k'}}, x, y_i) \in A$.

We claim that firt there is at least one i with $0 \leq i \leq t$, such that $(1^{\frac{3k'}{2}}, x, y_i) \in A$. Indeed A is correct on input $(1^{\frac{3}{2k'}}, x, y_i)$ when either $d(f(x), y_i) \leq \frac{1}{2} \cdot \frac{2}{3k'} = \frac{1}{3k'}$ or $d(f(x), y_i) > \frac{3}{2} \cdot \frac{2}{3k'} = \frac{1}{k'}$. Moreover we can suppose wlog that $f(x) \in [y_{i_0}, y_{i_0+1}]$; therefore $d(f(x), y_{i_0}) \leq \frac{1}{2} \cdot \frac{2}{3k'} = \frac{1}{3k'}$. Therefore $(1^{\frac{3k'}{2}}, x, y_{i_0}) \in A$. Second we prove that when $(1^{\frac{3k'}{2}}, x, y_i) \in A$ then $d(f(x), y_i) \leq \frac{1}{k'}$. But this is true because of the promise on A. Indeed if $(1^{\frac{3k'}{2}}, x, y_i) \in A$ then $d(f(x), y_i) \leq \frac{3}{2} \cdot (\frac{3k'}{2})^{-1} = \frac{1}{k'}$.

□

In fact we have a much stronger result than Theorem 2, namely that the same result holds under uniform many-one polynomial reduction.

Theorem 5 *The promise problem* $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$ *is* **prBPP**-*complete under* $\leq_{\mathrm{mo}}^{\mathrm{PP}}$ *reduction.*

Proof

Part i) is the same as in Theorem 2.

ii) $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$ is **prBPP**-hard.

We prove that $(\mathcal{Q}_{\mathrm{prBPP}}, \mathcal{L}_{\mathrm{prBPP}}) \leq_{\mathrm{mo}}^{\mathrm{PP}} (\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$ which proves the Theorem. Let A be a solution of $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$. Let $(N, x, 1^t)$ be where N is a probabilistic polynomial time Turing machine such that the promise $\mathcal{Q}_{\mathrm{prBPP}}(N, x, 1^t)$ holds. We construct a solution that computes whether N accepts x. The promise guarantees that N behaves **BPP**-wise on x. Therefore by repeated trials, and using standard Chernoff bounds, we get a probabilistic Turing machine N', such that $\Pr_w[N'(x) = 1] \geq 1 - 2^{-q(|x|)}$ ($\leq 1 - 2^{-q(|x|)}$ respectivly), $L(N') = L(N)$ whenever the promise holds, and such that the running time of N' is polynomial in the running time of N, for a certain polynomial q. Thus we have that $\mathcal{Q}_{\mathrm{prBPP}}(N', x, 1^{q(t)})$ holds. Consider $C_{N',x}$ a Boolean circuit computing $N'(x)$. Suppose wlog that N' accepts x, i.e. $f_{\mathrm{CAPP}} \in [1 - 2^{q(n)}, 1]$. Consider $k = \frac{1}{10}$. Consider the following partial polynomial time computable reduction

$$\begin{cases} red : \{s \in \{0,1\}^* | \ \mathcal{Q}_{\mathrm{prBPP}}(s) \to \{0,1\}^* \\ (N, x, 1^t) \mapsto (1^{10}, C_{N',x}, 1) \end{cases}$$

Consider the set B defined by

$$B(s) = \begin{cases} A(red(s)) & \text{if } \mathcal{Q}_{\mathrm{prBPP}}(s) \\ 0 & \text{otherwise} \end{cases}$$

B is a solution of $(\mathcal{Q}_{\mathrm{prBPP}}, \mathcal{L}_{\mathrm{prBPP}})$. Indeed since $d(f_{\mathrm{CAPP}}(C_{N,x}), 1) \leq 1 - 2^{-q(|x|)} \leq \frac{1}{2k}$, the promise for A holds, therefore $(1^{10}, N'(x), 1) \in A$ and B concludes that N' accepts x, which is correct.

□

We now construct a mapping between **APP** and **prBPP**.
Consider the following two mappings

$$\Psi : \begin{cases} \mathbf{APP} \to \mathbf{prBPP} \\ f \mapsto (\mathcal{P}_{\mathrm{APP}}, gr(f)) \end{cases} \quad \Phi : \begin{cases} \mathbf{prBPP} \to \mathbf{APP} \\ (Q,R) \mapsto f_{Q,R} \end{cases}$$

Where $f_{Q,R}$ is defined as follows; Let $\{M_i\}_{\{i\in\mathbb{N}\}}$ be an enumaration of all probabilistic Turing machines solving (Q,R). Let M' be the first (in lexicographical order). We define $f_{Q,R}(x) = \Pr_w[M'_w(x) = 1]$ The following result states that the two mappings Φ and Ψ map complete problems to complete problems.

Theorem 6 *Ψ maps every* **APP** $\lessapprox^{\mathrm{P}}_{\mathrm{mo}}$*-complete function f to a* **prBPP** $\leq^{\mathrm{PP}}_{\mathrm{mo}}$*-complete problem $(\mathcal{P}_{\mathrm{APP}}, gr(f))$, and Φ maps every* **prBPP** $\leq^{\mathrm{PP}}_{\mathrm{mo}}$*complete problem (Q,R) to a* **APP** $\leq^{\mathrm{P}}_{\mathrm{T}}$*-complete function $f_{Q,R}$.*

Proof

For Ψ the result immediately follows from Theorem 5. The Proof for Φ follows.

First we prove that Φ maps $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$ to a **APP** $\leq^{\mathrm{P}}_{\mathrm{T}}$-complete function. Denote $h = \Phi(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$. Let M be the first (in lexicographical order) probabilistic Turing machine solving $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$. We have $h(1^k, x, y) = \Pr[M_w(1^k, x, y) = 1]$.

Claim: h is **APP** $\leq^{\mathrm{P}}_{\mathrm{T}}$-complete.

Proof (of Claim). Let $g \in$ **APP** be any real-valued function, and let N be a probabilistic polynomial Turing machine witnessing this fact. We construct a deterministic polynomial time oracle Turing machine K, such that K^h computes g. Here is a description of K^h on input $(1^k, x)$. Let $red : \{0,1\}^* \to \{0,1\}^*$ be a reduction in **FP** such that $g(x) \overset{\frac{1}{2k}}{=} f_{\mathrm{CAPP}}(red(x))$

- Divide the interval $[0,1]$ into subintervals of size at most $\frac{1}{3k}$. Denote $y_0, y_1, \ldots, y_t$ the endpoints of those subintervals.
- For $i = 0, 1, \ldots, t$ querry $h(1^{3k}, red(x), y_i)$ with precision $\frac{1}{10}$. Output the first y_i satisfying

$$h(1^{3k}, red(x), y_i) \geq \frac{3}{4} - \frac{1}{10} \quad (1).$$

Let's prove the correctness of K^h. First we show that there is a y_i satsfying (1). Indeed we can suppose wlog that $f_{\mathrm{CAPP}}(red(x)) \in [y_j, y_{j+1}]$. Therefore wlog $d(f_{\mathrm{CAPP}}(red(x)), y_j) \leq \frac{1}{6k}$. But thanks to the promise, we know that M decides $(1^{3k}, red(x), y_j)$ correctly if $d(f_{\mathrm{CAPP}}(red(x)), y_j) \leq \frac{1}{2} \cdot \frac{1}{3k}$, which is true. Second we show that all y_i satisfying (1) are such that $d(y_i, g(x)) \leq \frac{1}{k}$. Indeed let y_i (where $0 \leq i \leq t$) be any y_i such that $h(1^{3k}, red(x), y_i) \geq \frac{3}{4} - \frac{1}{10}$. Therefore M accepts $(1^{2k}, red(x), y_i)$ which implies, thanks to the promise, that $d(f_{\mathrm{CAPP}}(red(x)), y_i) \leq \frac{3}{2} \cdot \frac{1}{3k} = \frac{1}{2k}$ which implies $d(y_i, g(x)) \leq \frac{1}{k}$.

Second we prove that Φ maps every complete problem to a complete function. So let (S,T) be any **prBPP**-complete language. Therefore let red_2 be a reduction from $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$

to (S,T). Let N be the first (in lexicographical order) probabilistic polynomial Turing machine that solves (S,T). The following probabilistic polynomial Turing machine M solves $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$. M on input x computes and outputs $N(red_2(x))$. The end of the proof is similar to the first case.

□

We now prove our main result, stating that relative to **P**, **APP** equals **prBPP**. We first give the definition of an oracle for **APP** and **prBPP**. An oracle for a function $f \in \mathbf{APP}$ is querried $(1^k, x)$ and answers y where $y \overset{\frac{1}{k}}{=} f(x)$. An oracle for a promise problem (Q, R) is querried x and answers $R(x)$ whenever the promise $Q(x)$ holds.

Theorem 7 $\mathbf{P}^{\mathbf{APP}} = \mathbf{P}^{\mathbf{prBPP}}$.

Proof

First we prove that $\mathbf{P}^{\mathbf{APP}} \subseteq \mathbf{P}^{(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}$. Let L be any language in $\mathbf{P}^{\mathbf{APP}}$ and let $M^{f_{\text{CAPP}}}$ be a deterministic polynomial time oracle machine deciding it. We construct a deterministic polynomial oracle machine $N^{(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}$ deciding L. $N^{(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}$ on input x simulates $M^{f_{\text{CAPP}}}(x)$. Suppose that during its computation, $M^{f_{\text{CAPP}}}(x)$ querries string $(1^k, C)$ to its oracle. Then divide the interval $[0,1]$ into subintervals of size at most $\frac{2}{3k}$, denote by $y_0, y_1, \dots, y_t$ the endpoints of those subintervals. Querry whether $(1^{\frac{3k}{2}}, C, y_i) \overset{?}{\in} (\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$ for $i = 0, 1, \dots, t$. Denote by y_j the first y_i such that $(1^{\frac{3k}{2}}, C, y_i) \in (\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$. Answer $M^{f_{\text{CAPP}}}$'s querry $(1^k, C)$ with y_j.

Second we prove the other inclusion. Let L be any language in $\mathbf{P}^{\mathbf{prBPP}}$ and let $M^{\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}$ be a deterministic polynomial time oracle machine deciding it. We construct a deterministic polynomial oracle machine $N^{f_{\text{CAPP}}}$ deciding L. $N^{f_{\text{CAPP}}}$ on input x simulates $M^{\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}(x)$. Suppose that during its computation, $M^{\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}(x)$ querries string $(1^k, C, y)$ to its oracle (i.e asking whether $f_{\text{CAPP}}(C) \overset{\frac{1}{k}}{=} y$). Then querry $(1^{2k}, C)$ to the oracle for f_{CAPP}, (denote the answer by $\tilde{y}$), and answer $M^{\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}$'s querry $(1^k, C, y)$ with "yes" iff $d(\tilde{y}, y) \leq \frac{1}{k}$. It is clear that $N^{f_{\text{CAPP}}}$ answers $M^{\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))}$ querries correctly inside the promise $\mathcal{P}_{\text{APP}}$.

□

5 Consequences for APP

Our results significantly simplify the proofs of important results on **APP**. First it is shown in [KRC00] that similarly to the case of **BPP**, the interval of error probability for functions in **APP** can be reduced from $[\frac{1}{2} - p(n), \frac{1}{2} + p(n)]$ to $[2^{-q(n)}, 1 - 2^{q(n)}]$, for any polynomial $p(n)$ and $q(n)$. We give a much simpler proof using the fact that error reduction is possible in **prBPP**.

Theorem 8 *Let $f = \{f_n\}_{n \geq 0} : \{0,1\}^* \to [0,1]$ be a familiy of real-valued functions such that, there exists a probabilistic, polynomial time transducer M and a polynomial p, such that, $\forall k, n \in \mathbb{N}$,*

$$\Pr_w[M_w(1^k, x) \overset{\frac{1}{k}}{=} f_n(x)] \geq \frac{1}{2} + \frac{1}{p(k+n)} \quad ,$$

then for any polynomial q, there exists a probabilistic, polynomial time transducer N, such that $\forall k, n \in \mathbb{N}$,

$$\Pr_{w'}[M_{w'}(1^k, x) \stackrel{\frac{1}{k}}{=} f_n(x)] \geq 1 - 2^{-q(k+n)} \quad .$$

Proof

Let $f = \{f_n\}_{n \geq 0} : \{0,1\}^* \to [0,1]$ be a familiy of real-valued functions such that, there exists a probabilistic, polynomial time transducer M and a polynomial p, such that, $\forall k, n \in \mathbb{N}$,

$$\Pr_{w}[M_{w}(1^k, x) \stackrel{\frac{1}{k}}{=} f_n(x)] \geq \frac{1}{2} + \frac{1}{p(k+n)} \quad .$$

It is clear that the $[\frac{1}{4}, \frac{3}{4}]$ interval in the definition of the promise problem $(\mathcal{Q}_{\text{prBPP}}, \mathcal{L}_{\text{prBPP}})$ is quite arbitrary and can be replaced by the interval $[\frac{1}{2} - \frac{1}{t(n)}, \frac{1}{2} + \frac{1}{t(n)}]$, where t is any fixed polynomial. Therfore applying Theorem 3, we have that $(\mathcal{P}_{\text{APP}}, gr(f))$ is solved by a probabilistic Turing machine N that accepts (respectivly rejects) with probability $\frac{1}{2} + \frac{1}{p(k+n)}$ whenever the promise $\mathcal{P}_{\text{APP}}$ holds. Now let q be any polynomial. Using standard error reduction technique for **BPP**, we get a probabilistic Turing machine N' such that when the promise $\mathcal{P}_{\text{APP}}$ holds, N' decides the same language as N and N' accepts (respectivly rejects) with probability $\geq 1 - 2^{-q(k+n)}$ (respectivly $\leq 2^{-q(k+n)}$). By Theorem 4 we obtain a probabilistic Turing machine M witnessing the fact that $f \in$ **APP** and that errs with probability $2^{-q(k+n)}$.

□

Second it is shown in [KRC00] that the function $f_{\text{CAPP}} : \{0,1\}^* \to [0,1]$ is **APP**-complete under polynomial time many-one approximate reduction. We cannot prove this directly from our results. Still we can prove a slightly weaker result, namely the completeness of f_{CAPP} under polynomial time Turing approximate reduction. For two real valued functions f, g in **APP**, we say that f is Turing approximate reducible in polynomial time to g (denoted $\leq_{\text{T}}^{\text{P}}$), if there exists a deterministic polynomial time oracle Turing machine N such that $N^g(1^k, x) \stackrel{\frac{1}{k}}{=} f(x)$.

Theorem 9 *The function f_{CAPP} is* **APP***-complete under polynomial Turing approximate reduction.*

Proof

Let $f = \{f_n\}_{n \geq 0} : \{0,1\}^* \to [0,1]$ be any familiy of real-valued functions in **APP**. By Theorem 3 we have that $(\mathcal{P}_{\text{APP}}, gr(f)) \in$ **prBPP**. Since the promise problem $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$ is **prBPP** complete under uniform polynomial many one reduction (Theorem 2), there exists a partial function in **FP**

$$\begin{cases} red : \{s \in \{0,1\}^* | \quad \mathcal{P}_{\text{APP}}(s)\} \to \{0,1\}^* \\ (N, x, 1^t) \mapsto (1^{10}, C_{N',x}, 1) \end{cases}$$

such that for any solution A of $(\mathcal{P}_{\text{APP}}, gr(f_{\text{CAPP}}))$, the set

$$B(s) = \begin{cases} A(red(s)) & \text{if } \mathcal{P}_{\text{APP}}(s) \\ 0 & \text{otherwise} \end{cases}$$

is a solution of $(\mathcal{P}_{\mathrm{APP}}, gr(f))$. So let A be a fixed solution of $(\mathcal{P}_{\mathrm{APP}}, gr(f_{\mathrm{CAPP}}))$. We construct a polynomial time deterministic oracle Turing machine N, such that $N^{f_{\mathrm{CAPP}}}$ computes f. Here is a description of $N^{f_{\mathrm{CAPP}}}$ on input $(1^k, x)$.

- Divide the interval $[0, 1]$ into $\frac{3k}{2}$ subintervals of size at most $\frac{2}{3k}$. Denote by $y_1, \dots, y_t$ the endpoints of those subintervals.
- Test whether $(1^{\frac{3k}{2}}, x, y_i) \in (P, gr(f))$ for $i = 1, 2, \dots t$ by computing $B(1^{\frac{3k}{2}}, x, y_i) = A(red(1^{\frac{3k}{2}}, x, y_i))$ for $i = 1, 2, \dots t$.
- Output the first y_i such that B accepts $(1^{\frac{3k}{2}}, x, y_i)$.

Let us check that $N^{f_{\mathrm{CAPP}}}$ computes $f(x)$ correctly. So suppose wlog that $f(x) \in [y_i, y_{i+1}]$. Therefore wlog $d(f(x), y_i) \leq \frac{1}{2} \cdot \frac{2}{3k} = \frac{1}{3k}$. Thus there is at least one y_i where $0 \leq i \leq t$ such that B accepts $(1^{\frac{3k}{2}}, x, y_i)$. Thanks to the promise, we know that B correctly rejects any $(1^{\frac{3k}{2}}, x, z)$ such that $d(f(x), z) > \frac{3}{2} \cdot (\frac{3k}{2})^{-1} = \frac{1}{k}$. Therefore $d(f(x), y_i) \leq \frac{1}{k}$.

□

It is shown in [For01] that **APP** = **AP** iff **prBPP** is easy. We say that **prBPP** is easy if for every promise problem (Q, R) in **prBPP**, there is a language $L \in \mathbf{P}$, such that L decides R when the promise holds, i.e. $[Q(x) \Rightarrow R(x) = L(x)]$.

Theorem 10 **APP** = **AP** *iff* **prBPP** *is easy.*

Proof
Easy consequence of Theorem 7.

□

6 Final remarks

It would be interesting to see whether it is possible, while using our results, to prove the **APP**-completeness of the function f_{CAPP}, under approximate many-one reduction (instead of Turing reduction). The main difficulty here is that even if you are able to compute the graphe of the function f_{CAPP}, there is no easy way to compute the image $f_{\mathrm{CAPP}}(x)$, asking only **one** querry to its graphe.

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