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Answer to the open problem of ECCC TR02-007

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In the paper of Pavel Pudlak: "Monotone complexity and the rank of matrices", ECCC TR02-007, the following open problem was stated.

Let A be a family of subsets of $[n] = \{1, \ldots, n\}$, and B be a family of k-tuples of subsets of $[n]$ such that for every $a \in A$ and every $(b_1, \ldots, b_k) \in B$, a has a nonempty intersection with exactly one b_i. Let $z_1, \ldots, z_k \in F$ be arbitrary elements of a field F. Define an $|A|$ by $|B|$ matrix R, with its rows indexed by sets $a \in A$ and its columns indexed by k-tuples in B, such that the entry corresponding to a given pair $a, (b_1, \ldots, b_k)$ is z_t, where t is such that $a \cap z_t \neq \emptyset$. The open problem was: Does there exist sets A, B as above, such that the rank of the associated matrix is larger than $n^{O(\log n)}$. We show that the answer to this question is no.

We can assume that only disjoint nonempty sets are allowed in each k-tuple of sets, thus $k <= n$. (Otherwise, if some j appears in several sets in some k-tuple, then no $a \in A$ can contain j, and we can leave j out from all the sets, without changing the matrix. One could try to make $k > n$ by allowing empty sets in the k-tuples, but that would essentially mean that the matrix is not really specified, as it is not clear where to place the empty sets within the k-tuples.)

We will prove that for $k = 2$ one can get rank at most $n^{O(logn)}$, and Pavel Pudlak has proved in the paper, that taking larger k can help at most by a factor of k.

Let R be the matrix considered in the paper (for $k = 2$). Consider the following $2n$ rectangles $R1_i$ and $R2_i$ for $i = 1, \ldots, n$. $R1_i$ is the rectangle with rows indexed by sets $a \in A$ such that i is contained in the set a, and columns indexed by pairs (b_1, b_2), such that i is contained in the set b_1. $R2_i$ is the rectangle with rows indexed by sets $a \in A$ such that i is contained in the set a, and columns indexed by pairs (b_1, b_2), such that i is contained in the set b_2. Then all these rectangles are monochromatic: each entry of $R1_i$ is z_1, and each entry of $R2_i$ is z_2. The condition implies that each $a \in A$ intersects at least one of the sets b_1 or b_2 for each column (in some point i). Thus, the $2n$ rectangles defined above form a monochromatic cover of R. By the theorem of [1] on deterministic vs. nondeterministic communication complexity and the rank lower bound on deterministic communication complexity [2], this means that the rank of R (over any field) is at most $n^{O(logn)}$.

References

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