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Computational Complexity on Computable Metric Spaces

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Abstract

We introduce a new Turing machine based concept of time complexity for functions on computable metric spaces. It generalizes the ordinary complexity of word functions and the complexity of real functions studied by Ko [19] et al. Although this definition of TIME as the maximum of a generally infinite family of numbers looks straightforward, at first glance, examples for which this maximum exists seem to be very rare. It is the main purpose of this paper to prove that, nevertheless, the definition has a large number of important applications. Using the framework of TTE [40], we introduce computable metric spaces and computability on the compact subsets. We prove that every computable metric space has a *c-proper* c-admissible representation. We prove that Turing machine time complexity of a function computable relative to c-admissible c-proper representations has a computable bound on every computable compact subset. We prove that computably compact computable metric spaces have *concise* c-proper c-admissible representations and show by examples that many canonical representations are of this kind. Finally, we compare our definition with a similar but not equivalent one by Labhalla et al. [22]. Several examples illustrate the concepts. By these results natural and realistic definitions of computational complexity are now available for a variety of numerical problems such as image processing, integration, continuous Fourier transform or wave propagation.

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1 Introduction

The study of computational complexity of decidable sets and computable functions is one of the central issues in theoretical computer science. The majority of the numerous investigations, however, concern countable structures such as natural numbers, finite words, finite trees or finite graphs.

For the complexity of real number computation various models have been proposed. A popular one is the real-RAM (real Random Access Machine) approach which is used, e.g., in algebraic complexity [7] and computational geometry [9]. Roughly speaking, a real-RAM is a flowchart program over the structure $(\mathbb{R}, 0, 1, +, -, \cdot, /, \leq)$. The computational complexity is the time (number of steps) of a computation measured as a function of the dimension (number of real variables) of the input. Computational complexity of real-RAMs is discussed in detail in [3]. Another model using real-RAM computations is "Information Based Complexity" (IBC) [36]. Here complexity is the number of function evaluations and possibly arithmetic operations) measured as a function of precision of the result. Since only in some applications the real-RAM model of computation is realistic ([40], Chap. 9), we do not believe that it is suitable as a general model for computability and complexity in analysis.

In a completely different model of computation, Turing machines are used to approximate real numbers and functions. In this "bit-oriented" model, Turing machine time complexity is the number of Turing steps to compute an approximate result of precision 2^{-k}. Complexity bounds of concrete real functions are proved e.g. in [6, 24, 25, 31, 26, 30, 27, 33, 34].

In his book, Ko [19] studies various aspects of real functions computable on Turing machines in polynomial time, in particular lower complexity bounds in connection to the famous **P-NP** problem. Further papers using this model are e.g. [1, 15, 38, 2, 20, 14, 22], see also Chap.7 in [40].

There are only very few papers studying complexity in the bit-model not only of real functions but also of other objects. Polynomially time approximable sets and polynomially time computable sets have been considered in [19, 20], see also [17, 16, 40]. In [22], Ko's characterization of polynomial time computable real functions (Cor. 2.21 in [19]) is generalized to $\mathcal{C}$-computable functions between complete "computable" metric spaces for appropriate discrete complexity classes $\mathcal{C}$ (like polynomial time) which, in particular, must be closed under composition. Real functions and operators can be considered as "higher type objects" in computer science. Computational complexity of higher type functions is defined in [23, 35, 8, 18]. In these approaches evaluation $(f, x) \mapsto f(x)$ is an additional primitive operation counting one step. Their significance for application in analysis, however, is not obvious.

Seemingly, computational complexity in analysis is so underdeveloped not least, since no satisfactory definitions have been available so far. Finding natural, realistic and generally applicable definitions of computational complexity is one of the current challenges in computable analysis.

In this paper we introduce a new "bit-oriented" kind of computational complexity of functions on metric spaces. It generalizes Ko's definition [19] and the definition via the signed digit representation of computational complexity of real functions [40]. In some examples it is equivalent to the definition by Labhalla et. al. [22].

As a general framework for studying computability and computational complexity in analysis on the "bit-level" we use Type-2 Theory of Effectivity, TTE for short [40]. TTE is rooted in a definition of computable real functions introduced by A. Grzegorczyk [12] and later work on the theory of representations by J. Hauck and others [13, 42, 21]. In TTE, computable functions on finite and infinite strings of symbols (Σ^* and Σ^ω, respectively) are defined explicitly by Type-2 machines (extended Turing machines), and then finite and infinite strings are used as "names" of other objects such as real numbers, closed subsets of Euclidean space or continuous real functions. TTE is consistent with the computability approaches by Pour-El and Richards and by Ko and with the domain approach [28, 19, 11], but seemingly is more expressive in analysis ([40], Chap. 9). A surjective partial function $\delta :\subseteq \Sigma^\omega \to X$ assigning infinite sequences of symbols as "names" to the elements of X is called a *representation*. A function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ is a (δ_1, δ_0)-realization of a function $f :\subseteq X_1 \to X_0$, iff $f \circ \delta_1(p) = \delta_0 \circ g(p)$ whenever $\delta_1(p) \in \mathrm{dom}(f)$.

Let M be a Type-2 machine such that f_M is a (δ_1, δ_0)-realization of a function $f :\in X_1 \to X_0$. We count the number of Turing steps of the machine M as a function of the input and of the output precision. Suppose, for $q \in \mathrm{dom}(\delta_0)$, $k \in \mathbb{N}$ and a prefix $z \in \Sigma^*$ of $q \in \Sigma^\omega$ we have defined the meaning of "z approximates $\delta_0(q)$ with precision 2^{-k} " ($(z, 0^k) \in \mathrm{dom}(\mathrm{app})$ in Def. 5.1 below). Then let

> $\mathrm{Time}_M(p)(k)$ be the number of steps which M needs on input $p \in \mathrm{dom}(\delta_1)$ to compute some finite prefix z of $f_M(p)$ approximating $\delta_0 \circ f_M(p)$ with precision 2^{-k}.

Here, complexity is a function $\mathrm{Time}_M(p) : \mathbb{N} \to \mathbb{N}$ depending on the δ_1-name p of an element $x \in X_1$. In order to get a name-independent concept we consider the maximum $\mathrm{TIME}_M^{\{x\}}(k) := \max_{\delta_1(p)=x} \mathrm{Time}_M(p)(k)$ or, more generally,

$$\mathrm{TIME}_M^K(k) := \max_{\delta_1(p)\in K} \mathrm{Time}_M(p)(k) \tag{1}$$

for suitable subsets $K \subseteq \mathrm{dom}(f)$. Thus, for any δ_1-name of a point $x \in K \subseteq \mathrm{dom}(f)$ as an input, the machine M computes an approximation of $f(x)$ of precision 2^{-k} in at most $\mathrm{TIME}_M^K(k)$ steps (provided the maximum $\mathrm{TIME}_M^K(k)$ exists). This straightforward definition has a very concrete meaning for a user who realizes a computable function on a Type-2 machine, and it is realistic, since Type-2 machines are realistic.

We could now start to study computational complexity of numerous computable functions and operators in analysis. Unfortunately, examples for which the definition is meaningful seem to be very rare. Although we know that for the signed digit representation ρ_{sd} and for the "Ko representation" ρ_{Ko} of the real numbers (see Ex. 4.2 below) the maximum in (1) exists for compact sets ([40], Thm. 7.1.5), for most representations of most spaces, most computable functions and most sets K (even most singletons), however, the definition is useless, since the maximum in (1) does not exist.

In this paper we show that, nevertheless, the above definition (1) of computational complexity has a large number of important quite natural applications. For this purpose we consider computable metric spaces with c-admissible representations. We show that every c-admissible representation δ_0 has a computable concept of "approximation". We show that every computable metric space has c-proper c-admissible representations δ_1. As a main theorem we prove that TIME_M^K has a computable upper bound, if δ_1 is c-proper and c-admissible and K is a computable compact set. Finally, we construct natural "concise" c-proper c-admissible representations for computable compact spaces and add some useful examples. Although we consider only time complexity, the results hold for space complexity accordingly.

We will use the framework of TTE as presented, e.g., in the textbook [40]. In particular, let Σ be a (sufficiently large) finite alphabet. For a word $w \in \Sigma^*$ let $|w|$ be its length. Let $\nu_\Sigma : \mathbb{N} \to \Sigma^*$ be some standard bijection. Let $\nu_\mathbb{N} :\subseteq \Sigma^* \to \mathbb{N}$ and $\nu_\mathbb{Q} :\subseteq \Sigma^* \to \mathbb{Q}$ be standard notations of the natural and rational numbers, respectively, and let $\rho_C :\subseteq \Sigma^\omega \to \mathbb{R}$ be the standard Cauchy representation of the real numbers (see [40]). On Cantor space Σ^ω define the metric d_{Σ^ω} by $d_{\Sigma^\omega}(p,q) := 2^{-\min\{n \mid p(n) \neq q(n)\}}$ (for $p \neq q$). Occasionally we use the *wrapping* function $\iota(a_1 \dots a_m) := 110a_10 \dots 0a_m011$ ($a_i \in \Sigma$). Standard tupling functions on $\mathbb{N}, \Sigma^*$ and Σ^ω are denoted by $\langle\ \ \rangle$. On the product of metric spaces, in particular on $\mathbb{R}^n$, we will consider the maximum metric. As a realistic model of computation on finite and infinite sequences we consider Type-2 machines, i.e., Turing machines with typed one-way input and output tapes where the type of a tape is either Σ^* or Σ^ω (see Sec. 2.1 in [40]).

2 Computable Metric Spaces

In this section we recall the definition of computable metric spaces and Cauchy representations (which are *admissible* [39, 40]) and illustrate them by a number of examples.

Definition 2.1 (computable metric space) *1. A computable metric space is a quadruple $\mathbf{X} = (X, d, A, \alpha)$ such that (X, d) is a metric space, A is a dense subset of X and $\alpha :\subseteq \Sigma^* \to A$ is a notation of A such that $\mathrm{dom}(\alpha)$ is recursive and*

$$\{(t,u,v,w) \in (\Sigma^*)^4 \mid \nu_\mathbb{Q}(t) < d(\alpha(u), \alpha(v)) < \nu_\mathbb{Q}(w)\} \ \text{ is r.e.}\,. \tag{2}$$

2. The Cauchy representation $\delta_C :\subseteq \Sigma^\omega \to X$ is defined by

$$\delta_C(p) = x :\iff \begin{cases} \text{there are words } w_0, w_1, \ldots \in \mathrm{dom}(\alpha) \\ \text{such that } p = w_0\#w_1\#\ldots \text{ and} \\ d(x, \alpha(w_i)) \le 2^{-i} \text{ for all } i\,. \end{cases} \tag{3}$$

3. *Call a representation δ of X* admissible *(w.r.t.* $\mathbf{X}$*), iff* $\delta \equiv_t \delta_C$*, and* c-admissible *(w.r.t.* $\mathbf{X}$*), iff* $\delta \equiv \delta_C$.

We tacitly assume $\mathrm{dom}(\alpha) \in (\Sigma \setminus \{\#\})^*$. In [40] Def. 8.1.2, $\mathrm{dom}(\alpha)$ needs not to be recursive. However, it can be shown easily that for every notation α in a computable metric space there is an equivalent notation α' with recursive domain such that the induced Cauchy representations are equivalent. So Def. 2.1 is no proper restriction.

Property (2) means that the distance on A is (α, α, ρ_C)-computable. In [39] instead of (2) the weaker condition "$\{(u,v,w) \in (\Sigma^*)^3 \mid d(\alpha(u), \alpha(v)) < \nu_{\mathbb{Q}}(w)\}$ is r.e." is used. We suggest to call such effective metric spaces *semi-computable.* The definition of "admissible" in 3. is in accordance with the more general concept in [40], Definition 3.2.7. Many well-known metric spaces become computable metric spaces by considering canonical notations of dense subsets.

Example 2.2 (some computable metric spaces)

1. Countable discrete spaces: (X, d, X, α) where $d(x, x') = 1$ for $x \neq x'$ and α is a notation of X with recursive equivalence problem (this follows from (2) for the discrete metric). Then $\alpha \equiv \delta_C$. Examples: $(X, \alpha) = (\mathbb{N}, \nu_{\mathbb{N}})$, $(\Sigma^*, \mathrm{id}_{\Sigma^*})$.

2. Rational numbers: $(\mathbb{Q}, d, \mathbb{Q}, \nu_{\mathbb{Q}})$, where $d(x,y) = |x - y|$. Then $\nu_{\mathbb{Q}} \leq \delta_C$ but $\delta_C \not\leq_t \nu_{\mathbb{Q}}$.

3. Euclidean space: $(\mathbb{R}^n, d, \mathbb{Q}^n, \nu_{\mathbb{Q}}^n)$, where d is the the Euclidean distance, $\delta_C \equiv \rho_C^n$.

4. Real line with binary notation: $(\mathbb{R}, |\,|, D, \nu_D)$, where $\nu_D(sa_k \dots a_0 \bullet a_{-1} \dots a_{-l}) := (-1)^s \sum_{i=k}^{-l} a_i 2^i$ $(s, a_i \in \{0, 1\},\ a_k \neq 0)$. Then $\delta_C = \rho_C$.

5. Cantor space: $(\Gamma^\omega, d_\Gamma, A, \alpha)$, where Γ is a finite alphabet and $\alpha(u) := uaaa\dots$ $(a \in \Gamma)$ for all $u \in \Gamma^*$, $\delta_C \equiv \mathrm{id}_{\Gamma^\omega}$.

6. Baire space: $(\mathbb{B}, d, A, \alpha)$, where $\mathbb{B} = \mathbb{N}^\omega$, $d(p,q) := 2^{-\min\{n \mid p(n) \neq q(n)\}}$ (for $p \neq q$) and $\alpha(0^{i_0}10^{i_1}1 \dots 10^{i_k}) := (i_0 i_1 \dots i_k 000 \dots)$. The Cauchy representation is equivalent to $\delta_{\mathbb{B}}$ defined by $\delta_{\mathbb{B}}(0^{i_0}10^{i_1}1\dots) = (i_0 i_1 \dots)$.

7. Real $L^2(\mathbb{R})$: $(L^2(\mathbb{R}), d_{L^2}, A, \alpha)$ where $d(f,g) = \left(\int (f(x) - g(x))^2\, dx\right)^{1/2}$ and α is some standard notation of the set A of all finite step functions with rational break points.

8. Continuous functions: $(C[0;1], d, A, \alpha)$, where $C[0;1]$ is the set of all continuous functions $f : [0;1] \to \mathbb{R}$, $d(f,g) := \max_x |f(x) - g(x)|$ and α is a standard notation of all finite polygon functions with rational break points or of all polynomials with rational coefficients.

9. Compact subsets of $\mathbb{R}^n$: $(\mathcal{K}(\mathbb{R}^n), d_H, A, \alpha)$, where $\mathcal{K}(\mathbb{R}^n)$ is the set of all non-empty compact subsets of $\mathbb{R}^n$, d_H is the Hausdorff distance and α is a canonical notation of all non-empty finite subsets of $\mathbb{Q}^n$.

$\square$

3 Compact Sets

As a technical preparation, in this section we prove a theorem not yet available in computable analysis: the function $(f, K) \mapsto f[K]$ is computable w.r.t. canonical representations for continuous functions f and compact sets K. Computability on the set of compact subsets of Euclidean space is studied in [5, 40]. For computable metric spaces, various representations of the compact subsets are compared in [4]. In this section we give some examples and generalize Theorem 6.2.4.4 from [40] to computable metric spaces.

Let $\mathbf{X} = (X, d, A, \alpha)$ be a computable metric space. Define a notation $\mathrm{I} :\subseteq \Sigma^* \to \mathcal{B}$ of the set of all open balls with center in A and rational radius (the "open rational balls") by $\mathrm{I}\langle u, v\rangle := B(\alpha(u), \nu_{\mathbb{Q}}(v))$ (abbreviation $\mathrm{I}_w := \mathrm{I}(w)$). Furthermore, let fs be some standard notation of the set of finite subsets of Σ^*. We assume tacitly $\mathrm{dom}(\mathrm{I}), \mathrm{dom}(\mathrm{fs}) \subseteq (\Sigma \setminus \{\#\})^*$. The notations I and fs have recursive domains. Let FS be the (recursive) set of all $w \in \mathrm{dom}(\mathrm{fs})$ such that $\mathrm{fs}(w) \subseteq \mathrm{dom}(I)$. In contrast to Euclidean space, in general neither $\mathrm{I}_w \subseteq \mathrm{I}_{w'}$ nor $\mathrm{I}_w \cap \mathrm{I}_{w'} = \emptyset$ is r.e. . As an alternative we call $\mathrm{dom}(I)$ the set of *formal balls*. We define the center and the radius of a formal ball $\langle u, v\rangle$ by $\mathrm{ct}\langle u, v\rangle := \alpha(u)$ and $\mathrm{rad}\langle u, v\rangle := \nu_{\mathbb{Q}}(v)$ and use in particular the syntactic relations $\prec$ ("formal inclusion") and $\bowtie$ ("formal disjointness") defined by

$$\prec \;:=\; \{(u, u') \mid u, u' \in \mathrm{dom}(\mathrm{I}),\;\; d(\mathrm{ct}(u), \mathrm{ct}(u')) + \mathrm{rad}(u) < \mathrm{rad}(u')\}, \quad (4)$$

$$\bowtie \;:=\; \{(u, u') \mid u, u' \in \mathrm{dom}(\mathrm{I}),\;\; d(ct(u), \mathrm{ct}(u')) > \mathrm{rad}(u) + \mathrm{rad}(u')\}. \quad (5)$$

Both relations are r.e. by (2). The relation $\prec$ is transitive and the closed ball $\overline{B}(\mathrm{ct}(u), \mathrm{rad}(u))$ is a subset of the open ball $B(\mathrm{ct}(u'), \mathrm{rad}(u'))$, if $u \prec u'$, and $\overline{B}(\mathrm{ct}(u), \mathrm{rad}(u)) \cap \overline{B}(\mathrm{ct}(u'), \mathrm{rad}(u')) = \emptyset$, if $u \bowtie u'$.

Let $\mathcal{K}$ be the set of compact subsets of X. We generalize Def. 5.2.4 in [40] by introducing two representations of $\mathcal{K}$ (denoted by δ_{cover} and $\delta_{min-cover}$, respectively, in [4]):

Definition 3.1 (representations of compact sets) *Define the* covering representation κ_c *and the* minimal covering representation κ_{mc} *of* $\mathcal{K}(X)$ *by:*

$$K = \kappa_c(p), \;\; \textit{iff } p = \#v_0\#v_1\#\ldots \;\; \textit{and } \{v_0, v_1 \ldots\} = \{v \in \mathrm{FS} \mid K \subseteq \bigcup \mathrm{I}[\mathrm{fs}(v)]\},$$

$$K = \kappa_{mc}(p), \;\; \textit{iff } \;\; p = \#v_0\#v_1\#\ldots \;\; \textit{and}$$

$$\{v_0, v_1 \ldots\} = \{v \in \mathrm{FS} \mid K \subseteq \bigcup \mathrm{I}[\mathrm{fs}(v)], \forall w \in \mathrm{fs}(v). K \cap \mathrm{I}_w \neq \emptyset\}.$$

Notice that κ_c and κ_{mc} are admissible standard representations of effective topological spaces according to Defs. 3.2.1/2 in [40]. Obviously, $\kappa_{mc} \leq \kappa_c$, but $\kappa_c \not\leq_t \kappa_{mc}$ for non-trivial spaces.

Example 3.2 In the following let κ_c^X and κ_{mc}^X be the representations according to Def. 3.1 of the set $\mathcal{K}(X)$ of compact subsets of the computable metric space $\mathbf{X}$ under consideration.

1. Consider the computable metric space $(\mathbb{N}, d, \mathbb{N}, \nu_{\mathbb{N}})$ of natural numbers from Ex. 2.2.1. A set $K \subseteq \mathbb{N}$ is compact, iff it is finite. Define $\nu_{f\mathbb{N}} :\subseteq \Sigma^* \to \mathcal{K}(\mathbb{N})$ by $\nu_{f\mathbb{N}}(a_0 \ldots a_m) := \{i \mid a_i = 1\}$ $(a_i \in \{0, 1\})$. Define a representation $\kappa_>^{\mathbb{N}}$ of $\mathcal{K}(\mathbb{N})$

by $\kappa_>^{\mathbb{N}}(p) = K$, iff $p = \#w_0\#w_1\ldots$ and $\{\nu_{f\mathbb{N}}(w_0), \nu_{f\mathbb{N}}(w_1)\ldots\} = \{B \subseteq \mathbb{N} \mid B$ finite and $K \subseteq B\}$. Then $\kappa_>^{\mathbb{N}} \equiv \kappa_c^{\mathbb{N}}$ and $\nu_{f\mathbb{N}} \equiv \kappa_{mc}^{\mathbb{N}}$.

2. Consider the computable metric space $(\Sigma^*, d, \Sigma^*, \mathrm{id}_{\Sigma^*})$ from Ex. 2.2.1. A set $K \subseteq \Sigma^*$ is compact, iff it is finite. Define a representation $\kappa_>^{\Sigma^*}$ of $\mathcal{K}(\Sigma^*)$ by $\kappa_>^{\Sigma^*}(p) = K$, iff $p = \#w_0\#w_1\ldots$ and $\{\mathrm{fs}(w_0), \mathrm{fs}(w_1)\ldots\} = \{B \subseteq \Sigma^* \mid B$ finite and $K \subseteq B\}$. Then $\kappa_>^{\Sigma^*} \equiv \kappa_c^{\Sigma^*}$ and $\mathrm{fs} \equiv \kappa_{mc}^{\Sigma^*}$.

3. Consider Cantor space from Ex. 2.2.5. A subset $K \subseteq \Sigma^\omega$ is compact, iff it is closed. Define a representation $\kappa_>^{\Sigma^\omega}$ of the set $\mathcal{K}(\Sigma^\omega)$ of compact subsets of Σ^ω by $\kappa_>^{\Sigma^\omega}(p) = K$, iff $\Sigma^\omega \setminus K = \bigcup\{w\Sigma^\omega \mid \iota(w)$ is a subword of $p\}$ (where $\iota(a_1 \ldots a_m) := 110a_10\ldots0a_m011$). Then $\kappa_>^{\Sigma^\omega} \equiv \kappa_c^{\Sigma^\omega}$ [4]. □

For further examples see [40, 4]. Classically, continuous functions map compact sets to compact sets. The following computable version of this fact generalizes Thm. 6.2.4.4 in [40].

Theorem 3.3 *Let $\mathbf{X}_i$ $(i = 1, 2)$ be computable metric spaces with Cauchy representations δ_C^i of their points and representations κ_c^i and κ_{mc}^i of their compact subsets (Def. 3.1), respectively. Let δ be a representation of partial continuous functions $f :\subseteq X_1 \to X_2$ such that* apply : $(f, x) \mapsto f(x)$ *is $(\delta, \delta_C^1, \delta_C^2)$-computable. Then the function*

$$(f, K) \mapsto f[K] \quad \textit{for compact} \quad K \subseteq \mathrm{dom}(f)$$

is $(\delta, \kappa_c^1, \kappa_c^2)$-computable and $(\delta, \kappa_{mc}^1, \kappa_{mc}^2)$-computable.

Proof: 1. $(f, K) \mapsto f[K]$ is $(\delta, \kappa_c^1, \kappa_c^2)$-computable:
It suffices to show that there is an r.e. set $Y \subseteq \Sigma^\omega \times \Sigma^\omega \times \Sigma^*$ (Sec. 2.4 in [40]) such that for all $q \in \mathrm{dom}(\delta)$ and $p \in \mathrm{dom}(\kappa_c^1)$ with $\kappa_c^1(p) \in \mathrm{dom}(\delta(q))$ and all $w \in \Sigma^*$,

$$(q, p, w) \in Y \iff \delta(q)[\kappa_c^1(p)] \subseteq \bigcup \mathrm{I}^2[\mathrm{fs}(w)]\,. \tag{6}$$

There is a Type-2 machine N such that $f_N :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ is a $(\delta, \delta_C^1, \delta_C^2)$-realization of the apply-function. Let FN be the set of all $(p, u, v) \in \Sigma^\omega \times \Sigma^* \times \Sigma^*$ such that on input $(p, u0^\omega)$ in some number of steps, the machine N writes v on its output tape reading at most u from the second input tape. The set FN is r.e. (see Sec. 2.4 in [40]).

Let d_i be a computable sequence of words such that $\nu_{\mathbb{Q}}(d_i) = 2^{-i}$. Define a computable function $h :\subseteq \Sigma^* \to \Sigma^*$ by $h(u) = \langle u_k, d_k\rangle$, if $u = u_0\#u_1\#\ldots\#u_k\#$ $(u_i \in (\Sigma \setminus \{\#\})^*)$, $h(u)$ is undefined otherwise. Call a sequence $u_0\#u_1\#\ldots$ strict, iff $u_i \in \mathrm{dom}(\alpha_1)$ and $d_1(\alpha_1(u_i), \alpha_1(u_j)) < 2^{-i-1}$ for all $i < j$. Obviously, $x \in \mathrm{I}^1 h(u_0\#\ldots\#u_k\#)$ for all k and any strict δ_C^1-name $u_0\#u_1\#\ldots$ of x. If $v_0\#v_1\#\ldots$ is a δ_C^1-name of x, then $v_2\#v_3\#\ldots$ is a strict δ_C^1-name of x, thus every (computable) point has a strict (computable) δ_C^1-name. Call a word $u \in \Sigma^*$ strict, iff $u \in \mathrm{dom}(h)$ and u is a prefix of some strict sequence. The set of strict words is r.e. by (2).

Assume $\delta(q) = f$, $\kappa_c^1(p) = K \subseteq \mathrm{dom}(f)$ and $\mathrm{I}^2[\mathrm{fs}(w)] = L \subseteq \mathcal{B}^2$. Then

$\delta(q)[\kappa_c^1(p)] \subseteq \bigcup \mathrm{I}^2[\mathrm{fs}(w)]$,
iff $\quad f[K] \subseteq \bigcup L$,

iff $\forall x \in K.\exists B \in L.f(x) \in B$,

iff $\forall x \in K.\exists u, v \in \mathrm{dom}(h).$
$\big(u \text{ is strict}, x \in \mathrm{I}^1 h(u), (q,u,v) \in \mathrm{FN}, \exists w' \in \mathrm{fs}(w).h(v) \prec_2 w'\big)$

iff $\exists u_1, v_1, \ldots, u_m, v_m \in \mathrm{dom}(h).K \subseteq \bigcup_{i=1}^m \mathrm{I}^1 h(u_i)$ and for all i
$\big(u_i \text{ is strict}, (q,u_i,v_i) \in \mathrm{FN}, \exists w' \in \mathrm{fs}(w).h(v_i) \prec_2 w'\big)$

iff $\exists u_1, v_1, \ldots, u_m, v_m \in \mathrm{dom}(h).$
$\exists w_1.\{h(u_1), \ldots, h(u_m)\} = \mathrm{fs}(w_1), K \subseteq \bigcup \mathrm{I}^1[\mathrm{fs}(w_1)]$ and for all i
$\big(u_i \text{ is strict}, (q,u_i,v_i) \in \mathrm{FN}, \exists w' \in \mathrm{fs}(w).h(v_i) \prec_2 w'\big)$

iff $\exists u_1, v_1, \ldots, u_m, v_m \in \mathrm{dom}(h).$
$\exists w_1.\{h(u_1), \ldots, h(u_m)\} = \mathrm{fs}(w_1)$, $\#w_1\#$ is a subword of p and
for all i, $\big(u_i \text{ is strict}, (q,u_i,v_i) \in \mathrm{FN}, \exists w' \in \mathrm{fs}(w).h(v_i) \prec_2 w'\big)$

$=_{\mathrm{def}}$ Q.

The 3rd "only if" holds, since every $x \in \mathrm{dom}(f)$ has a strict name and a finite computation of N guarantees $x \in \mathrm{I}^2(w') \in L$. Notice that $f[\mathrm{I}^1 h(u)] \subseteq \mathrm{I}^2(w')$. The 4th "iff" follows from compactness of K: already finitely many open balls $\mathrm{I}^1 h(u)$ cover K.

Now define Y by $(q,p,w) \in Y$, iff Q. Since $\mathrm{dom}(h)$ is recursive and the subword relation, strictness, FN and $\prec_2$ are r.e., Y is r.e. . The above equivalences show that Y satisfies (6).

2. $(f,K) \mapsto f[K]$ is $(\delta, \kappa^1_{mc}, \kappa^2_{mc})$-computable:
It suffices to show that there is an r.e. set $Y' \subseteq \Sigma^\omega \times \Sigma^\omega \times \Sigma^*$ [40] such that for all $q \in \mathrm{dom}(\delta)$, $p \in \mathrm{dom}(\kappa^1_{mc})$ with $\kappa^1_{mc}(p) \in \mathrm{dom}(\delta(q))$ and $w \in \Sigma^*$, $(q,p,w) \in Y'$, iff

$$\delta(q)[\kappa^1_{mc}(p)] \subseteq \bigcup \mathrm{I}^2[\mathrm{fs}(w)] \text{ and } \forall w' \in \mathrm{fs}(w).\delta(q)[\kappa^1_{mc}(p)] \cap \mathrm{I}^2(w') \neq \emptyset \qquad (7)$$

(cf. (6)). If p lists all minimal covers of K, we have

$f[K] \cap \mathrm{I}^2(w') \neq \emptyset$

iff $\exists x \in K.f(x) \in \mathrm{I}^2(w')$

iff $\exists w_1, w_2, u, v.\big(\#w_1\# \text{ is a subword of } p, w_2 \in \mathrm{fs}(w_1), (q,u,v) \in \mathrm{FN},$
$w_2 \prec_1 h(u), h(v) \prec_2 w'\big)$

$=_{\mathrm{def}}$ P.

Define Y' by $(q,p,w) \in Y' \iff (Q \text{ and } \forall w' \in \mathrm{fs}(w).P)$. Then Y' is r.e. and satisfies (7) □

Sometimes it is convenient to restrict a computable metric space to a subspace, e.g., $\mathbb{R}^2$ to $[0;1]^2$. If the subspace is sufficiently simple, the representations of points and compact sets of the subspace are equivalent to the restrictions of the original representations. The following can be proved straightforwardly.

Lemma 3.4 (representation of a subspace) *Consider computable metric spaces $\mathbf{X} = (X,d,A,\alpha)$ and $\mathbf{X}' = (X',d',A',\alpha')$ such that $X' \subseteq X$ and α' and d' are restrictions of α and d, respectively, with representations δ_C, δ'_C, κ_c and κ'_c of the points and the compact subsets, respectively. Then $\delta_C|^{X'} \equiv \delta'_C$ and $\kappa_c|^{\mathcal{K}(X')} \equiv \kappa'_c$.*

4 Proper Admissible Representations

In Section 1 we have defined $\mathrm{TIME}_M^K(k) := \max_{\delta_1(p)\in K} T_M(p)(k)$, where $T_M(p)(k)$ is the time which a machine M needs on input p to determine the result with precision 2^{-k}. Since $p \mapsto T_M(p)(n)$ is continuous, $\max_{p\in A} T_M(p)(n)$ exists, if $A \subseteq \Sigma^\omega$ is compact and $T_M(p)(n)$ exists for all $p \in A$. Schröder [32] calls a representation δ of a metric space *proper*, iff $\delta^{-1}[K]$ is compact for every compact subset K. If $\mathrm{dom}(\delta)$ is closed, then $\delta^{-1}[K]$ is a closed, i.e., compact subset of Σ^ω for compact K. But in general $\mathrm{dom}(\delta)$ is not closed. Schröder shows that every separable metric space has a proper admissible representation δ_p. In the following we introduce *c-proper* representations. In particular, we prove that c-proper c-admissible representations exist and that they are closed under Cartesian product. We still assume that $\mathbf{X} = (X, d, A, \alpha)$ is a computable metric space such that $\mathrm{dom}(\alpha)$ is recursive. Let $\kappa_>^{\Sigma^\omega}$ be the representation of the set of compact subsets of Cantor space from Ex. 3.2.3.

Definition 4.1 (c-proper representation) *Call a representation δ of X* c-proper*, iff the function $K \mapsto \delta^{-1}[K]$ for compact $K \subseteq X$ is $(\kappa_c, \kappa_>^{\Sigma^\omega})$-computable.*

Example 4.2

1. Consider the discrete computable metric space from Ex. 2.2.1. Define a representation δ_α of X by $\delta_\alpha(\iota(w)00\ldots) := \alpha(w)$ (where ι is the wrapping function, $\iota(a_1\ldots a_m) := 110a_10\ldots0a_m011$). Then $\alpha \equiv \delta_\alpha \equiv \delta_C$. If for some computable function $h :\subseteq \Sigma^* \to \Sigma^*$,

$$\alpha^{-1}[\{\alpha(w)\}] = \mathrm{fs}\circ h(w) \quad (w \in \mathrm{dom}(\alpha)), \tag{8}$$

then δ_α is c-proper (h computes the finite set of all names of $\alpha(w)$).

2. The *signed digit representation* $\rho_{\mathrm{sd}} :\subseteq \Sigma^\omega \to \mathbb{R}$ is defined by $\rho_{\mathrm{sd}}(a_k\ldots a_0{\bullet}a_{-1}a_{-2}\ldots) := \sum_{i=k}^{-\infty} a_i 2^i$, $a_i \in \{1,0,-1\}$ (abbreviate -1 by $\overline{1}$), $a_k \neq 0$ for $k \geq 0$ and $a_ka_{k-1} \neq 1\overline{1},\ \overline{1}1$ for $k \geq 1$ [42, 40]. The signed digit representation is equivalent to ρ_C and c-proper (Thm. 7.2.5 and Ex. 7.2.9 in [40]). In [40] it is used to define the complexity of real functions.

3. Consider Ex. 2.2.4. Let δ_{Ko} be the restriction of the Cauchy representation to names $w_0\#w_1\#\ldots$ such that $w_i \in \Sigma^*{\bullet}\Sigma^i$. Then δ_{Ko} is c-admissible and c-proper. Ko's complexity of real functions [19] is (essentially) based on this representation, which is very similar to the signed digit representation. □

In Sec. 6 we will construct c-proper representations with compact domains.

Lemma 4.3 *Let δ be a c-admissible representation of a computable metric space $\mathbf{X}$ with $\kappa_>^{\Sigma^\omega}$-computable compact domain. Then δ is c-proper.*

Proof: Consider $K = \kappa_c(q)$. Then

$$p \notin \delta^{-1}[K] \iff p \in \Sigma^\omega \setminus \mathrm{dom}(\delta) \ \text{ or } \ (p \in \mathrm{dom}(\delta) \text{ and } \delta(p) \notin K).$$

Let $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ be a computable translation from δ to δ_C. Then for any $p \in \mathrm{dom}(\delta)$, $\delta(p) \notin K$ $(= \kappa_c(q))$, iff there are words $u, w, w_0, \ldots, w_m$ such that $\#u\#$ is a subword of q, w is a prefix of p, $h(w\ldots) = (w_0\#\ldots\#w_m\#\ldots)$ and $v \bowtie \langle w_m, w'\rangle$ for all $v \in \mathrm{fs}(u)$ where $\nu_{\mathbb{Q}}(w') = 2^{-m+1}$. Let $Z \subseteq \Sigma^\omega \times \Sigma^\omega$ be the set of all pairs (p, q) such that: there are words $u, w, w_0, \ldots, w_m$ such that $\#u\#$ is a subword of q, w is a prefix of p, $h(w \ldots) = (w_0\# \ldots \#w_m\# \ldots)$ and $v \bowtie \langle w_m, w' \rangle$ for all $v \in \mathrm{fs}(u)$. Then Z is r.e. open and

$$p \in \Sigma^\omega \setminus \delta^{-1}[\kappa_c(q)] \iff p \in \Sigma^\omega \setminus \mathrm{dom}(\delta) \ \text{ or } \ (p, q) \in Z.$$

Since $\Sigma^\omega \setminus \mathrm{dom}(\delta)$ is r.e. open, $p \in \Sigma^\omega \setminus \delta^{-1}[\kappa_c(q)]$ is r.e. open and so there is some r.e. set $B \subseteq \Sigma^* \times \Sigma^*$ such that

$$\{(p, q) \mid p \in \Sigma^\omega \setminus \delta^{-1}[\kappa_c(q)]\} = \bigcup_{(u,v) \in B} \mathrm{I}_u \times \mathrm{I}_v .$$

There is a computable function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $g(q)$ is a list of all words $\iota(u)$ such that for some prefix v of q, $(u, v) \in B$. Then $\kappa_>^{\Sigma^\omega} \circ g(q) = \delta^{-1}[\kappa_c(q)]$. Therefore, δ is c-proper. □

For a computable metric space, Schröder's proper representation δ_p is even c-proper and c-admissible. In the following let $\nu_\alpha : \mathbb{N} \to A$ be a (total) numbering defined by $\nu_\alpha(i) := \alpha \circ \nu_\Sigma(i)$, if $\nu_\Sigma(i) \in \mathrm{dom}(\alpha)$ and $\nu_\alpha(i) := a$ otherwise (for some fixed $a \in A$). Then $\alpha \equiv \nu_\alpha \circ \nu_\Sigma^{-1}$ and (2) in Def. 2.1 holds for ν_α accordingly.

Theorem 4.4 (existence of c-proper c-admissible representations) *Define a representation $\delta_p :\subseteq \Sigma^\omega \to X$ as follows: $\mathrm{dom}(\delta_p) \subseteq \{0,1\}^\omega$ and for all $x \in X$ and all $p \in \{0,1\}^\omega$:*

$$\delta_p(p) = x \iff (\forall m, n) \begin{cases} p\langle m, n\rangle = 1 & \Longrightarrow \quad d(x, \nu_\alpha(m)) \leq 2^{-n} \\ p\langle m, n\rangle = 0 & \Longrightarrow \quad d(x, \nu_\alpha(m)) \geq 2^{-n-1} \end{cases}.$$

Then δ_p is c-proper and c-admissible.

Since $d(x, \nu_\alpha(m)) \leq 2^{-n}$ or $d(x, \nu_\alpha(m)) \geq 2^{-n-1}$ for all $m, n \in \mathbb{N}$, the representation δ_p is well-defined. However, the domain of δ_p is not closed in general. As an example, consider the computable metric space of natural numbers from Ex. 2.2.1. Suppose that $\Sigma^\omega \setminus \mathrm{dom}(\delta_p)$ is open. If $\delta_p(0^\omega) = x$, then $d(x, a) \geq 1/2$ for all $a \in \mathbb{N}$ which is impossible, therefore, $0^\omega \in \Sigma^\omega \setminus \mathrm{dom}(\delta_p)$. Then $0^k\Sigma^\omega \cap \mathrm{dom}(\delta_p) = \emptyset$ for some number k. Let $j \in \mathbb{N}$ be the smallest number not in $\nu_\alpha[\{m_0, \ldots, m_k\}]$ (where $i = \langle m_i, n_i \rangle$). Then $q \in 0^k\Sigma^\omega$ for some δ_p-name q of j, hence $q \in 0^k\Sigma^\omega \cap \mathrm{dom}(\delta_p)$ (contradiction).

Proof of Theorem 4.4: First, we show that there is some r.e. set $Y \subseteq \Sigma^\omega \times \Sigma^*$ such that

$$\forall q \in \mathrm{dom}(\kappa_c).\ \forall\, z \in \Sigma^*.\ \ (q, z) \in Y \iff z\Sigma^\omega \cap \delta_p^{-1}[\kappa_c(q)] = \emptyset. \tag{9}$$

Notice that $z\Sigma^\omega \cap \delta_p^{-1}[K] = \emptyset \iff \delta_p[z\Sigma^\omega] \cap K = \emptyset$. Let $K = \kappa_c(q)$ and $z = z_0 z_1 \ldots z_{k-1} \in \{0,1\}^k$. We have

$$\delta_p[z\Sigma^\omega] \cap K = \emptyset \iff \forall x \in X.\ \big(x \notin \delta_p[z\Sigma^\omega] \text{ or } x \notin K\big). \tag{10}$$

Since

$$x \notin \delta_p[z\Sigma^\omega] \iff \exists \langle m,n\rangle < |z|. \begin{cases} z_{\langle m,n\rangle} = 1 & \text{and} \quad d(x,\nu_\alpha(m) > 2^{-n} \\ & \text{or} \\ z_{\langle m,n\rangle} = 0 & \text{and} \quad d(x,\nu_\alpha(m)) < 2^{-n-1}, \end{cases}$$

we have $x \notin \delta_p[z\Sigma^\omega]$, iff for some $t \in \mathrm{dom}(\mathrm{I})$, $x \in \mathrm{I}(t)$ and

$$Q_1: \ \exists \langle m,n\rangle < |z|. \begin{cases} z_{\langle m,n\rangle} = 1 & \text{and} \quad d(\mathrm{ct}(t),\nu_\alpha(m) > 2^{-n} + \mathrm{rad}(t) \\ & \text{or} \\ z_{\langle m,n\rangle} = 0 & \text{and} \quad d(\mathrm{ct}(t),\nu_\alpha(m)) + \mathrm{rad}(t) < 2^{-n-1}. \end{cases}$$

Furthermore, $x \notin K$, iff for some $t \in \mathrm{dom}(\mathrm{I})$, $x \in \mathrm{I}(t)$ and

$$Q_2: \ \exists w \in \mathrm{FS}. \big(\#w\# \text{ is a subword of } q \text{ and } \forall t' \in \mathrm{fs}(w).\ t' \bowtie t\big)$$

(see(5)), since q lists arbitrarily narrow finite coverings of K. Therefore, $\delta_p[z\Sigma^\omega] \cap K = \emptyset$, iff for every $x \in X$ there is some open neighborhood $\mathrm{I}(t)$ of x such that (Q_1 or Q_2). Notice that $\mathrm{I}(t) \cap \delta_p[z\Sigma^\omega] \cap K = \emptyset$, if t satisfies (Q_1 or Q_2).

Since K is compact, already finitely many balls $\mathrm{I}(t)$ with (Q_1 or Q_2) cover K, iff $\delta_p[z\Sigma^\omega] \cap K = \emptyset$. Since q lists all finite collections of open basic balls covering K, we obtain $\delta_p[z\Sigma^\omega] \cap K = \emptyset$, iff

$$Q_3: \ \exists w \in \mathrm{FS}. \big(\#w\# \text{ is a subword of } q \text{ and } \forall t \in \mathrm{fs}(w).\ (Q_1 \text{ or } Q_2)\big).$$

Define $(q,z) \in Y :\iff (z \notin \{0,1\}^* \text{ or } Q_3)$. Then Y is r.e. and satisfies (9).

There is a computable function $g : \Sigma^\omega \to \Sigma^\omega$ such that $\iota(z)$ is a subword of $g(q)$, iff $(q,z) \in Y$ ($\iota(a_1 \ldots a_m) := 110a_10\ldots0a_m011$ is the standard wrapping). Then for $q \in \mathrm{dom}(\kappa_c)$, $g(q)$ lists all z such that $z\Sigma^\omega \cap Z = \emptyset$, $Z := \delta_p^{-1}[\kappa_c(q)]$, i.e. $\delta_p^{-1}[\kappa_c(q)] = \kappa_>^{\Sigma^\omega} \circ g(p)$. Therefore, δ_p is c-proper.

It remains to show $\delta_C \equiv \delta_p$.

Suppose $x = \delta_C(w_0\#w_1\#w_2\ldots)$. Then for all $n \in \mathbb{N}$, $d(x,\alpha(w_{n+3})) \le 2^{-n-3}$. For all $m,n \in \mathbb{N}$ determine $p\langle m,n\rangle$ as follows. Using (2) find rational numbers r,s such that $r < d(\alpha(w_{n+3},\nu_\alpha(m)) < s$ and $s - r < 2^{-n-3}$. Then $r - 2^{-n-3} < d(x,\nu_\alpha(m)) < s + 2^{-n-3}$. Choose $p\langle m,n\rangle := 1$, iff $s + 2^{-n-3} \le 2^{-n}$. Then $p\langle m,n\rangle := 1 \Longrightarrow d(x,\nu_\alpha(m)) < s + 2^{-n-3} \le 2^{-n}$ and $p\langle m,n\rangle := 0 \Longrightarrow d(x,\nu_\alpha(m)) > r - 2^{-n-3} > s - 2^{-n-2} > 2^{-n-1}$. Therefore, $\delta_p(p) = x$. There is some Type-2 machine computing p from $w_0\#w_1\#w_2\ldots \in \mathrm{dom}(\delta_C)$, hence $\delta_C \le \delta_p$.

On the other hand, let M be a Type-2 machine which on input $p \in \mathrm{dom}(\delta_p)$ determines a sequence $w_0\#w_1\#w_2\ldots$ as follows. For $n \in \mathbb{N}$ the machine searches for some $m \in \mathbb{N}$ such that $p\langle m,n\rangle = 1$. (Since A is dense, there is some m such that $d(x,\nu_\alpha(m)) < 2^{-n-1}$. Then $p\langle m,n\rangle = 0$ is false, hence $p\langle m,n\rangle = 1$ and so $d(x,\nu_\alpha(m)) \le 2^{-n}$.) Choose w_n such that $\alpha(w_n) = \nu_\alpha(m)$. We obtain $\delta_p(p) = \delta_C(w_0\#w_1\#w_2\ldots)$. Therefore, $\delta_p \le \delta_C$. □

In the following example, $K \mapsto \delta_p^{-1}[K]$ is not $(\kappa_{mc}, \kappa_{mc}^{\Sigma^\omega})$-computable, i.e. computable w.r.t. the minimal-cover representations of compact sets.

Example 4.5 Consider the real line $\mathbb{R}$ (Ex. 2.2.3). If the function $K \mapsto \delta_p^{-1}[K]$ for compact $K \subseteq \mathbb{R}$ is $(\kappa_{mc}, \kappa_{mc}^{\Sigma^\omega})$-continuous, then there is a continuous function $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that for every $q \in \mathrm{dom}(\kappa_{mc})$, $h(p)$ is a list of all $z \in \Sigma^*$ such that $\delta_p[z\Sigma^\omega] \cap \kappa_{mc}(q) \neq \emptyset$ (see [4]). Consider numbers $m_1, m_2 \in \mathbb{N}$ such that $\nu_\alpha(m_1) = 0 \in \mathbb{Q}$ and $\nu_\alpha(m_2) = 1/4 \in \mathbb{Q}$. There is some $p \in \mathrm{dom}(\delta_p)$ such that $\delta_p(p) = 1/2$ and $p\langle m_1, 0\rangle = 0$ and $p\langle m_2, 2\rangle = 1$. For $k = \max(\langle m_1, 0\rangle, \langle m_2, 2\rangle)$, $\delta[p_{\leq k}\Sigma^\omega] = \{1/2\}$. Let q be a κ_{mc}-name of $\{1/2\}$. Then h cannot be continuous in q. □

The representation δ_p from Thm. 4.4 is constructed artificially. By the following corollary, c-proper c-admissible representations can be obtained from c-admissible representations by restriction.

Corollary 4.6 *Every c-admissible representation has a c-proper c-admissible restriction.*

Proof: Since $\delta_p \equiv \delta_C$ by Thm. 4.4, $\delta_p \equiv \delta$. Let $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ be computable translations from δ_p to δ. Let γ be the restriction of δ to the subset $h[\mathrm{dom}(\delta_p)]$ of $\mathrm{dom}(\delta)$. Then $\gamma \equiv \delta_p$. Notice that h translates δ_p to γ. For any $K \subseteq X$, $\gamma^{-1}[K] = h \circ \delta_p^{-1}[K]$. By Thm. 3.3, the function $L \mapsto h[L]$ for compact $L \subseteq \mathrm{dom}(h)$ is $(\kappa_>^{\Sigma^\omega}, \kappa_>^{\Sigma^\omega})$-computable (let $\delta(p) := h$ for all $p \in \Sigma^\omega$ in Thm. 3.3). Therefore, by Thm. 4.4, $K \mapsto \gamma^{-1}[K]$ for compact $K \subseteq X$ is $(\kappa_c, \kappa_>^{\Sigma^\omega})$-computable. □

If a representation δ is c-proper, then it is proper (i.e., for every compact set $K \subseteq X$, $\delta^{-1}[K]$ is compact) and $\delta^{-1}[K]$ is $\kappa_>^{\Sigma^\omega}$-computable, if K is κ_c-computable. For $i = 1, 2$, let $\mathbf{X_i} = (X_i, d_i, A_i, \alpha_i)$ be a computable metric space. Define the product $\mathbf{X} = (X, d, A, \alpha)$ of $\mathbf{X_1}$ and $\mathbf{X_2}$ by $d((x_1, x_2), (y_1, y_2)) := \max(d_1(x_1, y_1), d_2(x_2, y_2))$, $A := A_1 \times A_2$ and $\alpha\langle u_1, u_2\rangle := (\alpha_1(u_1), \alpha_2(u_2))$. $\mathbf{X}$ is a computable metric space. Let δ_C^1, δ_C^2 and δ_C be the Cauchy representations, let I^1, I^2 and I be the notations of the open rational balls, and let κ_c^1, κ_c^2 and κ_c be the covering representations of the compact subsets of X_1, X_2 and X, respectively. Then $\mathrm{I}\langle\langle u_1, u_2\rangle, v\rangle = \mathrm{I}^1\langle u_1, v\rangle \times \mathrm{I}^2\langle u_2, v\rangle$ and $[\delta_C^1, \delta_C^2] \equiv \delta_C$. For representations δ_1 and δ_2 of X_1 and X_2, respectively, the representation $[\delta_1, \delta_2] : \Sigma^\omega \to X_1 \times X_2$ of the Cartesian product is defined by $[\delta_1, \delta_2]\langle p_1, p_2\rangle := (\delta_1(p_1), \delta_2(p_2))$, see Sec. 3.3 in [40]. If δ_1 and δ_2 are admissible, then $[\delta_1, \delta_2]$ is admissible. We conclude this section with a very useful theorem.

Theorem 4.7 (c-proper product) *If $\delta_i :\subseteq \Sigma^\omega \to X_i$ $(i = 1, 2)$ are c-proper c-admissible representations, then the product $[\delta_1, \delta_2] :\subseteq \Sigma^\omega \to X_1 \times X_2$ is c-proper c-admissible. This holds accordingly for $n > 2$ factors.*

Proof: Since $\delta_C \equiv [\delta_C^1, \delta_C^2] \equiv [\delta_1, \delta_2]$, the product $[\delta_1, \delta_2]$ is c-admissible.
Proposition 1: $\mathrm{pr}_1 : K \mapsto \{x_1 \mid \exists x_2.\ (x_1, x_2) \in K\}$ is (κ_c, κ_c^1)-computable.
Proof 1: If $\langle\langle u_1, u_2\rangle, v\rangle$ is a formal ball in a finite covering of K, then $\langle u_1, v\rangle$ is a formal ball in a finite covering of $\mathrm{pr}_1(K)$. A κ_c^1-name of $\mathrm{pr}_1(K)$ can be computed from a κ_c-name of K by substituting everywhere $\langle u_1, v\rangle$ for $\langle\langle u_1, u_2\rangle, v\rangle$.
Proposition 2: $\mathrm{sec} : (x_1, K) \mapsto \{x_2 \mid (x_1, x_2) \in K\}$ is $(\delta_1, \kappa_c, \kappa_c^2)$-computable.
Proof 2: Since $\delta_1 \equiv \delta_C^1$, there is a computable function $h :\subseteq \Sigma^\omega \times \Sigma^* \to \Sigma^*$ such that $\delta_1(p) \in \mathrm{I}^1 \circ h(p, 0^n)$ and $\mathrm{rad}(h(p, 0^n)) \leq 2^{-n}$ for all $p \in \mathrm{dom}(\delta^1)$ and $n \in \mathbb{N}$. There is

a Type-2 machine M which on input and $p \in \mathrm{dom}(\delta^1_C)$ and $\#w_0\#w_1\#\ldots \in \mathrm{dom}(\kappa_c)$ produces a sequence of all words w such that there exist $k, m, n \in \mathbb{N}$ and words $u_{11}, u_{21}, v_1, \ldots, u_{1m}, u_{2m}, v_m$ such that

$$\begin{array}{rcl} \mathrm{fs}(w_k) & = & \{\langle\langle u_{11}, u_{21}\rangle, v_1\rangle, \ldots, \langle\langle u_{1m}, u_{2m}\rangle, v_m\rangle\} \\ \mathrm{fs}(w) & = & \{\langle u_{21}, v_1\rangle, \ldots, \langle u_{2m}, v_m\rangle\} \\ h(p, 0^n) & \prec_1 & \langle u_{1i}, v_i\rangle \text{ for } i = 1, \ldots, m. \end{array}$$

Then f_M is a $(\delta_1, \kappa_c, \kappa_c^2)$-realization of sec. This proves Prop. 2.

Let $K = \kappa_c(q)$. Then for all $x_1 \in X_1$ and $x_2 \in X_2$,

$$(x_1, x_2) \notin K \iff x_1 \notin \mathrm{pr}_1(K) \text{ or } x_2 \notin \mathrm{sec}(x_1, K),$$

hence for all $p_1, p_2 \in \Sigma^\omega$,

$$\langle p_1, p_2\rangle \notin [\delta_1, \delta_2]^{-1}[K] \iff \begin{cases} p_1 \notin \delta_1^{-1}[\mathrm{pr}_1(K)] \text{ or} \\ p_1 \in \mathrm{dom}(\delta_1) \text{ and } p_2 \notin \delta_2^{-1}[\mathrm{sec}(\delta_1(p_1), K)]. \end{cases}$$

Since δ_1 is c-proper, by Prop. 1 there is a computable $(\kappa_c, \kappa_>^{\Sigma^\omega})$-realization f_1 of $K \mapsto \delta_1^{-1}[\mathrm{pr}_1(K)]$. Since δ_2 is c-proper, by Prop. 2 there is a computable $(\delta^1_C, \kappa_c, \kappa_>^{\Sigma^\omega})$-realization f_2 of $(x_1, K) \mapsto \delta_2^{-1}[\mathrm{sec}(x_1, K)]$. Therefore, for all $p_1, p_2 \in \Sigma^\omega$ and $q \in \mathrm{dom}(\kappa_c)$,

$$\langle p_1, p_2\rangle \notin [\delta_1, \delta_2]^{-1}[K] \iff \begin{cases} p_1 \notin \kappa_>^{\Sigma^\omega} \circ f_1(q) \text{ or} \\ p_1 \in \mathrm{dom}(\delta_1) \text{ and } p_2 \notin \kappa_>^{\Sigma^\omega} \circ f_2(p_1, q). \end{cases}$$

Proposition 3: For every computable function $f :\subseteq Y_1 \times \ldots \times Y_k \to \Sigma^\omega$ ($Y_i \in \{\Sigma^*, \Sigma^\omega\}$) there is a total computable function $g : Y_1 \times \ldots \times Y_k \to \Sigma^\omega$, such that $\kappa_> \circ f(x) = \kappa_>^{\Sigma^\omega} \circ g(x)$, if $f(x)$ exists.
Proof 3: Let M be a Type-2 machine computing f. There is a Type-2 machine N, which on input x writes every $\iota(w)$ which M writes on input x and additionally writes infinitely often the word 11 (in order to produce a result in Σ^ω). Remember that $\kappa_>^{\Sigma^\omega}(p)$ exists for all $p \in \Sigma^\omega$.

By Prop. 3 we may assume that f_1 and f_2 are total functions. Define $Y \subseteq \Sigma^\omega \times \Sigma^\omega$ by

$$(q, \langle p_1, p_2\rangle) \in Y \iff p_1 \notin \kappa_>^{\Sigma^\omega} \circ f_1(q) \text{ or } p_2 \notin \kappa_>^{\Sigma^\omega} \circ f_2(p_1, q).$$

Since $p \notin \kappa_>^{\Sigma^\omega}(p')$ is r.e. and f_1 and f_2 are total computable functions, Y is r.e.. The property

$$(q, \langle p_1, p_2\rangle) \in Y \iff \langle p_1, p_2\rangle \notin [\delta_1, \delta_2]^{-1}[\kappa_c(q)] \quad (p_1, p_2 \in \Sigma^\omega, q \in \mathrm{dom}(\kappa_c)) \quad (11)$$

can be proved straightforwardly. Since Y is r.e., there is a computable function $g :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^*$ such that $Y = \mathrm{dom}(g)$. By the smn-theorem, there is a computable function $r : \Sigma^\omega \to \Sigma^\omega$ such that $g(q, p) = \eta^{\omega*}_{r(q)}(p)$ (where $= \eta^{\omega*}$ is the standard representation of $F^{\omega*}$, see Sec. 2.3 in [40]). Therefore, $\langle p_1, p_2\rangle \in \mathrm{dom}(\eta^{\omega*}_{r(q)})$, iff $\langle p_1, p_2\rangle \notin [\delta_1, \delta_2]^{-1}[\kappa_c(q)]$, hence

$$\Sigma^\omega \setminus \mathrm{dom}(\eta^{\omega*}_{r(q)}) = [\delta_1, \delta_2]^{-1}[\kappa_c(q)].$$

Define a representation $\kappa_{\rm dom}$ of the compact subsets of Σ^ω by $\kappa_{\rm dom}(p) := \Sigma^\omega \setminus {\rm dom}(\eta_p^{\omega *})$. Then $[\delta_1,\delta_2]^{-1}[\kappa_c(q)] = \kappa_{\rm dom} \circ r(q)$, hence $K \mapsto [\delta_1,\delta_2]^{-1}[K]$ is $(\kappa_c, \kappa_{\rm dom})$-computable. Since $\kappa_{\rm dom}(p) \equiv \kappa_>^{\Sigma^\omega}$ (this is $\delta_{\rm dom} \equiv \delta_{union}$ for Cantor space in [4]), the function $K \mapsto [\delta_1,\delta_2]^{-1}[K]$ is $(\kappa_c, \kappa_>^{\Sigma^\omega})$-computable. Therefore $[\delta_1,\delta_2]$ is c-proper.

The proof for $n > 2$ can be reduced to the case $n = 2$ by induction. □

5 Complexity and Lookahead of Computations

In [41] and [40] (Def. 7.1.1) time complexity is introduced for functions on Cantor space. Simultaneously as another useful concept, *lookahead*, counting the number of input symbols, is introduced and studied. For Type-2 machines computing real functions w.r.t. the signed digit representation $\rho_{\rm sd}$, time complexity and lookahead are defined in [39] and [40] (Def. 7.2.6). In both cases complexity and lookahead are considered as functions of the number of output digits. On Cantor space, the first n digits determine the result with precision 2^{-n}, and for machines realizing real functions, the first n digits after the dot of a $\rho_{\rm sd}$-name determine the result with precision 2^{-n}. Therefore, in both cases complexity and lookahead are considered as functions of precision. In the following we generalize these definitions to computations of machines realizing functions $f :\subseteq X_1 \to X_2$ on computable metric spaces. First, we introduce approximation functions for the codomain X_0. Then we define the computational complexity and the lookahead of a realizing machine. As our main theorem we prove that for c-proper c-admissible representations of the input set, from every compact set $K \in {\rm dom}(f)$ of the realized function, upper bounds of complexity and lookahead can be computed. In particular, for computable compact sets, time and lookahead have computable upper bounds.

Definition 5.1 (approximation function) *An approximation function for a representation δ of a computable metric space $\mathbf{X} = (X, d, A, \alpha)$ is a function* app $:\subseteq \Sigma^* \times \Sigma^* \to \Sigma^*$ *such that for all $p \in {\rm dom}(\delta)$ and all $k \in \mathbb{N}$ there is a prefix z of p such that*

$$d(\delta(p), \alpha \circ {\rm app}(z, 0^k)) \le 2^{-k} \tag{12}$$

and $w = \varepsilon$, if $(z, 0^k) \in {\rm dom}({\rm app})$ and $(zw, 0^k) \in {\rm dom}({\rm app})$.

Therefore, the prefix z of p is sufficient to determine some point $a \in A$ such that $d(\delta(p), a) \le 2^{-k}$. For a Cauchy representation we can choose ${\rm app}(w_0\#w_1\#\ldots w_k\#, 0^k) := w_k$. For the signed digit representation $\rho_{\rm sd}$ (Ex. 4.2.2) of the computable metric space for the real line from Ex. 2.2.4, we can choose ${\rm app}(b_l \ldots b_0 \bullet b_{-1} \ldots b_{-k}, 0^k) := w$, such that $\nu_D(w) = \sum_{j=l}^{-k} b_j \cdot 2^j$.

As usual, we call a function md : $\mathbb{N} \to \mathbb{N}$ a modulus of continuity of δ at $p \in {\rm dom}(\delta)$, iff

$$d(\delta(p), \delta(q)) \le 2^{-k} \quad \text{whenever} \quad d_{\Sigma^\omega}(p, q) \le 2^{-{\rm md}(k)} \tag{13}$$

for all $k \in \mathbb{N}$ and $q \in {\rm dom}(f)$. And we call md a modulus of uniform continuity of δ on $Y \subseteq {\rm dom}(\delta)$, iff (13) for all $p, q \in Y$. On Cantor space, $d_{\Sigma^\omega}(p, q) \le j$, iff the first j symbols of p and q coincide. Therefore, if md(k) is the length of the unique prefix

z of p such that $(z, 0^k) \in \mathrm{dom(app)}$, then md is a modulus of continuity of δ at p. Notice that in general such a modulus function md is not minimal (e.g. for a Cauchy representation of a discrete space).

Lemma 5.2 *Every c-admissible representation δ of a computable metric space $\mathbf{X} = (X, d, A, \alpha)$ has a computable approximation function.*

Proof: Since δ is c-admissible, there is some Type-2 machine M translating δ to δ_C. There is a Type-2 machine N, which on input $(z, 0^k)$ simulates M on input $z0^\omega$ until a result of the form $w_0\#w_1\#\ldots w_k\#$ has been produced and then writes w_k, if exactly the word z has been read by M during this simulation, (and diverges otherwise). □

In the following we define the time and the lookahead of a machine to compute the result with "precision k" relative to a given approximation function app.

Definition 5.3 (time, lookahead of a machine) *For a Type-2 machine M of type $(\Sigma^\omega)^m \to \Sigma^\omega$ define time and lookahead w.r.t. an approximation function* app *by*

$$\mathrm{Time}_M(y)(k) \quad := \quad \begin{cases} \textit{the number of steps which } M \textit{ on input } y \textit{ needs to} \\ \textit{compute some } z \in \Sigma^* \textit{ such that } \ (z, 0^k) \in \mathrm{dom(app)}\ , \end{cases}$$

$$\mathrm{La}_M(y)(k) \quad := \quad \begin{cases} \textit{the maximal number of input symbols which } M \\ \textit{reads from some input tape until it has printed} \\ \textit{some } z \in \Sigma^* \textit{ such that } \ (z, 0^k) \in \mathrm{dom(app)}\ . \end{cases}$$

for all $y \in (\Sigma^\omega)^m$ and all $k \in \mathbb{N}$.

In applications, the approximation function app should be easily computable and the "precision test" dom(app) should be decidable very easily. Def. 7.1.1 in [40] corresponds to the special case $\mathrm{dom(app)} = \{(z, 0^k) \mid k = |z|\}$. The functions $(y, k) \mapsto \mathrm{Time}_M(y)(k)$ and $(y, k) \mapsto \mathrm{La}_M(y)(k)$ are computable, if the precision test dom(app) is recursive. Since reading a symbol requires at least one step of computation, $\mathrm{La}_M(y)(k) \le \mathrm{Time}_M(y)(k)$. Next, we define the complexity of a realized function on a set K.

Definition 5.4 (complexity of a realized function) *Let δ_0 be a representation of X_0 with approximation function* $\mathrm{app} \subseteq \Sigma^* \times \Sigma^* \to \Sigma^*$. *Let M be a Type-2 machine such that f_M is a $(\delta_1, \ldots, \delta_m, \delta_0)$-realization of a function $f :\subseteq X_1 \times \ldots \times X_m \to X_0$. Then M computes f on $K \subseteq \mathrm{dom}(f)$ in time $t : \mathbb{N} \to \mathbb{N}$ with lookahead $s : \mathbb{N} \to \mathbb{N}$, iff*

$$\mathrm{TIME}_M^K \in O(t), \tag{14}$$

$$\mathrm{LA}_M^K(k) \le s(k) \quad (\forall k \in \mathbb{N})) \tag{15}$$

where $\mathrm{TIME}_M^K(k) := \max_{(\delta_1, \ldots, \delta_m)(y) \in K} \mathrm{Time}_M(y)(k)$,
$\mathrm{LA}_M^K(k) := \max_{(\delta_1, \ldots, \delta_m)(y) \in K} \mathrm{La}_M(y)(k)$.

Remember that for a function $t : Z \to \mathbb{N}$, $f \in O(t) \iff \exists c. \forall z. f(z) \le c \cdot t(z) + c$. For time we consider membership in $O(t)$ in order to obtain a definition robust under the usual modifications of the Turing machine model. Notice that (14) is stronger than "$\mathrm{Time}_M^y \in O(t)$ for all $y \in (\delta_1, \ldots, \delta_m)^{-1}[K]$" (in (14), the constant c must be the same for all $y \in K$). Definition 7.2.6 (via Def. 7.1.1) in [40] is the special case of Def. 5.4 for the signed digit representation.

The above definition looks very natural. Unfortunately, it is meaningless in almost all situations, since for most spaces, for most representations and for most subsets K, the maxima in the definitions of TIME_M^K and LA_M^K do not exist. As the central result of this paper we show that the definition is meaningful for c-proper c-admissible representations of metric spaces and compact subsets K. Applying Thm. 3.3, we prove that under appropriate assumptions upper bounds of TIME_M^K and LA_M^K can be computed from the compact set K. Let $\delta_{\mathbb{B}}$ be the standard representation of Baire space (see Ex. 2.2.6).

Theorem 5.5 *For $i = 1, \ldots, m$ let δ_i be a c-proper c-admissible representation of the computable metric space $\mathbf{X}_i$, and let δ_0 be a c-admissible representation of the computable metric space $\mathbf{X}_0$ with approximation function* app *such that* dom(app) *is recursive. Let κ_c be the covering representation of the set of compact subsets $\mathcal{K}(X)$ of $X := X_1 \times \ldots \times X_m$. Let M be a Type-2 machine such that f_M is a $(\delta_1, \ldots, \delta_m, \delta_0)$-realization of a function $f :\subseteq X_1 \times \ldots \times X_m \to X_0$. Then the multi-valued functions $H_T :\subseteq \mathcal{K}(X) \rightrightarrows \mathbb{B}$ and $H_L :\subseteq \mathcal{K}(X) \rightrightarrows \mathbb{B}$,*

$$\begin{aligned} \mathrm{graph}(H_T) &= \{(K,t) \mid K \subseteq \mathrm{dom}(f),\ \forall k \in \mathbb{N}.\ \mathrm{TIME}_M^K(k) \le t(k)\}, \\ \mathrm{graph}(H_L) &= \{(K,s) \mid K \subseteq \mathrm{dom}(f),\ \forall k \in \mathbb{N}.\ \mathrm{LA}_M^K(k) \le s(k)\} \end{aligned}$$

are $(\kappa_c, \delta_{\mathbb{B}})$-computable.

Proof: By Thm. 4.7, $[\delta_1, \ldots, \delta_m]$ is a c-proper c-admissible representation of the the product space $\mathbf{X}_1 \times \ldots \times \mathbf{X}_m$. Therefore, the function $H_1 : K \mapsto [\delta_1, \ldots, \delta_m]^{-1}[K]$ for compact $K \subseteq X$ is $(\kappa_c, \kappa_c^{\Sigma^\omega})$-computable.

The function $G : (k, y) \mapsto \mathrm{Time}_M(y)(k)$, $k \in \mathbb{N}$, $y \in X$, is $(\nu_{\mathbb{N}}, [\mathrm{id}_{\Sigma^\omega}]^m, \nu_{\mathbb{N}})$-computable, therefore, $(\delta_C^{\mathbb{N}}, [\mathrm{id}_{\Sigma^\omega}]^m, \delta_C^{\mathbb{N}})$-computable, since $\nu_{\mathbb{N}} \equiv \delta_C^{\mathbb{N}}$ by Ex. 2.2.1. Define a representation $\delta_\to$ of partial functions $h :\subseteq \Sigma^\omega \to \mathbb{N}$ by

$$\delta_\to(p)\langle y\rangle := \mathrm{Time}_M(y)(\delta_C^{\mathbb{N}}(p)), \quad y = (y_1, \ldots, y_m), \quad y_i \in \Sigma^\omega. \tag{16}$$

Then for some computable $(\delta_C^{\mathbb{N}}, [\mathrm{id}_{\Sigma^\omega}]^m, \delta_C^{\mathbb{N}})$-realization r of G,

$$\begin{aligned} \mathrm{apply}(\delta_\to(p), \mathrm{id}_{\Sigma^\omega}\langle y\rangle) &= \delta_\to(p)\langle y\rangle &&= \mathrm{Time}_M(y)(\delta_C^{\mathbb{N}}(p)) \\ &= G(\delta_C^{\mathbb{N}}(p), [\mathrm{id}_{\Sigma^\omega}]^m\langle y\rangle) &&= \delta_C^{\mathbb{N}} \circ r(p, \langle y\rangle). \end{aligned}$$

Therefore, the apply function of $\delta_\to$ is $(\delta_\to, \mathrm{id}_{\Sigma^\omega}, \delta_C^{\mathbb{N}})$-computable and so $(\delta_\to, \delta_C^{\Sigma^\omega}, \delta_C^{\mathbb{N}})$-computable, since $\mathrm{id}_{\Sigma^\omega} \equiv \delta_C^{\Sigma^\omega}$ by Ex. 2.2.5. By Thm. 3.3, $H_2 : (h, K) \mapsto h[K]$ for compact $K \subseteq \mathrm{dom}(h)$ is $(\delta_\to, \kappa_c^{\Sigma^\omega}, \kappa_c^{\mathbb{N}})$-computable.

There is a computable function s such that $\nu_{\mathbb{N}}(w) = \delta_C^{\mathbb{N}} \circ s(w)$. Define a function $H_3 : k \mapsto h$ by $H_3 \circ \nu_{\mathbb{N}}(w) := \delta_\to \circ s(w)$. By (16), H_3 is well-defined. H_3 is $(\nu_{\mathbb{N}}, \delta_\to)$-computable. Therefore, the function

$$H : (K, k) \mapsto H_2(H_3(k), H_1(K))$$

is $(\kappa_c, \nu_{\mathbb{N}}, \kappa_c^{\mathbb{N}})$-computable.

Let $k \in \mathbb{N}$ and $y \in (\Sigma^\omega)^m$. There is some w such that $\nu_{\mathbb{N}}(w) = k$. Then

$$H_3(k)\langle y\rangle = H_3(\nu_{\mathbb{N}}(w))\langle y\rangle = \delta_\to(s(w))\langle y\rangle = \mathrm{Time}_M(y)(k).$$

Therefore, for compact $K \subseteq \mathrm{dom}(f)$,

$$H(K,k) = \{\mathrm{Time}_M(y)(k) \mid y \in (\delta_1, \dots, \delta_m)^{-1}[K]\}.$$

Finally, it suffices to find an upper bound of the finite set $H(K,k)$. Since $\kappa_c^{\mathbb{N}} \equiv \kappa_>^{\mathbb{N}}$ by Ex. 3.2.1, the function H is $(\kappa_c, \nu_{\mathbb{N}}, \kappa_>^{\mathbb{N}})$-computable. If $q_0 = \#w_0\#w_1\#\dots$ is a $\kappa_>^{\mathbb{N}}$-name of $H(K,k)$, then $H(K,k) \subseteq \nu_{f\mathbb{N}}(w_0)$, and so $l := \max(\nu_{f\mathbb{N}}(w_0))$ is an upper bound of $H(K,k)$.

Let $h :\subseteq \Sigma^\omega \times \Sigma^* \to \Sigma^*$ be a computable $(\kappa_c, \nu_{\mathbb{N}}, \kappa_>^{\mathbb{N}})$-realization of H. There is a Type-2 machine N which on input $p \in \mathrm{dom}(\kappa_c)$ such that $K := \kappa_c(p) \subseteq \mathrm{dom}(f)$ computes a sequence $0^{i_0}10^{i_1}1\dots$ such that $i_k = \max(\nu_{f\mathbb{N}}(w_0))$ where for some w with $\nu_{\mathbb{N}}(w) = k$, $h(p,w) = \#w_0\#w_1\#\dots$. If $t = \delta_{\mathbb{B}}(0^{i_0}10^{i_1}1\dots)$, then for all k, $\mathrm{TIME}_M^K(k) \le t(k)$. Therefore, f_N is a $(\kappa_c, \delta_{\mathbb{B}})$-realization of the multi-valued function H_T. The proof for F_L is almost the same. □

Since $\mathrm{La}_M(y)(k) \le \mathrm{Time}_M(y)(k)$, every upper time bound is an upper lookahead bound. The direct proof, however, might give much smaller lookahead bounds.

Corollary 5.6 *Under the assumptions of Thm. 5.5, on every κ_c-computable compact set $K \subseteq \mathrm{dom}(f)$, time and lookahead have computable bounds.*

Since points $x \in X_0$ can be identified with the functions $f : \{()\} \to X_0$ (where $()$ is the the tuple with 0 components), we obtain as a special case:

Definition 5.7 (complexity of a point) *Let M be a Type-2 machine computing a δ_0-name $p = f_M()$ of a point $x \in X_0$. Then M computes x in time $t : \mathbb{N} \to \mathbb{N}$, iff $\mathrm{Time}_M() \in O(t)$.*

In $\mathrm{Time}_M()(k)$ the machine M can determine x with precision 2^{-k}.

Example 5.8 Consider the signed digit representation ρ_{sd} (Ex. 4.2.2) of the real line for input and output, and choose $\{(u{\bullet}v, 0^{|v|}) \mid u, v \in \{0,1,-1\}^*\}$ as the precision test dom(app) for measuring the output precision. Then

1. addition is computable in time k with lookahead $k + c$ for some c,
2. multiplication and division are computable in time $k \cdot \log k \cdot \log\log k$ with lookahead $2k + c$ for some c,
3. sin and exp are computable in time $k \cdot \log^2 k \cdot \log\log k$

on every compact subset of its domain [6, 40]. □

Occasionally, time and lookahead may have computable bounds also for non-compact sets K (not contained in a compact set). As an example, let M compute a constant computable value $x_0 \in X_0$ without reading the input.

If a set $K \in \mathrm{dom}(f)$ is not compact but a K_σ-set, i.e., a countable union of compact sets, $K = \bigcup_{n\in\mathbb{N}} K_n$, then the complexity of a machine can be determined by a function of the index n and the output precision k; replace (14) by

$$\exists c.\forall k, n.\ \mathrm{TIME}_M^{K_n}(k) \le c \cdot t(n,k) + c$$

and so the bound is a function $t(n,k)$. Notice that every open subset of the real line is a K_σ-set (examples: $\mathbb{R} = \bigcup_{n\in\mathbb{N}}[-n;n]$, $\mathbb{R}_+ = \bigcup_{n\in\mathbb{N}}[1/n;n]$). By the following example, ordinary Type-1 complexity can be considered as a special case of the new concepts introduced here.

Example 5.9 (word functions) Consider the computable discrete metric space $(\Sigma^*, d, \Sigma^*, \mathrm{id}_{\Sigma^*})$. Then id_{Σ^*} is equivalent to the Cauchy representation (see Ex. 2.2.1). Define a representation δ of Σ^* by $\delta(p) = w \iff p = \iota(w)0^\omega$ (where $\iota(a_1 \ldots a_m) := 110a_10\ldots0a_m011$). Then $\delta \equiv \mathrm{id}_{\Sigma^*}$ and δ is c-proper (see Ex. 4.2.1). For measuring output precision define the approximation function app such that $\mathrm{dom(app)} := \{(\varepsilon,\varepsilon)\} \cup \{(\iota(w), 0^k) \mid w \in \Sigma^*, k \geq 1\}$. Let $f :\subseteq \Sigma^* \to \Sigma^*$ be a computable function. From a Turing machine M computing f one can construct easily a Type-2 machine N such that f_N is a (δ,δ)-realization of f and vice versa such that $O(T_M(w)) = O(\mathrm{TIME}_N^{\{w\}}(1))$.

In Type-1 theory, the complexity of a computable function $f : \Sigma^* \to \Sigma^*$ is usually measured as a function of the *length* of the input: $T'_M(n) := \max\{T_M(w) \mid w \in \Sigma^n\}$. The corresponding Type-2 concept is $\mathrm{TIME}_N^{\Sigma^n}(1)$, where N is a Type-2 machine realizing f w.r.t. δ. Notice that Σ^n is a compact subset of Σ^*. □

6 Concise Representations

Reading and writing symbols contributes to the time complexity of a computation. Since complexity is measured as a function of precision, and since for a modulus of uniform continuity $\mathrm{md} : \mathbb{N} \to \mathbb{N}$ of a representation δ, $\mathrm{md}(k)$ is the length of a prefix of p which determines $\delta(p)$ with precision 2^{-k}, the modulus of continuity of the representations contributes to the computational complexity.

In this section we consider compact metric spaces (X, d). We call a representation of X informally *concise*, iff it has a small modulus of uniform continuity. In general, an admissible representation of a compact space may have no uniform modulus of continuity. If it is proper, it has a uniform modulus of continuity (but it may not be concise). The following lemma is a computational version of this observation.

Lemma 6.1 *Let $\delta :\subseteq \Sigma^\omega \to X$ be a c-admissible c-proper representation of a computable metric space. Then the multi-valued function* $\mathrm{UMC} : \mathcal{K}(X) \rightrightarrows \mathbb{B}$*, defined by*

$$\mathrm{md} \in \mathrm{UMC}(K) \iff \begin{cases} \forall k \in \mathbb{N}\ .\forall p,q \in \delta^{-1}[K]. \\ \big(d_{\Sigma^\omega}(p,q) \leq 2^{-\mathrm{md}(k)} \Longrightarrow d(\delta(p),\delta(q)) \leq 2^{-k}\big) \end{cases}$$

for all compact $K \subseteq X$ and $\mathrm{md} : \mathbb{N} \to \mathbb{N}$*, is $(\kappa_c, \delta_{\mathbb{B}})$-computable.*

The proof is similar to that of Thm. 5.5 (use Lemma 5.2). In particular, the restriction of δ to any κ_c-computable compact set has a computable modulus of uniform continuity.

The example of δ_p (Thm. 4.4) shows that even names of c-admissible c-proper representations may be extremely redundant. The signed digit representation ρ_{sd} (Ex.

4.2.2), however, is concise. But in special applications, also $\rho_{\rm sd}$-names may contain unnecessary information.

As an example consider a real function with domain $K := [2^{100} - 1; 2^{100} + 1] \subseteq \mathbb{R}$ to be computed w.r.t. the signed digit representation. Then for every $\rho_{\rm sd}$-name of some $x \in K$, the 100 digits before the "$\bullet$" identify x as a member of K and the following digits after the "$\bullet$" localize x within the interval K. In this case it is more convenient to use a new "local" more concise representation δ_l, $\delta_l(p) := 2^{100} + \rho_{\rm sd}(p)$ where $\mathrm{dom}(\delta_l) := 0\bullet\{1, 0, \overline{1}\}^\omega$ in order to avoid reading unnecessary digits.

For our purpose, the crucial property of a compact set is its *width*. Remember that a metric space X is *totally bounded* (or *precompact* [10]), iff for every $\varepsilon > 0$, X can be covered by finitely many open balls of radius ε and that the space X is compact, iff it is totally bounded and complete.

Definition 6.2 (width of a totally bounded set) *Let (X, d) be a totally bounded metric space .*

1. *A set $E \subseteq X$ is called k-separated, iff $d(x, y) \geq 2^{-k}$ for $x, y \in E,\ x \neq y$.*
2. *A set $F \subseteq X$ is called k-spanning, iff $\forall x \in X.\ \exists y \in F.\ d(x, y) \leq 2^{-k}$.*
3. *A sequence $\mathrm{sep} : \mathbb{N} \to \mathbb{N}$ is a (lower) separation bound, iff for all k, X has a k-separated set of $\mathrm{sep}(k)$ elements.*
4. *A sequence $\mathrm{span} : \mathbb{N} \to \mathbb{N}$ is a (an upper) spanning bound, iff for all k, X has a k-spanning set of $\mathrm{span}(k)$ elements.*

The minimal spanning bound $\mathrm{wid} : \mathbb{N} \to \mathbb{N}$ is called width.

Sometimes the logarithm of the width is called metric entropy (cf. [37] and [29], p. 60). From the definitions we obtain immediately

$$\mathrm{sep}(k - 2) \leq \mathrm{wid}(k) \leq \mathrm{span}(k) \tag{17}$$

for every separation bound sep and spanning bound span of a totally bounded space.

Example 6.3

1. Cantor space $X = \{0, 1\}^\omega$: The set $\{0, 1\}^k 0^\omega$ is k-separated and k-spanning, $\mathrm{wid}(k) = 2^k$.
2. Unit interval $X = [0; 1] \subseteq \mathbb{R}$: $E_k := \{i \cdot 2^{-k} \mid 0 \leq i \leq 2^k\}$ is a k-separated set, and for $k \geq 1$, $F_k := \{(2i + 1)2^{-k} \mid 0 \leq i < 2^{k-1}\}$ is a k-spanning set, $\mathrm{wid}(k) = 2^{k-1}$.
3. Bounded subset of Baire space: A subset $L \subseteq \mathbb{B}$ is totally bounded, iff $\forall k \in \mathbb{N}.\ f(k) \leq r(k)$ for some $r \in \mathbb{B}$. For $r \in \mathbb{B}$, the subspace $L_r := \{f \in \mathbb{B} \mid \forall k.\ f(k) \leq r(k)\}$ has $\prod_{i=1}^{k}(r(i) + 1)$ as a separation bound and a spanning bound.

□

Every compact subset K of a metric space has a width which, however, may be non-computable even if the set K is κ_c-computable. But spanning bounds can be computed from κ_c-names. In particular, every κ_c-computable compact set has a computable spanning bound.

Lemma 6.4 *For every computable metric space* $\mathbf{X}$*, the multi-valued function* $S : K \rightrightarrows s$ *such that* $s : \mathbb{N} \to \mathbb{N}$ *is a spanning bound of the compact set* $K \subseteq X$*, is* $(\kappa_c, \delta_{\mathbb{B}})$*-computable.*

Proof : If $\kappa_c(p) = K$, then p is a list of all finite coverings of K with balls from $\mathcal{B}$ (see Sec. 3). For determining some $m \in s(k)$ from p and k find some covering of K with balls of radius 2^{-k}. Let m be its cardinality. $\square$.

The width supplies a lower bound for the modulus of uniform continuity of a continuous representation.

Theorem 6.5 (information theoretic bound) *Let* (X, d) *be a metric space with width* wid*, let* $\delta :\subseteq \Gamma^\omega \to X$ *be a representation of* X *with modulus* md_δ *of uniform continuity. Then*

$$\frac{\log_2 \mathrm{wid}(k)}{\log_2 |\Gamma|} \leq \mathrm{md}_\delta(k) \qquad (\textit{for all } k). \tag{18}$$

Proof : Notice that md_δ is a modulus of uniform continuity of δ, iff

$$\delta[B^c(p, 2^{-\mathrm{md}_\delta(k)})] \subseteq B^c(\delta(p), 2^{-k}) \quad (p \in \mathrm{dom}(\delta),\ k \in \mathbb{N}) \tag{19}$$

(where B^c denotes closed balls). The set of all $\delta[B^c(p, 2^{-\mathrm{md}_\delta(k)})]$ $(p \in \mathrm{dom}(\delta))$, covers X. At least $\mathrm{wid}(k)$ of these sets are necessary for covering X. Since $B^c(p, 2^{-\mathrm{md}_\delta(k)}) = B^c(p', 2^{-\mathrm{md}_\delta(k)})$, if the first $\mathrm{md}_\delta(k)$ symbols of p and p' coincide, there must be at least $\mathrm{wid}(k)$ words of length $\mathrm{md}_\delta(k)$, i.e., $|\Gamma|^{\mathrm{md}_\delta(k)} \geq \mathrm{wid}(k)$. $\square$

Let (X, d) be a compact metric space with spanning bound span. Then there is a sequence $i \mapsto F_i$ of subsets such that F_i is an i-spanning set of $\mathrm{span}(i)$ elements. Therefore, for every $x \in X$ there is a sequence $i \mapsto x_i$ such that $x_i \in F_i$ and $d(x, x_i) \leq 2^{-i}$ for all i. After appropriate encoding, such sequences can be used as "names" of x of a representation of X. By the next theorem, under sufficient computability assumptions, such a representation δ can be chosen to be c-admissible and c-proper with a modulus of continuity roughly bounded by $\log_2 \circ\, \mathrm{span}$. Remember that by Lemma 6.4, every κ_c-computable set K has a computable spanning bound.

Theorem 6.6 (existence of concise representations) *Let* (X, d, A, α) *be a compact computable metric space with computable spanning bound* span *such that* X *is* κ_c*-computable. Then* X *has a c-proper c-admissible representation* δ *with computable modulus* md_δ *of uniform continuity such that*

$$\mathrm{md}_\delta(k) \leq \sum_{i=0}^{k} \lceil \log_2(\mathrm{span}(i+1) + 1) \rceil. \tag{20}$$

Proof : For each i there is a set of at most $\mathrm{span}(i+1)$ closed balls of radius 2^{-i-1} covering X, hence there is a set of at most $\mathrm{span}(i+1)$ open balls from $\mathcal{B}$ of radius 2^{-i} which cover X. Using a computable κ_c-name $p = \#v_0\#v_1\#\ldots$ of X we can find a computable function $h : \mathbb{N} \to \Sigma^*$, such that $B_i := \mathrm{fs} \circ h(i)$ has at most $\mathrm{span}(i+1)$ elements and the set $F_i := \alpha[\mathrm{fs} \circ h(i)]$ is an i-spanning set for X (from p select an appropriate set of balls covering X and choose the centers).

For every i there is a bijective "coding" function $c_i : C_i \to B_i$ such that $C_i \subseteq \{0,1\}^*$ and $|w| < \lceil \log_2(\mathrm{span}(i+1)+1) \rceil$ for $w \in C_i$. Choose the functions $c_0, c_1, \ldots$ such that the coding function $c :\subseteq \mathbb{N} \times \Sigma^* \to \Sigma^*$, $c(i,w) = c_i(w)$, if $w \in C_i$, $c(i,w) = $ div otherwise, is computable and dom(c) is recursive. Define $\delta :\subseteq \Sigma^\omega \to K$ by

$$\delta(p) = x \quad :\Longleftrightarrow \quad \begin{cases} \text{there are words } \ w_0 \in C_0, \ w_1 \in C_1, \ \ldots \\ \text{such that } \ p = w_0\#w_1\# \ \ldots \ , \\ d(\alpha \circ c(i,w_i), \alpha \circ c(j,w_j) \leq 2^{-i} \ \text{ for } \ j > i \\ \text{and } \ x = \lim_{i\to\infty} \alpha \circ c(i,w_i) \end{cases} \tag{21}$$

(remember that $\alpha \circ c(i,w_i) \in F_i$ and F_i is i-spanning).
Prop.1: $\delta \equiv \delta_C$.
The computable function

$$w_0\#w_1\#\ldots \mapsto c(0,w_0)\#c(1,w_1)\#\ldots$$

translates δ to δ_C. Therefore, $\delta \leq \delta_C$.
On the other hand, suppose $\delta_C(w_0\#w_1\#\ldots) = x \in X$ and let $m \in \mathbb{N}$. Since F_m is m-spanning, there is some $u \in C_m$ such that $d(x, \alpha \circ c(m,u)) \leq 2^{-m}$, hence $d(\alpha(w_m), \alpha \circ c(m,u)) \leq 2 \cdot 2^{-m}$. By (2) from $w_0\#w_1\#\ldots$ and m some $u_m \in C_m$ can be computed such that $d(\alpha(w_m), \alpha \circ c(m,u_m)) < 3 \cdot 2^{-m}$. For $j > m$ we obtain

$$\begin{aligned} & d(\alpha \circ c(m,u_m), \alpha \circ c(m,u_j)) \\ \leq\ & d(\alpha \circ c(m,u_m), \alpha(w_m)) + d(\alpha(w_m), x) + d(x, \alpha(w_j)) + d(\alpha(w_j), \alpha \circ c(m,u_j)) \\ \leq\ & 3 \cdot 2^{-m} + 2^{-m} + 2^{-j} + 3 \cdot 2^{-j} \\ <\ & 2^{-m+3} \end{aligned}$$

Therefore, the computable function $w_0\#w_1\#\ldots \mapsto u_{m+3}\#u_{m+4}\#\ldots$ translates δ_C to δ and so $\delta_C \leq \delta$.
Prop.2: δ is c-proper
By Lemma 4.3 it suffices to show that dom(δ) is $\kappa_>^{\Sigma^\omega}$-computable. For any $p \in \Sigma^\omega$, $p \notin$ dom(δ), iff at least one of the following conditions holds (where $\Gamma := \Sigma \setminus \{\#\}$):
(1) $p \in (\Gamma^*\#)^k \Gamma^m \Sigma^\omega$ for some m, k such that $m \geq \lceil \log_2(\mathrm{span}(k+1)+1) \rceil$.
(2) $p \in (\Gamma^*\#)^k w\#\Sigma^\omega$ for some k and $w \in \Gamma^*$ such that $(k,w) \notin$ dom(c).
(3) $p \in (\Gamma^*\#)^i w_i\#(\Gamma^*\#)^{j-i-1} w_j\#\Sigma^\omega$ for some $i < j$ and $w_i, w_j \in \Gamma^*$
such that $d(\alpha \circ c(i,w_i), \alpha \circ c(j,w_j)) > 2^{-i}$.
Each of the three subsets of Σ^ω is r.e., therfore, dom(δ) is $\kappa_>^{\Sigma^\omega}$-computable.

It remains to show (20). For determining $\delta(w_0\#w_1\#\ldots$ with precision 2^{-k}, the prefix $w_0\#w_1\#\ldots w_k\#$ is sufficient.
Its length is bounded by $\sum_{i=0}^{k} \lceil \log_2(\mathrm{span}(i+1)+1) \rceil$. □.

In the following theorem we estimate the width of the Cartesian product, of the set of compact subsets and of the set of continuous functions with bounded modulus of continuity.

Theorem 6.7 *For $i = 1,2$ let (X_i, d_i) be compact metric spaces with k-separated set E_i, k-spanning set F_i, separation bound* sep_i *and spanning bound* span_i.

1. *$E_1 \times E_2$ is a k-separated set and $F_1 \times F_2$ is a k-spanning set of the product space $X_1 \times X_2$.*

2. Let $\mathcal{K}^*(X_1)$ be the space of non-empty compact subsets of X_1 with Hausdorff metric d_H. Then $\mathrm{NS}(E_1) := \{A \subseteq E_1 \mid A \neq \emptyset\}$ is a k-separated set and $\mathrm{NS}(F_1) := \{A \subseteq F_1 \mid A \neq \emptyset\}$ is a k-spanning set of this space.

3. Let $C(X_1, X_2, m)$ be the set of all continuous functions $f : X_1 \to X_2$ supplied with the sup metric $d(f, g) = \sup_{x \in X_1} d(f(x), g(x))$ with modulus of uniform continuity $m : \mathbb{N} \to \mathbb{N}$. Then $\mathrm{span}(k) := \mathrm{span}_2(k+2)^{\mathrm{span}_1 \circ m(k+2)}$ is a spanning bound of $C(X_1, X_2, m)$.

Proof: 1. If $(x_1, x_2), (y_1, y_2) \in E_1 \times E_2$ are different, then $x_1 \neq x_2$ or $y_1 \neq y_2$, hence $d((x_1, x_2), (y_1, y_2)) \geq 2^{-k}$. For $(x_1, x_2) \in X_1 \times X_2$ there are $y_i \in F_i$ such that $d_i(x_i, y_i) \leq 2^{-k}$ $(i = 1, 2)$ hence $d((x_1, x_2), (y_1, y_2)) \leq 2^{-k}$.

2. Let $A, B \in \mathrm{NS}(E_1)$, $A \neq B$. There is, w.l.g., some $a \in A \setminus B$. We obtain $d_1(a, b) \geq 2^{-k}$ for all $b \in B$, hence $d_H(A, B) \geq 2^{-k}$. Therefore, $\mathrm{NS}(E_1) := \{A \subseteq E_k \mid A \neq \emptyset\}$ is a k-separated set. Let $A \in \mathcal{K}^*(X_1)$. For all $a \in A$ there is some $b_a \in F_1$ such that $d_1(a, b_a) \leq 2^{-k}$. Let $B \subseteq F_1$ be the set of all these b_a $(a \in A)$. Then $d_H(A, B) \leq 2^{-k}$, therefore, $\mathrm{NS}(F_1) := \{B \subseteq F_k \mid B \neq \emptyset\}$ is a k-spanning set.

3. Let F be a $m(k+2)$-spanning subset of X_1 and let G be a $(k+2)$-spanning set of X_2. for every $h : F \to G$ define

$$H(h) := \{f \in C(X_1, X_2, m) \mid \forall a \in F.\ f[B^c(a, 2^{-m(k+2)})] \subseteq B^c(h(a), 2^{-k-1})\}.$$

Consider $f \in C(X_1, X_2, m)$ and $a \in F$. Then there is some $b \in G$ such that $d_2(f(a), b) \leq 2^{-k-2}$. If $d_1(a, x) \leq 2^{-m(k+2)}$ then $d_2(f(x), b) \leq d_2(f(x), f(a)) + d_2(f(a), b) \leq 2^{-k-1}$. Therefore, $f \in H(h)$ for some function $h : F \to G$. For $f, f' \in H(h)$ and $x \in X_1$ there is some $a \in F$ such that $d_1(x, a) \leq 2^{-m(k+2)}$ and so $d_2(f(x), f'(x)) \leq d_2(f(x), f(a)) + d_2(f(a), f'(x)) \leq 2^{-k}$. Therefore, $H(h)$ is contained in a closed ball of radius 2^{-k}. This shows that $C(X_1, X_2, m)$ can be covered by at most $|G|^{|F|}$ balls of radius 2^{-k}. □

If X_1 is connected (like the unit cube in $\mathbb{R}^n$) and X_2 is totally disconnected (like the Cantor space Σ^ω), then every continuous function is constant, and so $C(X_1, X_2, m)$ (see Thm. 6.7.3) and X_2 are isometric and have the same spanning bounds and separation bounds. For finding non-trivial separation bounds for $C(X_1, X_2, m)$ further assumptions are needed.

The restrictions to $[0; 1]$ of the signed digit representation ρ_{sd} (Ex. 4.2.2) and of Ko's representation δ_{Ko} (Ex. 4.2.3) are examples of c-admissible c-proper representations with small modulus of continuity.

Example 6.8 Let $\delta :\subseteq \Sigma^\omega \to [0; 1]$ be the restriction of the signed digit representation ρ_{sd} (Ex. 4.2.2) to $0{\bullet}1\Sigma^\omega$. Then $\mathrm{md}(k) = k + 2$ and $\mathrm{wid}(k) = 2^{k-1}$ (Ex. 6.3.2), i.e., the modulus is approximately $\log_2 \circ \mathrm{wid}$.

Let $\delta :\subseteq \Sigma^\omega \to [0; 1]$ be the restriction of the Cauchy representation from Ex. 2.2.4 to names $w_0 \# w_1 \# \ldots$ such that $w_i \in 0{\bullet}\{0, 1\}^i$ (cf. Ex. 4.2.3). Then the modulus of δ is approximately $\sum_{i=0}^{k} \lceil \log_2(\mathrm{wid}(i) + 1) \rceil$. If $\delta(w_0 \# w_1 \# \ldots) = x$, then the sequence $i \mapsto w_i$ is an *oracle* of x according to Ko's definition [19].

We conclude with two further examples of concise c-proper c-admissible representations satisfying (20), a representation of the closed subsets of $[0;1]^2$ (black and white images) and a representation of the Lipschitz bounded functions from $C[0;1]$. They are defined according to the idea from the proof of Thm. 6.6 and induce very natural concepts of computability and computational complexity.

Example 6.9 (compact subsets of $[0;1]^2$) Let $\mathcal{K}^*$ be the set of all non-empty compact subsets of the unit square $[0;1]^2$ with Hausdorff metric d_H. On the interval $[0;1]$ the set $L_k := \{i \cdot 2^{-k} \mid 0 \leq i \leq 2^k\}$ is k-spanning (cf. Ex. 6.3.2). By Theorem 6.7.1, $L_k \times L_k$ is k-spanning in $[0;1]^2$ and by Theorem 6.7.2, the set $F_k := \mathrm{NS}(L_k \times L_k)$ of its non-empty subsets is k-spanning in $\mathcal{K}^*$, hence $\mathrm{span}(k) = 2^{(2^k+1)^2}$ is a spanning bound of $\mathcal{K}^*$. Fig.1 shows a compact K set and an element $B \in F_5$ such that $d_H(K, B) \leq 2^{-5}$. Every $B \in F_k$ corresponds to a $(2^k+1) \times (2^k+1)$ matrix over $\{0, 1\}$.

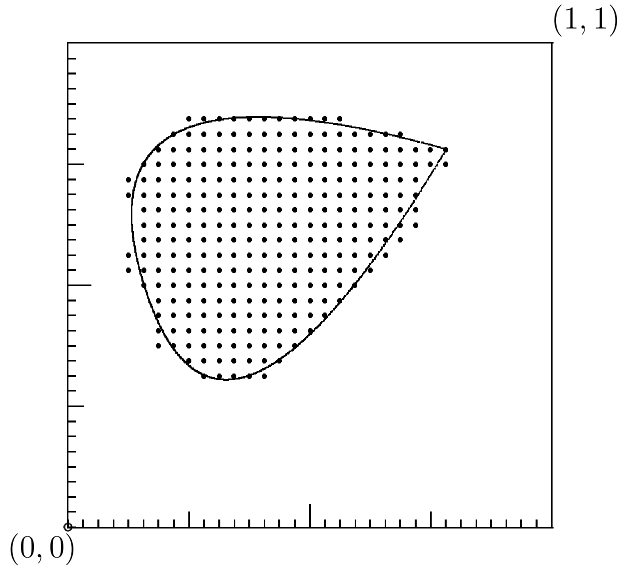

Figure 1: A compact set $K \subseteq [0;1]^2$ approximated by a subset of the 33×33 grid of points ($k = 5$).

For $w \in C_k := \{0,1\}^{(2^k+1)^2} \setminus \{0\}^*$ let $\beta_k(w) = B$, iff w is the "line by line" notation of this matrix. Extend the metric space to a computable metric space $(\mathcal{K}^*, d_H, F, \alpha)$ by $F := \bigcup_k F_k$ and $\alpha(0^k 1 w) := \beta_k(w)$. Define $\delta(p) = K$, iff $p = w_0 \# w_1 \# \ldots$ such that $w_k \in C_k$, $d_H(\beta_i(w_i), \beta_j(w_j)) \leq 2^{-i}$ for $i < j$ and $K = \lim_{i\to\infty} \beta_i(w_i)$.

The representation δ is c-admissible (it is equivalent to a restriction of the Cauchy representation κ_{mc} of the compact subsets of $\mathbb{R}^2$ (see Ex. 2.2.9). It is c-proper by Lemma 4.3, since $\mathrm{dom}(\delta)$ is $\kappa_>^{\Sigma^\omega}$-computable compact. By Ex. 6.3.2 and Thm. 6.7, $2^{2^{2k-2}} - 1$ is a spanning bound of the space $(\mathcal{K}^*, d_H)$. It is even minimal, and so

$$2^{2k-2} \leq \mathrm{md}_\gamma(k) \tag{22}$$

(for $k \geq 2$) for every representation γ of $\mathcal{K}^*$ by Thm. 6.5.

Since the prefix $w_0\#w_1\#\dots w_k\#$ of p suffices to determine $\delta(p)$ with precision 2^{-k},

$$\mathrm{md}_\delta(k) \le \sum_{i=0}^{k}(2^i+1)^2. \tag{23}$$

Since for sufficiently large k, $\sum_{i=0}^{k}(2^i+1)^2 \le 2^{2k+1}$, $\mathrm{md}_\delta(k) \le 2^{2k+1}$. By (22), the upper bound 2^{2k+1} is tight. Therefore, δ is concise.

Computational complexity of points (i.e. closed subsets of $[-;1]^2$) and functions induced by the representation δ is realistic. Notice, that no injective representation is equivalent to δ. □

Example 6.10 (Lipschitz bounded functions from $C[0;1]$) Let X be the set of continuous functions $f : [0;1] \to \mathbb{R}$ such that $f(0) = 0$ and $|f(x) - f(y)| \le |x - y|$ (Lipschitz bounded) supplied with the sup-metric. Consider $k \in \mathbb{N}$. For every element $a = a_1 a_2 \dots a_{2^k} \in \{0, 1, -1\}^{2^k}$ let $f_a : [0;1] \to \mathbb{R}$ be the polygon function with the vertices (x_i, y_i), $(i = 0, 1, \dots 2^k)$ such that $(x_0, y_0) = (0, 0)$, $x_i = x_{i-1} + 2^{-k}$ and $y_i = y_{i-1} + a_i \cdot 2^{-k}$ (therefore, $f(x_i) = \sum_{j=1}^{i} a_j \cdot 2^{-k}$). Then $f_a \in X$ for all a, $d(f_a, f_b) \ge 2^{-k}$ for $a \ne b$, and for every $f \in X$ there is some a such that $d(f, f_a) \le 2^{-k}$ (for $i = 1, \dots, 2^k$ choose a_i such that $\left|\sum_{j=1}^{i} a_i \cdot 2^{-k} - f(x_i)\right| \le 2^{-k}$, see Fig. 2). Therefore, the set $F_k := \{f_a \mid a \in \{1, 0, -1\}^{2^k}\}$ is k-separated and k-spanning, hence $k \mapsto 3^{2^k}$ is a separation bound as well as a spanning bound.

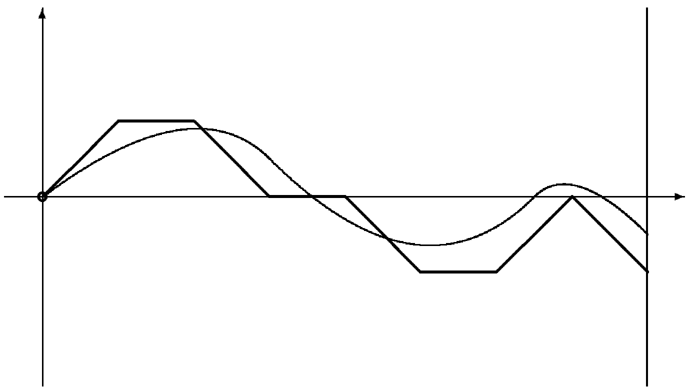

Figure 2: The polygon f_a, $a = (1, 0, -1, 0, -1, 0, 1, -1)$, and a function $f \in X$ such that $d(f_a, f) \le 2^{-3}$

According to the construction in the proof of Thm. 6.6, define functions $\beta_k : \{1, 0, -1\}^{2^k} \to X$ by $\beta_k(a) := f_a$ and a representation δ of X by $\delta(p) = f$, iff $p = w_0\#w_1\#\dots$ such that $w_i \in \{1, 0, -1\}^{2^i}$ and $d(\beta_i(w_i), \beta_j(w_k)) \le 2^{-i}$ for $i < j$ and $f = \lim_{i\to\infty} \beta_i(w_i)$. Then δ is equivalent to the standard representation $\delta_\to$ (see [40], Sec. 6.1), c-proper and concise.

Computational complexity of the representation δ of the set X of functions is realistic. For example, evaluation $(f, x) \mapsto f(x)$ and integration $(f, x) \mapsto \int_0^x f(\xi)\, d\xi$ are easily computable. □

7 Comparison with the LLM-Definition

In [22] Labhalla et al. define the computational complexity of functions on computable metric spaces. We reformulate their definition in our framework. Let $\mathbf{X}_i = (X_i, d_i, A_i, \alpha_i)$ be computable metric spaces. Let $\mathcal{C}$ be an "interesting" complexity class of word functions such as **PTIME**, the bounds of which in particular should be closed under composition. Then

(L) "$f : X_1 \to X_0$ is uniformly in $\mathcal{C}$", iff f has a modulus of uniform continuity in $\mathcal{C}$ (w.r.t. the unary notation $\nu_1 : 0^j \mapsto j$) and there is Turing machine M time bounded in $\mathcal{C}$ such that

$$d_0(f \circ \alpha_1(v), \alpha_0 \circ f_M(v, 0^k)) \leq 2^{-k} \text{ for all } v \in \mathrm{dom}(\alpha_1) \text{ and } k \in \mathbb{N}. \quad (24)$$

Definition (L) generalizes Ko's characterization of the real functions $f : [a; b] \to \mathbb{R}$ computable in polynomial time (Cor. 2.21 in [19]). According to Defs. 5.3 and 5.4, for representations δ_i of X_i,

(T) "$f : X_1 \to X_0$ is computable on $K \subseteq \mathrm{dom}(f)$ in time $t : \mathbb{N} \to \mathbb{N}$", iff for some Type-2 machine M, f_M realizes f w.r.t. (δ_1, δ_0) such that

$$\mathrm{TIME}_M^K(k) := \max_{\delta_1(p) \in K} \mathrm{Time}_M(p)(k) \quad \leq \quad t(k), \quad (25)$$

where $\mathrm{Time}_M(p)(k)$ is the time which the machine M on input p needs to compute a partial result z of precision 2^{-k} w.r.t. the given representation δ_0 of the codomain.

For easier comparison we consider $X_1 = K$ and the Cauchy representation for X_0 with the standard approximation $\mathrm{app}(w_0\#w_1\#\ldots w_k\#, 0^k) := w_k$ and modify (T) as follows:

(T') "$f : X_1 \to X_0$ is computable in time $t : \mathbb{N} \to \mathbb{N}$", iff there is a Type-2 machine M which on all inputs $(p, 0^k)$ ($p \in \mathrm{dom}(\delta_1)$) halts in at most $t(k)$ steps such that

$$d_0(f \circ \delta_1(p), \alpha_0 \circ f_M(p, 0^k)) \leq 2^{-k}. \quad (26)$$

In Def. (L), $(v, 0^k) \mapsto w$ ($v \in \Sigma^*$) such that $d_0(f \circ \alpha_1(v), \alpha_0(w)) \leq 2^{-k}$ must be in the complexity class $\mathcal{C}$. The realizing machine M operates (only on names of) the dense subset A_1 and on 0^k. The other points of X_1 are captured by uniform continuity the modulus of which must also be in the complexity class $\mathcal{C}$. If e.g. $\mathcal{C} = \mathbf{PTIME}$, longer α_1-names $v \in \Sigma^*$ allow more computation time, e.g., replacing α by α', $\alpha'(w\#2^{|w|}) := \alpha(w)$, is rewarded considerably.

In Def. (T') a function $(p, 0^k) \mapsto w$ ($p \in \Sigma^\omega$) such that $d_0(f \circ \delta_1(p), \alpha_0(w)) \leq 2^{-k}$ must be computed in time $t(k)$. A realizing machine works in time $t(k)$ uniformly on all δ_1-names (uniform continuity follows automatically, the modulus, however might not be bounded by t). Since the time is considered only as a function of precision, increasing the redundancy of δ_1-names artificially cannot reduce the complexity.

While Definition (L) is meaningful only (mainly ?) for complexity classes the bounds of which are closed under composition, Definition (T) is meaningful also for bounds t like k^3 or $k \log k$.

So far we can say that (L) and (T) are two non-equivalent definitions of computational complexity of computable functions on metric spaces which in some applications define the same complexity classes. For a more detailed comparison more concrete examples should be available.

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