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Tractable Constraint Satisfaction Problems on a 3-element set

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Abstract

The Constraint Satisfaction Problem (CSP) provides a common framework for many combinatorial problems. The general CSP is known to be NP-complete; however, certain restrictions on a possible form of constraints may affect the complexity, and lead to tractable problem classes. There is, therefore, a fundamental research direction, aiming to separate those subclasses of the CSP which are tractable, and those which remain NP-complete.

Schaefer gave an exhaustive solution of this problem for the CSP on a 2-element domain. In this paper we generalise this result to a classification of the complexity of CSPs on a 3-element domain. The main result states that every subclass of the CSP is either tractable or NP-complete, and the criterion separating them is that conjectured in [3, 7]. We also exhibit a polynomial time algorithm which, for a given set of allowed constraints, outputs if this set gives rise to a tractable problem class. To obtain the main result and the algorithm we extensively use the algebraic technique for the CSP developed in [17] and [3, 7].

1 Introduction

In the Constraint Satisfaction Problem (CSP) [24] we aim to find an assignment to a set of variables subject specified constraints. Many combinatorial problems appearing in computer science and artificial intelligence can be expressed as particular subclasses of the CSP. The standard examples include the propositional satisfiability problem, in which the variables must be assigned Boolean values [13], graph colorability, scheduling problems, linear systems and many others. One advantage of considering a common framework for all of these diverse problems is that it makes it possible to obtain generic structural results concerning the computational complexity of constraint satisfaction problems that can be applied in many different areas such as database theory [21, 31], temporal and spatial reasoning [29],

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machine vision [24], belief maintenance [9], technical design [26], natural language comprehension [1], programming language analysis [25], etc.

The general CSP is NP-complete; however, certain restrictions on the allowed form of the constraints involved may ensure tractability. Therefore, one of the main approaches in study of the CSP is identifying tractable subclasses of the general CSP obtained in this way [12, 14, 15, 17, 28]. Developments in this direction provide an efficient algorithm solving a particular problem, if the problem falls in one of the known tractable subclasses, or assist in speeding up of general superpolynomial algorithms [10, 11, 22]. To formalise the idea of restricting the allowed constraints, we make use of the notion of a *constraint language* [16], which is simply a set of possible relations that can be used to specify constraints in a problem. The ultimate goal of this research direction is to find the precise border between tractable and intractable constraint languages.

This goal was achieved by Schaefer [28] in the important case of Boolean constraints; he has characterised tractable constraint languages, and proved that the rest are NP-complete. Schaefer's result is known as Dichotomy Theorem for Boolean constraints. Dichotomy theorems are of particular interest in study of the CSP, because, on the one hand, they determine the precise complexity of constraint languages, and on the other hand, the a priori existence of a dichotomy result cannot be taken for granted. For more dichotomy results for Boolean CSPs, and a short survey of dichotomy results for other cases the reader is referred to [15].

The analogous problem for the CSP in which the variables can be assigned more than 2 values remains open since 1978, in spite of intensive efforts. For instance, Feder and Vardi, in [12], used database technique and group theory to identify some large tractable families of constraints; Jeavons and co-authors have characterised many tractable and NP-complete constraint languages using invariance properties of constraints [17, 18, 19]; in [7], a possible form of a dichotomy result for the CSP on finite domains was conjectured; in [6], a dichotomy result was proved for certain type of constraint languages on a 3-element domain. In this paper we generalise the results of [28] and [6], and prove the dichotomy conjecture from [7] for the constraint satisfaction problem on a 3-element domain. In particular, we completely characterise tractable constraint languages in this case, and prove that the rest are NP-complete. The main result will be precisely stated at the end of Section 2.

The dichotomy problem for a domain containing more than 2 elements, even for a 3-element domain, turns out to be much harder than that for 2-element case. Besides the obvious reason that Boolean CSPs closely relates to various problems from propositional logic, and therefore, are much better investigated, there is another deep reason. As is showed in [19, 20, 17], when studying the complexity of constraint languages we may restrict ourselves with a certain class of languages, so called *relational clones*. There are only countably many relational clones on a 2-element set, and all of them are known [27]. However, already the class of relational clones on a 3-element set contains continuum many elements, and is believed to be incomprehensible.

Another problem tackled here is referred to, in [15], as the *meta-problem*: given constraint language determine if it gives rise to a tractable problem class. Making use of the dichotomy theorem obtained we exhibit an effective algorithm solving the meta-problem for the CSP on a 3-element domain.

The technique used in this paper relies upon the idea that was developed in [7, 3, 17] (and also mentioned in [12] as a possible direction for future research), that algebraic invariance properties of constraints can be used for studying the complexity of the corresponding constraint satisfaction problems. The main advantage of this technique is that it allows us to employ structural results from universal algebra. The algebraic approach has proved to be very fruitful in identifying tractable classes of the CSP [2, 4, 18]. We strongly believe that the synthesis between complexity theory and universal algebra which we describe here is likely to lead to new results in both fields.

2 Algebraic structure of CSP classes

2.1 Constraint Satisfaction Problem

The set of all n-tuples with components from a set A is denoted A^n. Any subset of A^n is called an n-ary *relation* on A; and a *constraint language* on A is an arbitrary set of finitary relations on A.

Definition 1 *The* constraint satisfaction problem (CSP) *over a constraint language* Γ*, denoted* CSP(Γ)*, is defined to be the decision problem with instance* $(V, A, \mathcal{C})$*, where*

V *is a set of* variables*;*
A *is a set of* values *(sometimes called a* domain*); and*
$\mathcal{C}$ *is a set of* constraints, $\{C_1, \ldots, C_q\}$,

> *in which the constraint* $C_i \in \mathcal{C}$ *is a pair* $\langle s_i, R_i \rangle$ *with* s_i *is a tuple of variables of length* m_i*, called the* constraint scope*, and* R_i *an* m_i*-ary relation on* A*, called the* constraint relation.

The question is whether there exists a solution *to* $(V; A; C)$*, that is, a function from* V *to* A*, such that, for each constraint in* $\mathcal{C}$*, the image of the constraint scope is a member of the constraint relation.*

We shall be concerned with distinguishing between those constraint languages which give rise to *tractable* problems (i.e., problems for which there exists a polynomial-time solution algorithm), and those which do not.

Definition 2 *A constraint language, Γ is said to be* tractable, *if* $\mathrm{CSP}(\Gamma')$ *is tractable for each finite subset* $\Gamma' \subseteq \Gamma$. *It is said to be* NP-complete, *if* $\mathrm{CSP}(\Gamma')$ *is NP-complete for some finite subset* $\Gamma' \subseteq \Gamma$.

In [28], Schaefer has classified Boolean constraint languages with respect to the complexity. This result is known as Schaefer's Dichotomy theorem.

Theorem 1 *A Boolean constraint language, Γ, is tractable if and only if one of the following conditions holds:*

1. *every R in Γ contains $(0, \ldots, 0)$.*
2. *every R in Γ contains $(1, \ldots, 1)$.*
3. *every R in Γ is definable by a CNF formula in which each clause has at most one negated variable.*
4. *every R in Γ is definable by a CNF formula in which each clause has at most one unnegated variable.*
5. *every R in Γ is definable by a CNF formula in which each clause has at most two literals.*
6. *every R in Γ is the solution space of a linear system over* $\mathrm{GF}(2)$.

Otherwise Γ is NP-complete.

More examples of both tractable and NP-complete constraint languages will appear later in this paper and can also be found in [7, 8, 12, 19]. It follows from Theorem 1 that every Boolean constraint language is either tractable or NP-complete; and so, there is no language of intermediate complexity. The classification problem for larger domains is still open and seems to be very interesting and hard [12].

Problem 1 *Characterise all tractable constraint languages on finite domains.*

2.2 Algebraic structure of problem classes

Schaefer's technique heavily uses representation of Boolean relations by propositional formulas. Such a representation does not exist for larger domains. Instead, we shall use algebraic properties of relations. In our algebraic definitions we mainly follow [23].

Definition 3 *An* algebra *is an ordered pair $\mathbb{A} = (A, F)$ such that A is a nonempty set and F is a family of finitary operations on A. The set A is called the* universe *of $\mathbb{A}$, the operations from F are called* basic. *An algebra with a finite universe is referred to as a* finite algebra.

Every constraint language on a set A can be assigned an algebra with the universe A.

Definition 4 *An n-ary operation f* preserves *an m-ary relation R (or f is a* polymorphism *of R, or R is* invariant *under f) if, for any $(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn}) \in R$, the tuple $(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn}))$ belongs to R as well.*

The set of all polymorphisms of a family Γ of relations is denoted $\mathsf{Pol}\ \Gamma$; and the set of all relations invariant under all operations from a set F is denoted $\mathsf{Inv}\ F$.

Given a constraint language, Γ, on A, the algebra $(A, \mathsf{Pol}\ \Gamma)$ is called the algebra associated with Γ, *and is denoted $\mathbb{A}_\Gamma$.*

Conversely, for any finite algebra $\mathbb{A} = (A; F)$, there are a constraint language associated with $\mathbb{A}$, for example, $\mathsf{Inv}\ F$, and the associated problem class $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathsf{Inv}\ F)$. A connection between the complexity of a constraint language and the associated algebra is provided by the following theorem.

Theorem 2 ([17]) *A constraint language Γ on a finite set is tractable (NP-complete) if and only if $\mathsf{Inv}\ \mathsf{Pol}\ (\Gamma)$ is tractable (NP-complete).*

Informally speaking, Theorem 2 says that the complexity of Γ is determined by the algebra $\mathbb{A}_\Gamma$. We, therefore, make the following definition: for a constraint language Γ, the algebra $\mathbb{A}_\Gamma$ is said to be *tractable* (*NP-complete*) if Γ is tractable (NP-complete).

In [18, 19], Jeavons and co-authors have identified certain types of algebras which give rise to tractable problem classes.

Definition 5 *Let A be a finite set. An operation f on A is called*

- *a* projection *if there is $i \in \{1, \ldots, n\}$ such that $f(x_1, \ldots, x_n) = x_i$ for any $x_1, \ldots, x_n \in A$;*
- essentially unary *if $f(x_1, \ldots, x_n) = g(x_i)$, for some unary operation g, and any $x_1, \ldots, x_n \in A$;*
- *a* constant *operation if there is $c \in A$ such that $f(x_1, \ldots, x_n) = c$, for any $x_1, \ldots, x_n \in A$;*
- idempotent *if $f(x, \ldots, x) = x$ for any $x \in A$.*
- *a* semilattice *operation*[1]*, if it is binary and satisfies the following three conditions:*
  (a) $f(x, f(y, z)) = f(f(x, y), z)$ (Associativity), (b) $f(x, y) = f(y, x)$ (Commutativity), (c) $f(x, x) = x$ (Idempotency), for any $x, y, z \in A$;
- *a* majority *operation if it is ternary, and $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$, for any $x, y \in A$;*
- affine *if $f(x, y, z) = x - y + z$, for any $x, y, z \in A$, where $+, -$ are the operations of an Abelian group.*

[1] Note that in some earlier papers [17, 18] the term *ACI operation* is used.

For a finite algebra $\mathbb{A}$, an operation from $\mathsf{Pol}\ \mathsf{Inv}\ F$ is said to be a *term* operation[2] of $\mathbb{A}$. If Γ is a constraint language, the term operations of $\mathbb{A}_\Gamma$ are the polymorphisms of Γ.

Proposition 1 ([18, 19]) *If a finite algebra $\mathbb{A}$ has a term operations which is constant, semilattice, affine, or majority, then $\mathbb{A}$ is tractable.*

The 2-element algebras associated with Schaefer's six constraint languages have a constant term operation 0 or 1 in cases (1),(2); a semilattice term operation $\vee$ or $\wedge$ in cases (3),(4); the majority term operation $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ in case (5); and the affine term operation $x - y + z$ in case (6).

An algebra is said to be a *G-set* if every its term operation is essentially unary, and the corresponding unary operation is a permutation.

Proposition 2 ([18, 19]) *A finite G-set is NP-complete.*

By combining those two results, and the classical result of E.Post [27], the algebraic version of Schaefer's theorem can be derived [7].

Theorem 3 (Schaefer) *A constraint language Γ on a 2-element set is tractable if and only if $\mathbb{A}_\Gamma$ is not a G-set. Otherwise Γ is NP-complete.*

2.3 Algebraic constructions and the complexity of constraint languages

Certain transformations of constraint languages preserve the complexity. Let Γ be a constraint language on A, and g a unary polymorphism of Γ such that $g(g(x)) = g(x)$. By $g(\Gamma)$ we denote the set $\{g(R) \mid R \in \Gamma\}$ where $g(R) = \{(g(a_1), \ldots, g(a_n)) \mid (a_1, \ldots, a_n) \in R\}$; and by Γ^+ the set $\Gamma \cup \{\{(a)\} \mid a \in A\}$. If $g \in \mathsf{Pol}\ \Gamma$ is a unary operation range whose range is minimal among ranges of unary operations from $\mathsf{Pol}\ \Gamma$, then the constraint language $g(\Gamma)^+$ will be denoted Γ_g^{id}.

Proposition 3 ([7]) *Let Γ be a constraint language on A, and $g \in \mathsf{Pol}\ \Gamma$ a unary operation on A with a minimal range and such that $g(g(x)) = g(x)$. Then Γ is tractable [NP-complete] if and only if Γ_g^{id} is tractable [NP-complete].*

If Γ and g satisfy the conditions of Proposition 3 then the algebra $\mathbb{A}_{\Gamma_g^{\mathrm{id}}}$ is *idempotent*, that is, all its basic operations are idempotent. The complexity of the constraint language Γ_g^{id} does not depend on the choice of g, and we shall denote every such language Γ^{id}.

Due to Theorem 2 and Proposition 3 the study of the complexity of constraint languages is completely reduced to the study of properties of idempotent algebras.

[2] Every term operation can be obtained from operations of F by superposition.

Definition 6 *Let $\mathbb{A} = (A, F)$ be an algebra, and B a subset of A such that, for any $f \in F$ (n-ary), and for any $b_1, \ldots, b_n \in B$, we have $f(b_1, \ldots, b_n) \in B$. Then the algebra $\mathbb{B} = (B, F|_B)$, where $F|_B$ consists of restrictions of operations from F to B, is called a* subalgebra *of $\mathbb{A}$. The universe of a subalgebra of $\mathbb{A}$ is called a* subuniverse *of $\mathbb{A}$. A subalgebra $\mathbb{B}$ (a subuniverse B) is said to be* proper *if $\mathbb{B} \neq \mathbb{A}$ ($B \neq A$).*

An equivalence relation $\theta \in \mathsf{Inv}\, F$ is said to be a congruence *of $\mathbb{A}$. The θ-class containing $a \in A$ is denoted a^θ, the set $A/_\theta = \{a^\theta \mid a \in A\}$ is said to be the* factor-set, *and the algebra $\mathbb{A}/_\theta = (A/_\theta; F^\theta)$, $F^\theta = \{f^\theta \mid f \in F\}$ where $f^\theta(a_1^\theta, \ldots, a_n^\theta) = (f(a_1, \ldots, a_n))^\theta$, is said to be the* factor-algebra.

Proposition 4 ([7]) *Let Γ be a tractable constraint language on A, $B \subseteq A$ a subuniverse of $\mathbb{A}_\Gamma$, and θ an equivalence relation invariant under $\mathsf{Pol}\,\Gamma$. Then*
(1) the subalgebra $\mathbb{B} = (B; (\mathsf{Pol}\,\Gamma)|_B)$ of $\mathbb{A}_\Gamma$ is tractable;
(2) $\mathbb{A}_\Gamma/_\theta$, and the set $\Gamma^\theta = \{R^\theta \mid R \in \Gamma\}$ where $R^\theta = \{(a_1^\theta, \ldots, a_n^\theta) \mid (a_1, \ldots, a_n) \in R\}$, are tractable.

Hence, every subalgebra and every factor-algebra of a tractable algebra is tractable. Furthermore, every *factor* of a tractable algebra, that is, a factor-algebra of a subalgebra, is tractable and cannot be a G-set. Thus, every tractable algebra $\mathbb{A}$ satisfies the condition

$$\text{none of the factors of } \mathbb{A} \text{ is a } G\text{-set.} \qquad \text{(No-G-Set)}$$

Moreover, all known examples of NP-complete subclasses of the CSP have a G-set behind. We, therefore, make the following conjecture.

Conjecture 1 *A constraint language Γ on a finite set A is tractable if and only if $\mathbb{A}_{\Gamma^{\mathrm{id}}}$ satisfies* (No-G-Set). *Otherwise it is NP-complete.*

By Theorem 3, the conjecture holds for constraint languages on a 2-element set. The main result of this paper is that the conjecture holds for constraint languages on a 3-element set.

Theorem 4 *A constraint language Γ on a 3-element set is tractable if and only if the algebra $\mathbb{A}_{\Gamma^{\mathrm{id}}}$ satisfies* (No-G-Set). *Otherwise Γ is NP-complete.*

3 Algorithms

The necessity of the condition (No-G-Set) for the tractability of a finite algebra follows from Propositions 2,4. To show that the condition is sufficient, we have to exhibit, for any 3-element algebra $\mathbb{A}$ satisfying (No-G-Set), an algorithm that solves the problem CSP($\mathbb{A}$) in polynomial time. We split the proof into two parts. The first part is 'algebraic'; we show that, for any

3-element algebra $\mathbb{A} = (A; F)$ satisfying (NO-G-SET), the relations from $\mathsf{Inv}\ F$ satisfy one of 10 properties. The second part is 'algorithmic'; for each of those 10 properties, we exhibit a polynomial time algorithm that solves the constraint satisfaction problem rising from a set of relations satisfying this property. The algorithms will be constructed in this section, while the algebraic part based on an elaborate case analysis is done in Section 5.

It turns out, that we need only three types of algorithms: the first one is based on finding partial solutions, the second one reduces CSP($\mathbb{A}$) to the case of a 2-element domain, and the third is a generalised Gaussian elimination.

3.1 Partial solutions and bounded width

Let R be an n-ary relation, and $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$; then R_I denotes the k-ary relation $\{\mathbf{a}_I \mid \mathbf{a} \in R\}$ where $\mathbf{a}_I = (a_{i_1}, \dots, a_{i_k})$. We will often consider relations whose coordinate positions are indexed by not necessarily natural numbers, but elements of some arbitrary set, for example, the coordinate positions of constraint relations will be supposed to be indexed by variables.

Definition 7 *Let $\mathcal{P} = (V; A; \mathcal{C})$ be a constraint satisfaction problem, and $W \subseteq V$. The* restricted *problem $\mathcal{P}_W$ is defined to be $(W; A; \mathcal{C}_W)$ where, for each $\langle s, R\rangle \in \mathcal{C}$, there is $\langle s \cap W; R_{s\cap W}\rangle$ in $\mathcal{C}_W$. A solution to $\mathcal{P}_W$ is said to be a* partial solution, *and the set of all such solutions is denoted $\mathcal{S}_W$.*

*The problem $\mathcal{P}$ is said to be k-*minimal *if, for any k-element subset $W \subseteq V$, any $\langle s, R\rangle \in \mathcal{P}_W$, and any $\mathbf{a} \in R$, the tuple $\mathbf{a}_{s\cap W}$ is a part of a solution from $\mathcal{S}_W$.*

Any constraint satisfaction problem instance $\mathcal{P}$ can be modified to obtain a k-minimal problem instance $\mathcal{P}'$ without changing the set of solutions by repeating the following procedure until the instance stays unchanged: solve all subproblems involving k variables, and then remove from each constraint $\langle s, R\rangle$ all tuples $\mathbf{a} \in R$ such that $\mathbf{a}_{s\cap W}$ is a part of no partial solution for certain k-element set of variables W. This procedure is called 'establishing k-minimality', and $\mathcal{P}'$ is said to be the k-minimal instance associated with $\mathcal{P}$.

Definition 8 *A class $\mathbf{C}$ of constraint satisfaction problems is said to be of* width[3] *k if any problem instance $\mathcal{P}$ from $\mathbf{C}$ has a solution if and only if the k-minimal problem associated with $\mathcal{P}$ contains no empty constraint.*

Every class of finite width is tractable, because, assuming k fixed, establishing k-minimality takes polynomial time.

[3] There appear several notions of the width of a problem class. For instance, Feder and Vardi [12] characterised this concept in terms of Datalog programs. In this paper we use the weakest version, which, therefore, gives the widest possible family of problem classes.

3.2 Multi-sorted constraints satisfaction problem

In [5], an algebraic approach to a generalised version of the constraint satisfaction problem was developed. In this generalised version every variable is allowed to have its own domain. In this paper we need the notion of *multi-sorted* constraint satisfaction problem, and some results from [5] as an auxiliary tool.

Definition 9 *For any collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$, and any list of indices $(i_1, i_2, \ldots, i_m) \in I^m$, a subset R of $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$, together with the list $(i_1, i_2, \ldots, i_m)$, will be called an [m-ary]* relation *over $\mathcal{A}$ with* signature $(i_1, i_2, \ldots, i_m)$*. For any such relation R, the jth component of the signature of R will be denoted $\sigma(j)$.*

The ith component of a tuple $\overline{a}$ will be denoted by $\overline{a}[i]$.

Definition 10 *The* multi-sorted constraint satisfaction problem *is the combinatorial decision problem with*

INSTANCE: *a quadruple $(V; \mathcal{A}; \delta; \mathcal{C})$ where*

- *V is a set of* variables*;*
- *$\mathcal{A} = \{A_i \mid i \in I\}$ is a collection of sets of values [*domains*];*
- *δ is a mapping from V to I, called the* domain function*;*
- *$\mathcal{C}$ is a set of* constraints.
- *Each constraint $C \in \mathcal{C}$ is a pair $\langle s, \varrho \rangle$, where*
  - *$s = (v_1, \ldots, v_{m_C})$ is a tuple of variables of length m_C, called the* constraint scope*;*
  - *R is an m_C-ary relation over $\mathcal{A}$ with signature $(\delta(v_1), \ldots, \delta(v_{m_C}))$, called the* constraint relation.

QUESTION: *does there exist a* solution, *i.e. a function φ, from V to $\bigcup_{A \in \mathcal{A}} A$, such that, for each variable $v \in V$, $\varphi(v) \in A_{\delta(v)}$, and for each constraint $\langle s, R \rangle \in \mathcal{C}$, with $s = (v_1, \ldots, v_m)$, the tuple $(\varphi(v_1), \ldots, \varphi(v_m))$ belongs to R?*

It is possible to introduce the algebraic structure of the multi-sorted CSP in a very similar way as for the usual one.

Definition 11 *Algebras $\mathbb{A}_1 = (A_1, F_1)$, $\mathbb{A}_2 = (A_2, F_2)$ are said to be* similar *(or of the* same type*) if there exists a set I such that $F_1 = \{f_i^1 \mid i \in I\}$, $F_2 = \{f_i^2 \mid i \in I\}$ and, for all $i \in I$, f_i^1, f_i^2 are of the same arity.*

Thus, a class of similar algebras can be viewed as a collection of sets, and a set of *operation symbols* such that each operation is assigned the *arity* and has an *interpretation* in each set from the collection, that is an operation of the arity assigned. Let $\mathcal{A}$ be a class of similar algebras. We say that an (n-ary) operation symbol f *preserves* an (m-ary) multi-sorted relation R over the collection of the universes of algebras from $\mathcal{A}$ (or R is *invariant* with respect to f) if, for any $(a_{11},\ldots,a_{m1}),\ldots,(a_{1n},\ldots,a_{mn}) \in R$ we have

$$f\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} f^{\mathbb{A}_{\sigma(1)}}(a_{11},\ldots,a_{1n}) \\ \vdots \\ f^{\mathbb{A}_{\sigma(m)}}(a_{m1},\ldots,a_{mn}) \end{pmatrix} \in R$$

where σ is the signature of R. The set of all invariants of a set of terms C is denoted by $\mathsf{mInv}\ C$.

Definition 12 *For a given collection of similar finite algebras, $\mathcal{A}$,* CSP($\mathcal{A}$) *is defined to be the decision problem with*

INSTANCE: *An instance, $\mathcal{P} = (V;\mathcal{B};\delta;\mathcal{C})$, of the multi-sorted constraint satisfaction problem, in which*

- *for each variable v, the domain $A_{\delta(v)}$ is the universe of an algebra $\mathbb{A}_{\delta(v)} \in \mathcal{A}$;*
- *for each constraint $\langle s, R\rangle$, the relation R is invariant with respect to all operation symbols of $\mathcal{A}$.*

QUESTION: *Does $\mathcal{P}$ have a solution?*

A class $\mathcal{A}$ of similar finite algebras is said to be *tractable* if $\mathsf{mInv}\ F$ is tractable where F denotes the set of operation symbols of $\mathcal{A}$. If, for every algebra from a class, there is a operation symbol which is interpreted in the algebra as one of the operations listed in Proposition 1, then the corresponding CSP-class is tractable.

Theorem 5 ([5]) *Let $\mathbb{A}_1,\ldots,\mathbb{A}_l,\mathbb{B}_1,\ldots,\mathbb{B}_n$ be similar finite algebras, each $\mathbb{B}_i$ have either a constant term, or a semilattice or near-unanimity term, and let $\mathbb{A}_1,\ldots,\mathbb{A}_l$ have an affine term whose interpretations on other algebras are idempotent. Then $\{\mathbb{A}_1,\ldots,\mathbb{A}_l,\mathbb{B}_1,\ldots,\mathbb{B}_n\}$ is tractable.*

A class of similar algebras naturally arises when we consider the collection of all factors of an algebra. We say that a problem instance $\mathcal{P} = (V;A;\mathcal{C}) \in$ CSP($\mathbb{A}$) where $\mathbb{A}$ is a 3-element algebra is *2-valued* if, for any $v \in V$, $\mathcal{S}_{\{v\}}$ contains at most 2 elements. Such a problem can be treated as a multi-sorted problem over the family of all proper subalgebras of $\mathbb{A}$. If $\mathbb{A}$ satisfies (NO-G-SET), then by Theorem 5 and Schaefer's theorem, $\mathcal{P}$ can be solved in polynomial time.

Corollary 1 *If a 3-element idempotent algebra satisfies* (NO-G-SET) *then any 2-valued problem instance from* CSP($\mathbb{A}$) *can be solved in polynomial time.*

Most of the 'good' properties of relations below allows us, first, to reduce an arbitrary problem instance to a 2-valid problem instance, and, second, to solve the obtained instance as a multi-sorted problem instance by making use of the algorithms from [5].

3.3 'Good' properties of relations

Throughout the rest of this section $\mathbb{A} = (A; F)$ is a 3-element algebra satisfying (NO-G-SET). Certain properties of $\mathbb{A}$ make it possible to reduce any problem instance from CSP($\mathbb{A}$) to a 2-valued problem instance. The ith component of a tuple $\mathbf{a}$ will be denoted $\mathbf{a}[i]$. By $\underline{n}$, for a natural number n, we will denote the set $\{1, \ldots, n\}$.

Definition 13 *Let B be a 2-element subuniverse of $\mathbb{A}$, $a \in A - B$, and $b \in B$. The algebra $\mathbb{A}$ satisfies the $(a-b)$-*replacement property *if, for any (n-ary) $R \in \mathsf{Inv}\, F$, and any $\mathbf{a} \in R$, there is $\mathbf{b} \in R$ with*

$$\mathbf{b}[i] = \begin{cases} b, & \text{if } \mathbf{a}[i] = a \text{ and } a, b \in R_{\{i\}}, \\ \mathbf{a}[i], & \text{otherwise.} \end{cases}$$

Definition 14 *The algebra $\mathbb{A}$ satisfies the* partial zero property *if there are a set of its subuniverses, Z, and $z_B \in B$ for each $B \in Z$, such that (a) $A \in Z$; (b) for any relation $R \in \mathsf{Inv}\, F$, and any $\mathbf{a} \in R$, there is $\mathbf{b} \in R$ with*

$$\mathbf{b}[i] = \begin{cases} z_B, & \text{if } R_{\{i\}} = B \in Z, \\ \mathbf{a}[i], & \text{otherwise.} \end{cases}$$

For a relation $R \in \mathsf{Inv}\ F$, and a subuniverse B of $\mathbb{A}$ we denote $\theta_B(R)$ the equivalence relation on $W = \{i \mid B \subseteq R_{\{i\}}\}$ generated by the set

$$\{(i,j) \in W^2 \mid \text{for any } \mathbf{a} \in R,\ \mathbf{a}[i], \mathbf{a}[j] \in B \text{ or } \mathbf{a}[i], \mathbf{a}[j] \notin B\}.$$

Definition 15 *Let B be a 2-element subuniverse of $\mathbb{A}$. The algebra $\mathbb{A}$ is said to be B-*rectangular, *if for any relation $R \in \mathsf{Inv}\, F$,*
$R_W \cap B^{|W|} = (R_{W_1} \cap B^{|W_1|}) \times \ldots \times (R_{W_k} \cap B^{|W_k|})$ where $W = \{i \mid B \subseteq R_{\{i\}}\}$, and $W_1, \ldots, W_k$ are the classes of $\theta_B(R)$;
for any $\mathbf{a} \in R$ such that $\mathbf{a}[i] \in B$ whenever $i \in W$, there is $\mathbf{b} \in R$ with

$$\mathbf{b}[i] = \begin{cases} \mathbf{a}[i], & \text{if } i \in W \text{ or } R_{\{i\}}; \\ c, & \text{if } i \notin W,\ \{c\} = B \cap R_{\{i\}}. \end{cases}$$

Definition 16 *Let B be a 2-element subuniverse of $\mathbb{A}$. The algebra $\mathbb{A}$ is B-*semirectangular *if the equivalence relation η with classes B and $A - B = \{c\}$ is a congruence of $\mathbb{A}$, and, for any (n-ary) relation $R \in \mathsf{Inv}\ F$, any tuple*

$\mathbf{b} \in R$, *and any* $\mathbf{a}_i \in R_{W_i} \cap B^{|W_i|}$, $i \in \underline{k}$, $W_1, \ldots, W_k$ *are the classes of* $\theta_B(R)$, *the tuple* $\mathbf{a}$ *with*

$$\mathbf{a}[i] = \begin{cases} \mathbf{b}[i], & \text{if } B \not\subseteq R_i; \\ c, & \text{if } B \subseteq R_i \text{ and } \mathbf{b}[i] = c; \\ \mathbf{a}_j[i], & \text{if } i \in W_j \text{ and } \mathbf{b}[i] \in B \end{cases}$$

belongs to R.

Definition 17 *The algebra* $\mathbb{A}$ *satisfies the* splitting *property if any (n-ary) relation* $R \in \mathsf{Inv}\ F$ *can be represented in the form* $R_W \times R_{\underline{n}-W}$ *where* $W = \{i \in \underline{n} \mid R_i = A\}$, *and* $R_W = A^{|W|}$.

Recall that the *graph* of a mapping $f\colon A \to B$ is the binary relation $\{(a, f(a)) \mid a \in A\}$. Then, a relation R is said to be *irreducible* if, for no pair i, j of coordinate positions, the projection $R_{\{i,j\}}$ is the graph of a mapping.

Definition 18 *Let* B *be a proper subuniverse of* $\mathbb{A}$. *Then* $\mathbb{A}$ *satisfies* B-semisplitting *property if, for any irreducible (n-ary) relation* $R \in \mathsf{Inv}\ F$, $W = \{i \in \underline{n} \mid R_i = A\}$, *we have (i)* $(R_W \cap B^{|W|}) \times R_{\underline{n}-W} \subseteq R$; *and (ii) for any* $i, j \in W$ *and any* $(a_i, a_j) \in R_{\{i,j\}} \cap B^2$, *there is a tuple* $\mathbf{a} \in R_W \cap B^{|W|}$ *such that* $\mathbf{a}[i] = a_i, \mathbf{a}[j] = a_j$.

Definition 19 *Let* $B \subseteq A$ *be a 2-element subuniverse of* $\mathbb{A}$. *The algebra* $\mathbb{A}$ *satisfies the* B-extendibility property *if, for any (n-ary) relation* $R \in \mathsf{Inv}\ F$, *if* W *denotes the set* $\{i \mid B \subseteq R_i\}$, *then*

- *for any* $k \in W$ $(k, l \in W)$, *and any* $a \in B$ $(\begin{pmatrix} a \\ b \end{pmatrix} \in R_{\{k,l\}})$, *there is* $\mathbf{a} \in R$ *such that* $\mathbf{a}[i] \in B$ *for all* $i \in W$, *and* $\mathbf{a}[k] = a$ $(\mathbf{a}[k] = a, \mathbf{a}[l] = b)$;
- *for any* $\mathbf{a} \in B^{|W|}$ *such that* $\begin{pmatrix} \mathbf{a}[i] \\ \mathbf{a}[j] \end{pmatrix} \in R_{\{i,j\}}$, *for any* $i, j \in W$, *there is* $\mathbf{b} \in R$ *such that*

$$\mathbf{b}[i] = \begin{cases} \mathbf{a}[i], & \text{if } i \in W \text{ or } |R_{\{i\}}| = 1, \\ a, & \text{otherwise, } a \in R_i \cap B. \end{cases}$$

A ternary operation f is said to be *Mal'tsev* if the identities $f(x, y, y) = f(y, y, x) = x$ hold. A standard example of a Mal'tsev operation is provided by the operation $x - y + z$ of an Abelian group, or the operation $xy^{-1}z$ of an arbitrary group. A binary operation $f(x, y)$ on the set A is said to be *conservative commutative* if, for any $x, y \in A$, $f(x, y) = f(y, x)$, and $f(x, y) \in \{x, y\}$.

Theorem 6 *If an idempotent 3-element algebra $\mathbb{A} = (A; F)$ satisfies* (No-G-Set), *then there is a set F' of its term operations such that the algebra $\mathbb{A}' = (A; F')$ satisfies* (No-G-Set) *and one of the following conditions holds.*

(1) $\mathbb{A}'$ satisfies the $(a{-}b)$-replacement property for a 2-element subuniverse B, and $a \in A - B$, $b \in B$.
(2) $\mathbb{A}'$ satisfies the partial zero property.
(3) $\mathbb{A}'$ satisfies the B-extendibility property for a 2-element subuniverse $B \subseteq A$.
(4) $\mathbb{A}'$ is B-rectangular for a 2-element subuniverse B.
(5) $\mathbb{A}'$ satisfies the splitting property.
(6) $\mathbb{A}'$ satisfies the B-semisplitting property for a 2-element subuniverse B, and $\mathbb{B}$ has a majority term operation.
(7) $\mathbb{A}'$ satisfies the B-semirectangular property for a 2-element subuniverse B.
(8) $\mathbb{A}'$ has a majority term operation.
(9) $\mathbb{A}'$ has a conservative commutative term operation.
(10) $\mathbb{A}'$ has a Mal'tsev term operation.

3.4 Why 'good' properties are good

We conclude this section by showing that every algebra satisfying one of those properties is tractable.

3.4.1 Relations invariant with respect to a special operation

In (8), the tractability of $\mathbb{A}$ follows from Proposition 1. In (9), CSP($\mathbb{A}$) is of width 3, as is proved in [4]. The result of [2] states that any finite algebra with a Mal'tsev operation is tractable, and the solution algorithm is a generalised version of Gaussian elimination.

3.4.2 The partial zero property

In this case any problem instance can be reduced to a 2-valued one. Indeed, if $\mathbb{A}$ satisfies the partial zero property, and a 1-minimal problem instance $\mathcal{P} = (V; A; \mathcal{C}) \in \mathrm{CSP}(\mathbb{A})$ has a solution φ, then $\mathcal{P}$ also has the solution ψ such that $\psi(v) = \varphi(v)$ if $\mathcal{S}_v \notin Z$, and $\psi(v) = z_{\mathcal{S}_{\{v\}}}$ otherwise. Thus, to solve $\mathcal{P}$ we assign the value $z_{\mathcal{S}_{\{v\}}}$ to each variable $v \in V$ with $\mathcal{S}_{\{v\}} \in Z$. Since $A \in Z$, the obtained problem instance is 2-valued.

3.4.3 The replacement property

In this case any problem instance can also be reduced to a 2-valued one. If $\mathbb{A}$ satisfies the $(a-b)$-replacement property, and a 1-minimal problem instance $\mathcal{P} = (V; A; \mathcal{C}) \in \mathrm{CSP}(\mathbb{A})$ has a solution φ, then the mapping $\psi: V \to A$

such that $\psi(v) = b$ if $a, b \in \mathcal{S}_{\{v\}}$, $\varphi(v) = a$, and $\psi(v) = \varphi(v)$ otherwise, is a solution to $\mathcal{P}$. We therefore, may reduce $\mathcal{P}$ to a 2-valued problem instance $\mathcal{P}' = (V; A; \mathcal{C}')$ where for each $C = \langle s, R\rangle \in \mathcal{C}$ there is $C' = \langle s, R'\rangle$ such that $\mathbf{a} \in R'$ if and only if $\mathbf{a} \in R$ and $\mathbf{a}[v] \neq a$ whenever $a, b \in \mathcal{S}_{\{v\}}$.

3.4.4 The extendibility property

We prove that in this case CSP($\mathbb{A}$) is of width 3. Suppose that $\mathbb{A}$ satisfies the B-extendibility property, and take a 3-minimal problem instance $\mathcal{P} = (V; A; \mathcal{C})$.

Lemma 1 *Let $W \subseteq V$ be the set $\{v \in V \mid B \subseteq \mathcal{S}_{\{v\}}\}$. There is $\mathbf{a} =\in B^{|W|}$ such that $\begin{pmatrix} \mathbf{a}[v] \\ \mathbf{a}[w] \end{pmatrix} \in \mathcal{S}_{\{v,w\}}$, for any $v, w \in W$.*

Proof. Since $\mathcal{P}$ is 3-minimal, for any $v, w \in W$, there is a constraint $\langle s, R\rangle \in cC$ such that $v, w \in s$. Furthermore, since $\mathbb{A}$ satisfies the B-extendibility property, for any $a \in B$, there are $b \in B$ and $\mathbf{b} \in R$ such that $\mathbf{b}[v] = a, \mathbf{b}[w] = b$. Therefore, $\mathcal{S}_{\{v,w\}} \cap B^2 \neq \varnothing$. Let $W' \subseteq W$ be a maximal set such that there is $\mathbf{a} =\in B^{|W'|}$ with $\begin{pmatrix} \mathbf{a}[v] \\ \mathbf{a}[w] \end{pmatrix} \in \mathcal{S}_{\{v,w\}}$, for any $v, w \in W'$. If $W' = W$ then we are done, otherwise take $w \in W - W'$. Let $B = \{a, b\}$. The maximality of W' means that there is $u, v \in W'$ such that $\begin{pmatrix} \mathbf{a}_{W'}[u] \\ a \end{pmatrix} \notin \mathcal{S}_{\{u,w\}}$, $\begin{pmatrix} \mathbf{a}_{W'}[v] \\ b \end{pmatrix} \notin \mathcal{S}_{\{v,w\}}$. Since $\mathcal{P}$ is 3-minimal, there is a constraint $\langle s, R\rangle \in \mathcal{C}$ such that $u, v, w \in s$. We have $\begin{pmatrix} \mathbf{a}_{W'}[u] \\ \mathbf{a}_{W'}[v] \end{pmatrix} \in R_{\{u,v\}}$, and by the B-extendibility property, for some $\mathbf{b} \in R$, $\mathbf{b}[u] = \mathbf{a}_{W'}[u], \mathbf{b}[v] = \mathbf{a}_{W'}[v]$, and $\mathbf{b}[w] \in B$, that contradicts the assumptions made. □

Finally, the B-extendibility property of $\mathbb{A}$ implies that the mapping $\varphi: V \to A$ where

$$\varphi(v) = \begin{cases} \mathbf{a}[v], & \text{if } v \in W; \\ c, & \text{if } \{c\} = \mathcal{S}_{\{v\}} \cap B \end{cases}$$

is a solution to $\mathcal{P}$.

3.4.5 The rectangularity and semirectangularity

Suppose that $\mathbb{A}$ is B-rectangular or B-semirectangular, and $\{c\} = A - B$. We show that any problem instance in this case can be reduced to a 2-valued one. Take a problem instance $\mathcal{P} = (V; A; \mathcal{C}) \in$ CSP($\mathbb{A}$). Without loss of generality we may assume that $\mathcal{P}$ is 3-minimal. Let W denote the set of all variables $v \in V$ with $B \subseteq \mathcal{S}_{\{v\}}$. Let $\theta(\mathcal{P})$ be the equivalence relation generated by $\bigcup_{\langle s,R\rangle\in\mathcal{C}} \theta_B(R)$. Notice that, since $\mathcal{P}$ is 3-minimal, for any $\langle s, R\rangle \in \mathcal{C}$, any $u, v \in s \cap W$ such that $(u, v) \in \theta(\mathcal{P})$, and any $\mathbf{a} \in R$, either $\mathbf{a}[u], \mathbf{a}[v] \in B$, or $\mathbf{a}[u] = \mathbf{a}[v] = c$. Repeat the following procedure until the obtained problem instance coincides with the previous one.

- `For each class` W' `of` $\theta(\mathcal{P})$`, solve the restricted problem` $\mathcal{P}'_{W'} = (W'; B; \mathcal{C}')$ `where, for each` $\langle s, R\rangle \in \mathcal{C}$`, we make the constraint` $\langle s \cap W', (R_{s\cap W'}) \cap B^{|s\cap W'|}\rangle \in \mathcal{C}'$.
- `If, for a class` W' `of` $\theta(\mathcal{P})$`, the problem instance` $\mathcal{P}'_{W'}$ `has no solution then,`
  `for each constraint` $\langle s, R\rangle \in \mathcal{C}$`, remove from` R `all the tuples` $\mathbf{a}$ `such that` $\mathbf{a}[v] \in B$ `for some` $v \in s \cap W'$.
- `Replace the obtained problem instance with the associated 3-minimal problem instance` $\mathcal{P}$.
- `Remove from` W `those variables` v `for which` $\mathcal{S}_{\{v\}}$ `no longer equals` A `or` B.
- `Calculate the relation` $\theta(\mathcal{P})$ `for the obtained problem instance and the set` W.

Obviously, the obtained problem instance $\mathcal{P}$ has a solution if and only if the original problem instance has.

Suppose first that $\mathbb{A}$ is B-rectangular. Then, if $\mathcal{P}$ has no empty constraint, then there is a solution φ to $\mathcal{P}$ such that $\varphi(v) \in B$ whenever $B \subseteq \mathcal{S}_{\{v\}}$. Indeed, let $W_1, \ldots, W_k$ be the classes of $\theta(\mathcal{P})$, and ψ_i a solution to $\mathcal{P}'_{W_i}$, $i \in \{1, \ldots, k\}$. It follows straightforwardly from the B-rectangularity that the mapping $\varphi: V \to A$ where

$$\varphi(v) = \begin{cases} \psi_i(v), & \text{if } v \in W_i; \\ a, & \text{if } \mathcal{S}_{\{v\}} = \{a, c\},\ a \in B; \\ b, & \text{if } \mathcal{S}_{\{v\}} = \{b\},\ b \in A, \end{cases}$$

is a solution to $\mathcal{P}$.

Now, suppose that $\mathbb{A}$ satisfies the B-semirectangular property. Denote by $\mathcal{P}^\theta$ the *factor-problem*, that is the problem $(V; \{C_0 = A/_\theta\} \cup \{C_i \mid i \in I\}; \delta; \mathcal{C}')$ where

- C_i, $i \in I$ are the subuniverses of $\mathbb{A}$;
- $\delta(v) = i$ if and only if $\mathcal{S}_{\{v\}} = C_i$, and $\delta(v) = 0$ if $\mathcal{S}_{\{v\}} = A$;
- for each $\langle s, R\rangle \in \mathcal{C}$, there is $\langle s, R^\theta\rangle \in \mathcal{S}^\theta$ where $\mathbf{b} \in R^\theta$ if and only if there is $\mathbf{a} \in R$ such that

$$\mathbf{b}[v] = \begin{cases} \mathbf{a}[v], & \text{if } R_v \neq A; \\ (\mathbf{a}[v])^\theta, & \text{if } R_v = A. \end{cases}$$

It is not hard to see that if $\mathcal{P}$ has a solution, then $\mathcal{P}^\theta$ has a solution (see also [4]). Let φ be a solution to $\mathcal{P}^\theta$, and $\psi_1, \dots, \psi_k$ solutions to $\mathcal{P}_{W_1}, \dots, \mathcal{P}_{W_k}$. The mapping ψ where

$$\psi[v] = \begin{cases} \varphi(v), & \text{if } B \not\subseteq \mathcal{S}_{\{v\}}; \\ \psi_i(v), & \text{if } B \subseteq \mathcal{S}_{\{v\}},\ v \in W_i, \text{ and } \varphi(v) \neq c^\theta; \\ c, & \text{if } B \subseteq \mathcal{S}_{\{v\}},\ v \in W_i, \text{ and } \varphi(v) = c^\theta. \end{cases}$$

is a solution to $\mathcal{P}$. Indeed, take a constraint $\langle s, R\rangle \in \mathcal{C}$. Since for each $i \in \{1, \dots, k\}$ such that $\varphi(v) = B$, $v \in W_i$, ψ_i is a solution to $\mathcal{P}_{W_i}$, the tuple $(\psi(v))_{v \in s \cap W_i}$ belongs to $R_{v \in s \cap W_i}$. Moreover, φ is a solution to the factor-problem, therefore, there is $\mathbf{b} \in R$ such that $\mathbf{b}[v] = \varphi(v)$ when $v \in s - W$, $\mathbf{b}[v] = c$ when $\varphi(v) = c^\theta$, $\mathbf{b}[v] \in B$ when $\varphi(v) = B$. The semirectangularity of $\mathbb{A}$ implies that $(\psi(v))_{v \in s} \in R$.

Finally, the factor-problem is 2-valued, and therefore, can be solved in polynomial time by Corollary 1.

3.4.6 The splitting and semisplitting property

If $\mathbb{A}$ satisfies the splitting property, then for any 1-minimal problem instance $\mathcal{P} = (V; A; \mathcal{C})$, denote W the set $\{v \in V \mid \mathcal{S}_{\{v\}} = A\}$, $W' = V - W$, and notice that $\mathcal{P}_{W'}$ is 2-valued and any solution to $\mathcal{P}_{W'}$ can be arbitrarily extended to a solution to $\mathcal{P}$.

Suppose that B is a 2-element subuniverse of $\mathbb{A}$, there is a term operation f such that $f|_B$ is the majority operation, and $\mathbb{A}$ satisfies the B-semisplitting property. A problem instance is said to be *irreducible* if every its constraint relation is irreducible. Every 3-minimal problem instance $\mathcal{P} = (V; A; \mathcal{C})$ can be reduced to an equivalent irreducible problem instance in polynomial time.

Indeed, denote by η the binary relation on V such that $(u, v) \in \eta$ if and only if $\mathcal{S}_{\{u,v\}}$ is the graph of a bijective mapping $\pi_{u,v}\colon \mathcal{S}_{\{u\}} \to \mathcal{S}_{\{v\}}$. Since $\mathcal{P}$ is 3-minimal, $\mathcal{S}_{\{u,v\}} \circ \mathcal{S}_{\{v,w\}} \supseteq \mathcal{S}_{\{u,w\}}$, for any $u, v, w \in V$, where $\circ$ denotes the multiplication of binary relation; hence, η is an equivalence relation. Choose a representative from each class of η. Then, for no pair $v, w \in W$ of variables, $\mathcal{S}_{\{v,w\}}$ is the graph of a bijective mapping, and for any $v \in V - W$, there is $v' \in W$ such that $\mathcal{S}_{\{v',v\}}$ is the graph of a bijective mapping. We transform $\mathcal{P}$ in three steps.

- For each constraint $\langle s, R\rangle \in \mathcal{C}$ and any $\mathbf{a} \in R$, replace $\mathbf{a}$ with $\mathbf{b}$ where $\mathbf{b}[v] = \pi_{v,v'}(\mathbf{a}[v])$, $v \in s$ and $v' \in W$ is the representative of the η-class containing v.

- For each constraint $\langle s, R\rangle \in \mathcal{C}$ and each $v \in s$, replace v with v'.

- Replace every constraint $\langle s, R\rangle \in \mathcal{C}$ with $\langle s \cap W, R_{s \cap W}\rangle$.

Now, let $\mathcal{P} = (V; A; \mathcal{C}) \in \mathrm{CSP}(\mathbb{A})$ be a 3-minimal irreducible problem instance, and consider the instance $\mathcal{P}' = (V; A; \mathcal{C}')$ where, for each $\langle s, R\rangle \in \mathcal{C}$, there is $\langle s, R'\rangle \in \mathcal{C}'$ with $R' = \{\mathbf{a} \in R \mid \mathbf{a}[v] \in \{0, 2\}$ for all $v \in s$ such that $R_v = A\}$. The problem instance $\mathcal{P}'$ is 2-valued, therefore, we just have to show that $\mathcal{P}, \mathcal{P}'$ are equivalent.

Clearly, if $\mathcal{P}'$ has a solution then $\mathcal{P}$ has a solution. Conversely, let $\mathcal{P}$ have a solution, and set $W = \{v \in V \mid \mathcal{S}_{\{v\}} = A$, and $W' = V - W$. By condition (i) of the definition of the semisplitting property, $\mathcal{P}'$ has a solution if and only if both $\mathcal{P}'_W$ and $\mathcal{P}'_{W'}$ have solutions. Since $\mathcal{P}'_{W'} = \mathcal{P}_{W'}$, the instance $\mathcal{P}'_{W'}$ has a solution. By condition (ii), for any $v, w \in W$, $\mathcal{S}'_{\{v,w\}} = \mathcal{S}_{\{v,w\}} \cap B^2$ where $\mathcal{S}'_{\{v,w\}}$ denotes the set of partial solutions to $\mathcal{P}'_W$ for $\{v, w\}$. Moreover, since $\mathcal{P}$ is 3-minimal, for any $u \in W$ every such partial solution can be extended to a solution from $\mathcal{S}'_{\{v,w,u\}}$. The last property is called *strong 2-consistency* [18]. Recall that any relation $R \in \mathsf{Inv}\ F$ such that $R \subseteq B^n$ is invariant with respect to a majority operation m, in particular, all the constraint relations of $\mathcal{P}_W$ satisfy this condition. By Theorem 3.5 of [18], if $\mathcal{S}_{\{v,w\}} \neq \varnothing$ for any $v, w \in W$ then strong 2-consistency ensures existence of a solution to $\mathcal{P}_W$.

4 Recognising tractable cases

In a practical perspective, we need a method that allows us to recognise if a given constraint language Γ is tractable. The following problem is, therefore, very tempting.

TRACTABLE-LANGUAGE. Is a given finite constraint language Γ on a finite set tractable?

Schaefer's Dichotomy Theorem [28] does not solve this problem satisfactory. Indeed, it can be easily verified if a relation is of type (1) or (2), however, the way of recognising the types (3)–(6) is not obvious (see also [21]). Theorem 3, the algebraic version of Schaefer's result, fills this gap: to check the tractability of a Boolean constraint language one just have to check if all relations from the language are invariant under one of the 6 Boolean operations.

In the general case, such a method can hopefully be derived from a description of tractable algebras. For example, in [3], a polynomial time algorithm has been exhibited that checks if a finite algebra, whose basic operations are given explicitly by their operation tables, satisfies (NO-G-SET). Therefore, if Conjecture 1 holds then the tractability of an algebra can be tested in polynomial time. In particular, this algorithm sounds in the case of 3-element algebras.

However, the algorithm does not solve TRACTABLE-LANGUAGE even under the assumption of Conjecture 1, because in this problem we are given a constraint language, not an algebra. Actually, we need to solve the problem

NO-G-SET-LANGUAGE. Given finite constraint language Γ on a finite set, does the algebra $\mathbb{A}_{\Gamma^{\rm id}}$ satisfy (NO-G-SET)?

By the results of [3], this problem is NP-complete. However, its restricted version remains tractable.

NO-G-SET-LANGUAGE(k). Given finite constraint language Γ on a finite set A, $|A| \le k$, does the algebra $\mathbb{A}_{\Gamma^{\rm id}}$ satisfy (NO-G-SET)?

This means that the tractability of a constraint language on a 3-element set can be tested in polynomial time.

Theorem 7 *There is a polynomial time algorithm that given a constraint language Γ on a 3-element set determines if Γ is tractable.*

An example of such an algorithm is provided by the general algorithm from [3]. That algorithm employs some deep algebraic results and sophisticated constructions. In the particular case of a 3-element domain, we may avoid using of hard algebra, and apply a simpler and easier algorithm.

To this end, notice that if a 3-element algebra $\mathbb{A}$ has a 2-element subuniverse or a nontrivial congruence, and there is a term operation f which is not a projection on the subalgebra or the factor-algebra, then f witnesses that $\mathbb{A}$ itself is also not a G-set. We, therefore, have two cases to consider.

CASE 1. $\mathbb{A}$ has no 2-element subuniverse, and no proper congruence. Such an algebra is said to be *strictly simple.* There is a complete description of finite strictly simple algebras [30]. In particular, if a strictly simple algebra satisfies (NO-G-SET) then one of the following operations is its term operation: a majority operation, the Mal'tsev operation $x - y + z$ of an Abelian group, or the operation

$$t_0(x,y) = \begin{cases} 0, & \text{if } 0 \in \{x,y\}, \\ x, & \text{otherwise} \end{cases}$$

for some element $0 \in A$. In the last case, $\mathbb{A}$ satisfies the partial zero property for $Z = \{B \mid B \text{ is a subuniverse of } \mathbb{A}, \text{ and } 0 \in B\}$, and $z_B = 0$ for $B \in Z$.

CASE 2. $\mathbb{A}$ has either a 2-element subalgebra, or a proper congruence. In this case, $\mathbb{A}$ satisfies (NO-G-SET) if and only if every its 2-element subalgebra and every proper factor-algebra (which is also 2-element) is not a G-set. In its turn, the latter condition holds if and only if, for any 2-element subuniverse B of $\mathbb{A}$, and any congruence θ, there is a polymorphism f of Γ such that $f\big|_B$ (or f^θ) is one of the 4 Boolean operations: $\wedge$, $\vee$, the majority operation $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$, the affine operation $x - y + z$ (since $\mathbb{A}$ is idempotent, a constant cannot be its term operation).

As is well known [23], the subuniverses and congruences of a k-element algebra are completely determined by its k-ary term operations. Hence, we may restrict ourselves with finding ternary polymorphisms of a constraint language Γ. In the first 3 steps, the algorithm below construct the language $\Gamma^{\rm id}$.

Algorithm
INPUT A finite constraint language Γ on a 3-element set A.
OUTPUT "YES" if Γ is tractable, "NO" otherwise.

- `Find all the unary operations on` A `that preserves each relation from` Γ`.`
- `If there is a unary non-identity operation` f `such that` $f(f(x)) = f(x)$ `then take one with a minimal range, and replace` Γ `with` $f(\Gamma)$`, and` A `with` $f(A)$`.`
- `Add the relations` $\{(a)\}$, $a \in A$, `to` Γ`.`
- `Find the set` F `of all ternary operations preserving each relation from` Γ`.`
- `Find the set` S `of all 2-element subsets from` A `and the set` C `of all proper equivalence relations invariant under operations from` F`.`
- `If` $S = C = \varnothing$ `then`
  - `if` F `contains either a majority operation, or the operation` t_a `for some` $a \in A$`, or the Mal'tsev operation` $x - y + z$ `of an Abelian group, then output ''YES'';`
  - `otherwise output ''NO''.`
- `Otherwise, for each` $B \in S$ `(each` $\theta \in C$`), do`
  - `check if there is` $f \in F$ `such that` $f|_B$ `(`f^θ`) is one of the Boolean operations` $\wedge$, $\vee$, $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$, $x + y + z$`;`
  - `if not then output ''NO''.`
- `Output ''Yes''.`

This algorithm is polynomial time, because the hardest step, finding the set F, requires inspecting of all ternary operations on a 3-element set; and, since their number does not depend on Γ, takes cubic time.

5 Proof of Theorem 6

Everywhere in this section $\mathbb{A} = (A; F)$, $A = \{0, 1, 2\}$, is a 3-element algebra satisfying the conditions of Theorem 6.

5.1 Prerequisites

In this section we state and prove some auxiliary statements. An element $a \in A$ is said to be a *right-zero* [*left-zero*] with respect to the operation $t(x, y)$ if $t(x, a) = a$ [$t(a, x) = a$] for any $x \in A$. An operation of any of the following two types guarantees the tractability of an algebra.

Definition 20 *Let $A = \{a, b, c\}$. A binary operation f is said to be $(a-b)$-operation if b, c are left- (right-)zeroes, and $\{f(a, a), f(a, b), f(a, c)\} = \{a, b\}$ ($\{f(a, a), f(b, a), f(c, a)\} = \{a, b\}$).*

Definition 21 *Let $A = \{a, b, c\}$. A binary operation f is said to be* zero-operation *if one of the following conditions holds*

- *a is a right-zero, $a \in \{f(a, b), f(c, b)\}$, and $\{f(a, c), f(b, c)\} \cap \{a, b\} \neq \varnothing$;*
- *a is a left-zero, $a \in \{f(b, a), f(b, c)\}$, and $\{f(c, a), f(c, b)\} \cap \{a, b\} \neq \varnothing$.*

Lemma 2 *Let an algebra $\mathbb{A} = (\{a, b, c\}, F)$ have a term $(a - b)$-operation. Then $\mathbb{A}$ satisfies the $(a - b)$-replacement property.*

Proof. Suppose first that b, c are left-zeroes. Take a subuniverse R of $\mathbb{A}^n$, and a tuple $\mathbf{a} \in R$. We prove that, for all $k \leq n$, a tuple $\mathbf{a}_k \in R$ with

$$\mathbf{a}_k[i] = \begin{cases} \mathbf{a}[i], & \text{if } \mathbf{a}[i] \in \{b, c\}; \\ b, & \text{if } \mathbf{a}[i] = a,\ i \leq k, \end{cases}$$

if $f(a, b) = b$; and

$$\mathbf{a}_k[i] = \begin{cases} \mathbf{a}[i], & \text{if } \mathbf{a}[i] \in \{b, c\}; \\ b, & \text{if } \mathbf{a}[i] = a,\ i \leq k, \text{ and } R_i = A, \end{cases}$$

otherwise.

The tuple $\mathbf{a}_0$ can be set to be $\mathbf{a}$. Suppose that a tuple $\mathbf{a}_k$ is already found. Then set $\mathbf{a}_{k+1} = \mathbf{a}_k$ if $\mathbf{a}_k[k+1] \in \{b, c\}$, or $R_{k+1} = \{a, b\}$ and $f(a, b) = a$; otherwise, set $\mathbf{a}_{k+1} = f(\mathbf{a}_k, \mathbf{b})$ where $\mathbf{b} \in R$ is such that $\mathbf{b}[k+1] = b$ if $f(a, b) = b$, and $\mathbf{b}[k+1] = c$ if $f(a, c) = b$.

The case when b, c are right-zeroes is quite similar. □

Lemma 3 *Let an algebra $\mathbb{A} = (\{a, b, c\}, F)$ have a term zero-operation. Then $\mathbb{A}$ satisfies the partial zero property.*

Proof. Suppose first that a is a left-zero with respect to f. Then Z is the set consisting of A and all the 2-element subuniverses of $\mathbb{A}$, on which f is a semilattice operation. Notice that if a belongs to such a subuniverse, then

$f(a,x)=f(x,a)=a$, for any x from this subuniverse. Therefore, $z_C=a$ for any $C\in Z$ with $a\in C$, and $z_{\{b,c\}}=f(b,c)$ if $\{b,c\}\in Z$.

Take a subuniverse R of $\mathbb{A}^n$, and a tuple $\mathbf{a}\in R$. We prove that, for all $k\le n$, a tuple $\mathbf{a}_k$ with

$$\mathbf{a}_k[i]=\begin{cases} z_{R_i}, & \text{if } R_i\in Z \text{ and } i\le k;\\ \mathbf{a}[i], & \text{if } R_i\notin Z\end{cases}$$

belongs to R.

Set $\mathbf{a}_0=\mathbf{a}$, and suppose that $\mathbf{a}_k$ is already found. Then

$$\mathbf{a}_{k+1}=\begin{cases} \mathbf{a}_k, & \text{if } R_{k+1}\notin Z, \text{ or } \mathbf{a}_k[k+1]=z_{R_{k+1}};\\ f(\mathbf{a}_k,\mathbf{b}), & \text{if } \mathbf{a}_k[k+1]=b,\ a\in R_{k+1},\ \text{and } \mathbf{b}\in R \text{ is such}\\ & \text{that } \mathbf{b}[k+1]=x \text{ and } f(b,x)=a;\ \text{or}\\ & \mathbf{a}_k[k+1]=c,\ a\in R_{k+1},\ a\in\{f(c,a),f(c,b)\},\\ & \text{and } \mathbf{b}\in R \text{ is such that } \mathbf{b}[k+1]=x \text{ and}\\ & f(c,x)=a;\\ f(f(\mathbf{a}_k,\mathbf{b}),\mathbf{c}), & \text{if } \mathbf{a}_k[k+1]=c,\ a\in R_{k+1},\ \text{and } \mathbf{b},\mathbf{c}\in R \text{ are such}\\ & \text{that } \mathbf{b}[k+1]=x,\ \mathbf{c}[k+1]=y \text{ where } f(c,x)=b,\\ & f(b,y)=a;\\ f(\mathbf{a}_k,\mathbf{b}), & \text{if } \{b,c\}\in Z,\ x=f(b,c),\ \{y\}=\{b,c\}-\{x\},\\ & \mathbf{a}_k[k+1]=y,\ \text{and } \mathbf{b}\in R \text{ is such that}\\ & \mathbf{b}[k+1]=x.\end{cases}$$

□

An important particular type of zero-operations rises from the following definition.

Definition 22 *An element $a\in A$ is said to be a* zero-element *with respect to a binary operation $f(x,y)$, if $f(a,x)=f(x,a)=a$, for any $x\in A$.*

Lemma 4 *If a is a zero-element with respect to $f(x,y)$, then f is a zero-operation.*

We also need two simple observations that will be frequently used.

Lemma 5 *(1) If $f(x,y)$ is an idempotent operation on a 2-element set then f is either a projection or a semilattice operation.*

(2) If $f(x,y)$ is an idempotent operation on a 2-element set then $f(x,f(y,x))$ [$f(f(y,x),y)$] is a semilattice operation if f is a semilattice operation, and is the first [second] projection otherwise.

We consider 4 cases depending on the number of 2-element subalgebras and nontrivial homomorphic images of the algebra $\mathbb{A}$. Recall that an algebra is said to be *simple* if it has only two homomorphic images: one-element, and the algebra itself.

5.2 Strictly simple algebras

A *strictly simple algebra* is a simple algebra that has no subalgebras but one-element, and the algebra itself. Finite strictly simple surjective algebras have been described by Á. Szendrei [30]. To formulate Á.Szendrei's result we need some notation.

Let G be a permutation group on a set A. By $\mathrm{R}(G)$ we denote the set of operations on A preserving each relation of the form $\{(a, g(a)) \mid a \in A\}$ where $g \in G$, and $\mathrm{F}(G)$ denotes the set of idempotent members of $\mathrm{R}(G)$.

Let ${}_F\overline{A} = (A; +, F)$ be a finite vector space over a finite field F, $T(\overline{A})$ the group of translations $\{x + a \mid a \in A\}$, $\mathsf{End}\ {}_F\overline{A}$ the endomorphism ring of ${}_F\overline{A}$. Then one can consider $\overline{A}$ as a module over $\mathsf{End}\ {}_F\overline{A}$. This module is denoted by ${}_{(\mathsf{End}\ {}_F\overline{A})}\overline{A}$.

Finally, F^0_k denotes the set of all operations preserving the relation

$$X^0_k = \{(a_1, \ldots, a_k) \in A^k \mid a_i = 0 \text{ for at least one } i,\ 1 \le i \le k\}$$

where 0 is some fixed element of A, and let $\mathrm{F}^0_\omega = \bigcap_{k=2}^{\infty} \mathrm{F}^0_k$.

Algebras are said to be *term equivalent* if their sets of term operations are equal.

Theorem 8 *[30] Let $\mathbb{A}$ be a finite idempotent strictly simple algebra. Then it is term equivalent to one of the following algebras:*

(a°) $(A, \mathrm{F}(G))$ for a permutation group G on A such that every nonidentity member of G has at most one fixed point;

(b°) (A, F) where F is the set of all idempotent term operations of ${}_{(\mathsf{End}\ {}_K\overline{A})}\overline{A}$ for some vector space ${}_K\overline{A}$ over a finite field K;

(d) $(A, \mathrm{F}(G) \cap \mathrm{F}^0_k)$ for some k $(2 \le k \le \omega)$, some element $0 \in A$ and some permutation group G on A such that 0 is the unique fixed point of every nonidentity member of G;

(e) (A, F) where $|A| = 2$ and F contains a semilattice operation;

(f) a two-element algebra with empty set of basic operations.

In [7], tractable strictly simple algebras have been characterised: a finite idempotent strictly simple algebra is tractable if and only if it is of type (a°), (b°), (d), (e); otherwise it is NP-complete. As is easily seen, exactly those algebras satisfy the condition of Theorem 6. In the same paper we noticed that in the case (a°) the dual discriminator operation, that is the majority operation

$$d(x, y, z) = \begin{cases} y, & \text{if } y = z, \\ x, & \text{otherwise;} \end{cases}$$

and in the case (b°) the Mal'tsev operation $x - y + z$ of the vector space are term operations of the algebra. In the case (d) the algebra has the term operation

$$d(x,y) = \begin{cases} 0, & \text{if } 0 \in \{x,y\}, \\ x, & \text{otherwise,} \end{cases}$$

and 0 is a zero-element with respect this operation; in the case (e) the algebra has a term semilattice operation. Since a set of relations closed under a majority operation is of width 3 ([19]), and any semilattice has a zero-element, Theorem 6 holds for 3-element strictly simple algebras.

5.3 Simple algebras with 1 or 2 subalgebras

In this subsection we assume that $\mathbb{A}$ has the subalgebra $\mathbb{B} = \{0,1\}$, but at least one of $\{0,2\}, \{1,2\}$ is not.

We need several particular operations on the 3-element set:

$$(1)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 2 \end{array},\ (2)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 2 \end{array},\ (3)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 \end{array},\ (4)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 2 \end{array},$$

$$(5)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 2 \end{array},\ (6)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \end{array},\ (7)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array},\ (8)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array},$$

$$(9)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 2 \end{array},\ (10)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array},\ (11)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array},\ (12)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array},$$

$$(13)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array},\ (14)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array},\ (15)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \end{array},\ (16)\ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{array}.$$

Lemma 6 *If $\{0,1\}$ is a subuniverse of $\mathbb{A}$, but at least one of $\{0,2\}, \{1,2\}$ is not, then $\mathbb{A}$ either has a binary term operation f and a zero-element with respect to f, or a zero-operation, or a $(2-a)$-operation, $a \in \{0,1\}$, or one of the operations*
(a) (1),(2),(4),(6),(8),(10),(11),(12),(14),(16) if $\{1,2\}$ is not a subuniverse;
(b) (1),(3),(5),(7),(8),(9),(11),(13),(15) if $\{0,2\}$ is not a subuniverse.

Proof. Suppose that $\{1,2\}$ is not a subuniverse. In the case when $\{0,2\}$ is not a subuniverse, the proof is dual in the sense that 0 and 1 are swopped. Since $\{1,2\}$ is not a subuniverse of $\mathbb{A}$, there is a term operation $f(x_1,\dots,x_n)$, and $a_1,\dots,a_n \in \{1,2\}$ such that $f(a_1,\dots,a_n) = 0$. Without loss of generality we may assume that $a_1 = \dots = a_k = 1$, $a_{k+1} = \dots = a_n =$

2. Then the operation $f(\underbrace{x,\dots,x}_{k},y,\dots,y)$ also destroys the set $\{1,2\}$, and therefore, f can be chosen to be binary.

On the other hand, $B=\{0,1\}$ is a subuniverse. Hence, f preserves B, and we have four cases depending on what is the restriction of f onto B, a semilattice operation, the first or the second projection.

CASE 1.

f	0	1	2
0	0	0	
1	0	1	0
2			2

, that is $f|_B$ is a semilattice operation, $0 \geq 1$.

SUBCASE 1.1. $f(2,1)=0$.
If $f(2,0)=0$ then f is a zero-operation; if $f(2,0)=1$ then $f(x,f(x,y))$ is a zero-operation. If $f(2,0)=2$ then $g(x,y)=f(x,f(x,y))$ is the operation (10) in the case $f(0,2)\in\{0,1\}$, and 2 is a zero-element with respect to $g(g(x,y),y)$ in the case $f(0,2)=2$.

SUBCASE 1.2. $f(2,1)=1$.
In this case we may get Subcase 1.1 by substituting $f(f(x,y),f(y,x))$.

SUBCASE 1.3. $f(2,1)=2$.

SUBCASE 1.3.1. $f(2,0)=0$. The operation f is a zero-operation.

SUBCASE 1.3.2. $f(2,0)=1$.
If $f(0,2)\in\{0,1\}$ then 0 is a zero-element with respect to the operation $f(f(x,y),y)$. In the case $f(0,2)=2$, 0 is a zero-element with respect to $f(f(x,y),x)$.

SUBCASE 1.3.3 $f(2,0)=2$.
If $f(0,2)\in\{0,1\}$ then $f(x,y)$ is one of the operations (10),(11). If $f(0,2)=2$ then 2 is a zero-element with respect to $f(f(x,y),y)$.

CASE 2.

f	0	1	2
0	0	1	
1	1	1	0
2			2

, that is $f|_B$ is a semilattice operation, $1 \geq 0$.

SUBCASE 2.1. $f(2,1)\in\{0,1\}$.

SUBCASE 2.1.1. $f(2,0)\in\{0,1\}$.
In this case, 1 is a zero-element with respect to $g(x,y)=f(x,f(x,y))$ if $f(2,1)=1$, and with respect to $g(g(x,y),y)$ if $f(2,1)=0$.

SUBCASE 2.1.2. $f(2,0)=2$.
If $f(2,1)=1$ then 1 is a zero-element with respect to the operation $f(x,f(x,y))$. In the case, $f(2,1)=0$, $f(f(x,y),y)$ is a zero-operation if $f(0,2)=0$, or 1 is a zero-element with respect to $f(f(y,x),y)$ if $f(0,2)=1$, or f is the operation (6) if $f(0,2)=2$.

SUBCASE 2.2. $f(2,1)=2$.
If $f(2,0)\in\{0,1\}$ then by substituting $f(x,f(y,x))$ we get Subcase 2.1. Consider the case when $f(2,0)=2$. If $f(0,2)=0$ then f is the operation (12); if $f(0,2)=1$ then $f(x,y)$ is the operation (8); and if $f(0,2)=2$ then 2 is a zero-element with respect to $f(f(x,y),y)$.

CASE 3.

f	0	1	2
0	0	0	
1	1	1	0
2			2

, that is $f|_B$ is the first projection.

SUBCASE 3.1. $f(2,1) = 1$.

SUBCASE 3.1.1. $f(0,2) = 0$. In this case, f is a zero-operation.

SUBCASE 3.1.2. $f(0,2) = 1$.

If $f(2,0) = 0$ then, for the operation $g(x,y) = f(f(x,y),x)$, $g(x,g(x,y))$ is the operation (1). If $f(2,0) \in \{1,2\}$ then $f(f(x,y),y)$ is a $(2-1)$-operation.

SUBCASE 3.1.3. $f(0,2) = 2$.

If $f(2,0) = 0$ then $f(x,f(y,x))$ is a $(2-0)$-operation. If $f(2,0) = 1$ then $f(y,f(y,x))$ is a zero-operation. If $f(2,0) = 2$ then f is the operation (4).

SUBCASE 3.2. $f(2,1) = 0$.

SUBCASE 3.2.1. $f(0,2) = 0$. In this case, $f(y,x)$ is a zero-operation.

SUBCASE 3.2.2. $f(0,2) = 1$. The operation $f(f(x,y),y)$ is a $(2-0)$- or a $(2-1)$-operation.

SUBCASE 3.2.3. $f(0,2) = 2$.

If $f(2,0) = 0$ then $f(f(x,y),x)$ is the operation (14); if $f(2,0) = 1$ then $f(f(x,y),x)$ is a zero-operation. Finally, in the case $f(2,0) = 2$, set $h(x,y) = f(f(x,y),y)$. Then 2 is a zero-element with respect to the operation $h(x,h(x,y))$.

SUBCASE 3.3. $f(2,1) = 2$.

SUBCASE 3.3.1. $f(2,0) \in \{0,1\}$.

In this case, the operation table of $f(x,f(y,x))$ is

	0	1	2
0	0	0	
1	1	1	0
2		0/1	2

, and therefore, we get one of Subcases 3.1, 3.2.

SUBCASE 3.3.2. $f(2,0) = 2$.

If $f(0,2) = 0$ then f is the operation (14). In the case $f(0,2) = 1$, f is the operation (16). Finally, if $f(0,2) = 2$ then 2 is a zero-element with respect to $f(f(x,y),y)$.

CASE 4.

f	0	1	2
0	0	1	
1	0	1	0
2			2

, that is $f|_B$ is the second projection.

SUBCASE 4.1. $f(2,1) = 1$.

SUBCASE 4.1.1. $f(2,0) = 0$.

If $f(0,2) \in \{0,2\}$ then f is a $(2-0)$-operation. If $f(0,2) = 1$ then $f(f(x,y),y)$ is the operation (1).

SUBCASE 4.1.2. $f(2,0) = 1$. In this case, f is a zero-operation.

SUBCASE 4.1.3. $f(2,0) = 2$.

If $f(0,2) = 0$ then $f(f(x,y),x)$ is a $(2-0)$-operation; if $f(0,2) = 1$ then f is a zero-operation; and if $f(0,2) = 2$ then f is the operation (4).

SUBCASE 4.2. $f(2,1)=0$.

Consider the operation $g(x,y) = f(x,f(x,y))$. Its operation table is

g	0	1	2
0	0	1	
1	0	1	0
2		$f(2,0)$	2

. Therefore, if $f(2,0)=1$ or 2 then we get Subcase 4.1 or Subcase 4.3 respectively. If $f(2,0)=0$ then $g(2,0)=g(2,1)=0$, and g is a zero-operation.

SUBCASE 4.3. $f(2,1)=2$.

SUBCASE 4.3.1. $f(2,0)=0$. In this case, f is a zero-operation.

SUBCASE 4.3.2. $f(2,0)\in\{1,2\}$.

Set $g(x,y) = f(x,f(x,y))$. Its operation table is

	0	1	2
0	0	1	$f(0,2)$
1	0	1	0
2	2	2	2

. If $f(0,2)=0$ then $g(x,g(y,x))$ is the operation (14). If $f(0,2)=2$ then g is a zero-operation. Finally, in the case $f(0,2)=1$, the operation $f(x,f(y,x))$ is the operation (16). □

If $\mathbb{A}$ has a zero-element with respect to a binary term operation, or a zero-operation, or a $(2-a)$-operation, then by Lemmas 3,2, $\mathbb{A}$ satisfies either the partial zero-property, or the $(2-a)$-replacement property. We, therefore, have to show that if $\mathbb{A}$ has one of the numbered term operations, then $\mathbb{A}$ satisfies one of the properties listed in Theorem 6.

Lemma 7 *Let $\mathbb{A}$ have a term operation f which is one of the operations (1), (2), (3), $B=\{0,1\}$ if f is (1), $B=\{0,2\}$ if f is (2), $B=\{1,2\}$ if f is (3), and g a term operation which is either a semilattice, or majority, or minority operation on B. Then*
– if $g|_B$ is a semilattice operation then $\mathbb{A}$ has a semilattice operation, and therefore, satisfies the partial zero property;
– if $g|_B$ is a majority operation then $\mathbb{A}$ satisfies the B-extendibility property;
– if $g|_B$ is a minority operation then $\mathbb{A}$ is $\{0,1\}$-rectangular.

Proof. We prove the lemma in the case when f is the operation (1); the other 2 cases are quite similar.

If a term operation g is such that $g|_{\{0,1\}}$ is a semilattice operation, then $g(f(x,y),f(y,x))$ is a semilattice operation on A.

Now, suppose that g is a majority operation on B, and $R\in \mathsf{Inv}\ F$ an (n-ary) relation. We show first, that, for any $k,l\in \underline{n}$ with $R_k=R_l=A$ and $a\in B$ there is $b\in B$ such that $\begin{pmatrix} a \\ b \end{pmatrix} \in R_{\{k,l\}}$, and moreover, for any

$\begin{pmatrix} a \\ b \end{pmatrix} \in R_{\{k,l\}}$ such that $a, b \in B$, there is $\mathbf{a} \in R$ with $\mathbf{a}[k] = a, \mathbf{a}[l] = b$, and $\mathbf{a}[i] \in B$ for all i such that $R_i = A$. Without loss of generality we may assume that $k = 1, l = 2$, and $\mathbf{a}_1 \in R$ is such that $\mathbf{a}[1] = a$. Then, there is $\mathbf{b} \in R$ with $\mathbf{b}[2] \in B$; set $\mathbf{c} = f(\mathbf{b}, \mathbf{a}_1)$. As is easily seen, $\mathbf{c}[1] = \mathbf{a}[1] = a$, $\mathbf{c}[2] \in B$. Further, let $\begin{pmatrix} a \\ b \end{pmatrix} \in R_{\{1,2\}} \cap B^2$, and let $\mathbf{a}_2 \in R$ be such that $\mathbf{a}[1] = a, \mathbf{a}[2] = b$. Then, for any $k \leq n$, there is $\mathbf{a}_k \in R$ such that $\mathbf{a}[1] = a, \mathbf{a}[2] = b$, and $\mathbf{a}[i] \in B$ for all $i \leq k$ such that $R_i = A$. Indeed, suppose that $\mathbf{a}_{k-1}$ is already found. Then $\mathbf{a}_k$ can be chosen to be $\mathbf{a}_{k-1}$ if $\mathbf{a}_{k-1}[k] \in B$ or $R_k \neq A$. Otherwise, there is $\mathbf{b} \in R$ with $\mathbf{b}[k] \in B$, and the tuple $\mathbf{a}_k = f(\mathbf{b}, \mathbf{a}_{k-1})$ satisfies the required conditions.

Furthermore, denote $W = \{i \in \underline{n} \mid R_i = A\}$, and take $\mathbf{a} \in B^{|W|}$ such that $\begin{pmatrix} \mathbf{a}[i] \\ \mathbf{a}[j] \end{pmatrix} \in R_{\{i,j\}}$, for any $i, j \in W$. By what was proved above, for any $k, l \in W$, there is $\mathbf{a}_{k,l} \in R$ with $\mathbf{a}_{k,l}[k] = \mathbf{a}[k], \mathbf{a}_{k,l}[l] = \mathbf{a}[l]$, and $\mathbf{a}_{k,l}[i] \in B$ whenever $i \in W$. Since g is a majority operation on B, by Theorem 3.5 from [18], there is $\mathbf{b} \in R$ such that $\mathbf{b}|_W = \mathbf{a}$.

Let $R = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$, and

$$\mathbf{c} = f(\ldots f(\mathbf{c}_1, \mathbf{c}_2), \ldots, \mathbf{c}_m).$$

Then, $\mathbf{c}[i] = 0$ if $R_i = \{0, 2\}$, and $\mathbf{c}[i] = 1$ if $R_i = \{1, 2\}$; and for the tuple $\mathbf{d} = f(\mathbf{c}, \mathbf{b})$, we have $\mathbf{d}|_W = \mathbf{b}|_W = \mathbf{a}$, $\mathbf{d}[i] = \mathbf{c}[i] = a$ if $R_i = \{a, 2\}$, $a \in B$.

Let g be a term operation which is a minority operation on $\{0, 1\}$, $R \in$ Inv an (n-ary) relation, and $W = \{u \mid 0, 1 \in R_u\}$, $W' = \underline{n} - W$, and $W_1, \ldots, W_k \subseteq W$ be the blocks of $\delta(R)$. We have to prove that

$$R_W \cap \{0,1\}^{|W|} = (R_{W_1} \cap \{0,1\}^{|W_1|}) \times \cdots \times (R_{W_k} \cap \{0,1\}^{|W_k|}).$$

Notice first, that replacing g with $g(x, f(x, y), z)$, we may assume that $g(x, 2, z) = z$ whenever $x, z \in \{0, 1\}$.

Let $\mathbf{a}_i \in R|_{W_i} \cap \{0,1\}^{|W_i|}$ for $i \in \underline{k}$, and let us suppose we have proved that, for any $l-1$-element subset $I \subseteq \underline{k}$ there is $\mathbf{a} \in R$ with $\mathbf{a}|_{W_i} = \mathbf{a}_i$ whenever $i \in I$. Take an l-element subset $J \subseteq \underline{k}$; without loss of generality we may assume $J = \{1, \ldots, l\}$ and $U = W_1 \cup \ldots \cup W_l$. There exists a tuple $\mathbf{b} \in R$ such that $\mathbf{b}_{W_1} \in \{0,1\}^{|W_1|}, \mathbf{b}_{W_l} = (2, \ldots, 2)$, or vice versa. It will not be loss of generality if we suppose that $\mathbf{b}_{W_l} = (2, \ldots, 2)$, and there is $m < l$ such that, for any $i \leq l$, $\mathbf{b}_{W_i} = (2, \ldots, 2)$ if and only if $i \geq m$. There also exist $\mathbf{a}, \mathbf{c} \in R$ such that $\mathbf{a}_{W_i} = \mathbf{a}_i$ for all $1 \leq i < m$, and $\mathbf{c}_{W_i} = \mathbf{a}_i$ for all $m \leq i \leq k$. By rearranging the coordinate positions, the tuples $\mathbf{a}|_U$, $\mathbf{b}|_U$, $\mathbf{c}|_U$ can be viewed as consisting of 4 parts: the first one, U_1, includes those coordinate positions in which all the tuples have 0 or 1, this part is a subset of $W_1 \cup \ldots \cup W_{m-1}$ and is nonempty; the second part U_2 equals to $(W_1 \cup \ldots \cup W_{m-1}) - U_1$, and consists of those positions in which $\mathbf{a}, \mathbf{b}$ have $0, 1$ while $\mathbf{c}$ has 2; the third part, $U_3 \subseteq W_m \cup \ldots \cup W_l$, contains those positions in which $\mathbf{a}$ is $0, 1$; finally, the last part, U_4, contains the remaining coordinate positions, and in each such a position $\mathbf{a}, \mathbf{b}$ equal 2.

So, we may represent the tuples in the form $\begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \mathbf{a}^3 \\ \overline{2} \end{pmatrix}$, $\begin{pmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \overline{2} \\ \overline{2} \end{pmatrix}$, and $\begin{pmatrix} \mathbf{c}^1 \\ \overline{2} \\ \mathbf{c}^3 \\ \mathbf{c}^4 \end{pmatrix}$ respectively where $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3 \mathbf{b}^1, \mathbf{b}^2, \mathbf{c}^1, \mathbf{c}^3, \mathbf{c}^4$ consist of 0s and 1s. Then set $\mathbf{d} = f(\mathbf{c}, \mathbf{b}) = \begin{pmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \mathbf{c}^3 \\ \mathbf{c}^4 \end{pmatrix}$, and $\mathbf{a}' = f(\mathbf{c}, \mathbf{a}) = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \mathbf{a}^3 \\ \mathbf{c}^4 \end{pmatrix}$. Finally, we have

$$\mathbf{e} = g(\mathbf{a}', \mathbf{b}, \mathbf{d}) = g\left(\begin{pmatrix} ba^1 \\ \mathbf{a}^2 \\ \mathbf{a}^3 \\ \mathbf{c}^4 \end{pmatrix}, \begin{pmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \overline{2} \\ \overline{2} \end{pmatrix}, \begin{pmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \mathbf{c}^3 \\ \mathbf{c}^4 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \mathbf{c}^3 \\ \mathbf{c}^4 \end{pmatrix}.$$

The tuple $\mathbf{e}$ satisfies the condition $\mathbf{e}|_{W_i} = \mathbf{a}_i$ whenever $i \in J$ as required. □

Lemma 8 *Let $\mathbb{A}$ have a term operation f which is one of the operations (4),(5),(6),(9). Then*
– *$\mathbb{A}$ satisfies $(2-0)$-replacement property if f is (4);*
– *$\mathbb{A}$ satisfies $(2-1)$-replacement property if f is the operation (5);*
– *$\mathbb{A}$ satisfies the $\{1, 2\}$-extendibility property if f is (6);*
– *$\mathbb{A}$ satisfies the $\{0, 2\}$-extendibility property if f is (9).*

Proof.
(4) Let f is the operation (4). Take an (n-ary) relation $R \in \mathsf{Inv}\ F$. We prove that, for any $\mathbf{a} \in R$, and any $k \leq n$, there is $\mathbf{a}_k \in R$ such that

$$\mathbf{a}_k[i] = \begin{cases} 1, & \text{if } \mathbf{a}_k[i] = 1; \\ 0, & \text{if } \mathbf{a}_k[i] \in \{0, 2\},\ i \leq k, \text{ and } R_i = \{0, 1, 2\}; \\ 2, & \text{if } i \leq k, \text{ and } R_i = \{0, 2\}; \\ 2, & \text{if } 1 \notin R_i \text{ and } \mathbf{a}[i] = 2; \\ 0 \text{ or } 2, & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{a}_n$ is the tuple required in the $(2-0)$-replacement property.

Since $\mathbf{a}_0$ can be set to be $\mathbf{a}$, we have the base case of induction. Further, suppose that there is $\mathbf{a}_k \in R$ with the required properties.

CASE 1. $\mathbf{a}_k[k+1] = 1$, or $\mathbf{a}[k+1] = 2$ and $1 \notin R_{k+1}$, or $\mathbf{a}_k[k+1] = 0$ and $R_{k+1} \in \{\{0, 1, 2\}, \{0, 1\}, \{0\}\}$.
In this case, set $\mathbf{a}_{k+1} = \mathbf{a}_k$.

CASE 2. $\mathbf{a}_k[k+1] = 0$ and $R_i = \{0,2\}$.
There is $\mathbf{b} \in R$ with $\mathbf{b}[k+1] = 2$. By the induction hypothesis there is $\mathbf{b}_k \in R$. It can be straightforwardly verified that the tuple $\mathbf{a}_{k+1} = f(\mathbf{b}_k, \mathbf{a}_k)$ satisfies the required conditions.

CASE 3. $\mathbf{a}_k[k+1] = 2$ and $1 \in R_{k+1}$.
This case is very similar to the previous one, but $\mathbf{b}$ is to be chosen such that $\mathbf{b}[k+1] = 1$.

(6) Let f denote the operation (6). We prove that $\mathbb{A}$ satisfies the $\{1,2\}$-extendibility property. To this end, notice first that the operation $g(x,y,z) = f(f(x, f(y,z)), f(f(x,y),z))$ is the majority operation on $\{1,2\}$. Moreover, $f(f(x,y),y)$ is the operation (3); and the required result follows from Lemma 7.

The arguments for the operations (5) and (9) are quite similar. □

Recall that we denote θ the equivalence relation whose classes are $\{0,1\}$ and $\{2\}$.

Lemma 9 *(a) If a simple algebra $\mathbb{A}$ has a term operation which is one of the operations (7),(10),(12),(13),(14),(15), then it has a binary term operation destroying θ.*

(b) If $\mathbb{A}$ is simple and has a term operation which is one of the operations (8),(11),(16) then $\mathbb{A}$ has a binary term operation destroying θ, or an operation $g(x,y,z)$ such that

each of the operations $g(x,y,2), g(x,2,y), g(2,x,y)$ on $\{0,1\}$ either preserves the set $\{0,1\}$, or is the constant operation 2, or has the operation table

	0	1
0	0	2
1	2	1

;

each of the operations $g(x,2,2), g(2,x,2), g(2,2,x)$ on $\{0,1\}$ either preserves the set $\{0,1\}$, or is the constant operation 2.

Proof. Since $\mathbb{A}$ is simple there is an operation $g(x_1,\dots,x_n)$ and $\mathbf{a},\mathbf{b} \in A^n$ such $(\mathbf{a}[i],\mathbf{b}[i]) \in \theta$, but $(g(\mathbf{a}[1],\dots,\mathbf{a}[n]), g(\mathbf{b}[1],\dots,\mathbf{b}[n])) \notin \theta$. It is not hard to see that g, $\mathbf{a},\mathbf{b}$ can be chosen such that $\mathbf{a},\mathbf{b}$ differ only in one coordinate position. Without loss of generality, let $\mathbf{a}[1] \neq \mathbf{b}[1]$, $\mathbf{a}[2] = \mathbf{b}[2] = \dots = \mathbf{a}[p] = \mathbf{b}[p] = 0$, $\mathbf{a}[p+1] = \mathbf{b}[p+1] = \dots = \mathbf{a}[r] = \mathbf{b}[r] = 1$, $\mathbf{a}[r+1] = \mathbf{b}[r+1] = \dots = \mathbf{a}[n] = \mathbf{b}[n] = 2$. Then the operation

$$h(x,y,z,t) = f(x, \underbrace{y,\dots,y}_{p-1 \text{ times}}, \underbrace{z,\dots,z}_{r-p \text{ times}}, \underbrace{t,\dots,t}_{n-r \text{ times}})$$

also destroys θ.

Notice that if $(h(0,0,0,2), h(1,0,1,2)) \notin \theta$ then the operation $h(x,y,x,z)$ destroys θ. Otherwise, since $(h(1,0,1,2), h(0,0,1,2)) \notin \theta$, we have $(h(0,0,0,2), h(0,0,1,2)) \notin \theta$, and the operation $h(y,y,x,z)$ destroys θ. Thus, in each case there is an operation $g(x,y,z)$ such that $(g(0,1,2), g(1,1,2)) \notin \theta$.

Consider the operation $g'(x,y) = g(x,y,2)$. Since $(g'(0,1), g'(1,1)) \notin \theta$, we have 8 cases. Let f denote one of the operations listed in Lemma 9.

CASE 1.

g'	0	1
0	0/1	
1		2

, or

g'	0	1
0	2	
1		0/1

.

In this case $g(x,x,y)$ destroys θ, and we have a binary operation with this property.

CASE 2.

g'	0	1
0	0	2
1		0

.

For the operation $h(x,y) = g(g(x,x,y),x,y)$ we have $h(0,2) = 0$, $h(1,2) = 2$. We get a binary operation destroying θ.

CASE 3.

g'	0	1
0	1	2
1		1

.

For the operation $h(x,y) = g(x, g(x,x,y), y)$, we have $h(0,2) = 2$, $h(1,2) = 1$. We again get a binary operation destroying θ.

CASE 4.

g'	0	1
0	0/1	2
1	0/1	0/1

.

If f is one of (8),(10),(11),(12),(14),(16) then $f(0,2) \in \{0,1\}$, $f(1,2) = 0$, and, for the operation $h(x,y) = g(f(x,y),x,y)$, we have $h(1,2) = g(0,1,2) = 2$, and $h(0,2) = g(0/1,0,2) \in \{0,1\}$, i.e. h destroys θ. If f is one of (7),(13),(15) then $f(0,2) = f(1,2) = 1$, for the operation $h(x,y) = f(x, f(x,y), y)$, we have $h(0,2) = g(0,1,2) = 2$, and $h(1,2) = g(1,1,2) \in \{0,1\}$, i.e. again h destroys θ.

CASE 5.

g'	0	1
0	2	0/1
1	2	2

.

This case is quite analogous to the previous one.

CASE 6.

g'	0	1
0	2	0/1
1	0/1	2

.

If f is one of (10),(12),(14) then $f(0,2) = f(1,2) = 0$, and, for the operation $h(x,y) = g(f(x,y),x,y)$, we have $h(0,2) = g(0,0,2) = 2$, $h(1,2) = g(0,1,2) \in \{0,1\}$, and h destroys θ. Analogously, if f is one of (7),(13),(15), and so $f(0,2) = f(1,2) = 1$, then for h obtained in the same way we have $h(0,2) \in \{0,1\}$, $h(1,2) = 2$, i.e. h destroys θ. Finally, if $f(0,2) = 1$, $f(1,2) = 0$ then by substituting $g(f(x,z),y,z)$ we obtain Case 8.

CASE 7.

g'	0	1
0	1	2
1	2	0

The operation $g(g(x,x,z),g(y,y,z),z)$ satisfies the conditions of the Case 8.

CASE 8.

g'	0	1
0	0	2
1	2	1

If $f(0,2) = f(1,2) = 0$, that is f is one of the operations (10),(12), (14), then the operation $h(x,y) = g(f(x,y),x,y)$ satisfies the conditions $h(0,2) = g(0,0,2) = 0$, $h(1,2) = g(0,1,2) = 2$, and therefore, destroys θ. If $f(0,2) = f(1,2) = 1$, that is f is one of the operations (7),(13),(15), then h satisfies the conditions $h(0,2) = g(1,0,2) = 2$, $h(1,2) = g(1,1,2) = 1$.

Finally, if $f(0,2) = 1$, $f(1,2) = 0$, that is f is one of the operations (8),(11),(16), then each of the operations $g(2,y,z), g(x,2,z)$ either satisfies the same conditions as $g(x,y,2)$, or preserves $\{0,1\}$, or is the constant operation 2, or a binary operation destroying θ can be derived. Analogously, each of the operations $g(2,2,x), g(2,x,2), g(x,2,2)$ either preserves $\{0,1\}$, or is the constant operation 2, or a binary operation destroying θ can be derived. The lemma is proved. □

Lemma 10 *If $\mathbb{A}$ has a binary operation destroying the equivalence relation θ and one of the operations (7),(8),(10),(11),(12),(13),(14),(15),(16) then either 2 is a zero-element with respect to some binary term operation of $\mathbb{A}$, or $\mathbb{A}$ has a binary conservative commutative term operation, or a zero-operation, or a $(2-a)$-operation, $a \in \{0,1\}$, or a $(1-2)$-operation and $\{0,2\}$ is a subuniverse of $\mathbb{A}$.*

Proof.

Let g be the term operation destroying θ, $(g(0,2), g(1,2)) \notin \theta$, and f denote one of the operations listed.

Suppose, first, $g(0,2) = 0, g(1,2) = 2$, and consider 4 cases.

CASE A.

g	0	1	2
0	0	1	0
1	0	1	2
2			2

SUBCASE A.1. $g(2,0), g(2,1) \in \{0,1\}$.

The operation $g(x, g(x,y))$ is either a zero-operation or a $(2-0)$-operation.

SUBCASE A.2. $g(2,0) = 0$, $g(2,1) = 2$. In this case, g is a zero-operation itself.

SUBCASE A.3. $g(2,0) = 2$, $g(2,1) = 1$.

If f is one of the operations (7),(8),(10),(11),(12),(13), that is $f|_{\{0,1\}}$ is a semilattice, then $f(g(x,y),x)$ satisfies the conditions of Case C or Case D. Otherwise, by substituting $g(x, f(y,x))$ we get Subcase A.4 or Subcase A.1.

Subcase A.4. All other cases.
The operation $g(x, g(x, y))$ is a zero operation.

Case B.

g	0	1	2
0	0	0	0
1	1	1	2
2			2

.

Subcase B.1. $0 \in \{g(2,0), g(2,1)\}$. In this case g is a zero-operation.

Subcase B.2. $g(2,0) = g(2,1) = 1$ or $g(2,0) = 1, g(2,1) = 2$.
If $g(2,0) = g(2,1) = 1$ then $g(x, g(y, x))$ is a $(2-1)$-operation. Then, if $g(2,0) = 1, g(2,1) = 2$ then by substituting $g(x, g(x, y))$ we get Subcase B.4.

Subcase B.3. $g(2,0) = 2, g(2,1) = 1$.
In this case, if $f(0,2) = f(1,2)$, that is f is one of (7),(10),(12),(13),(14),(15), then $g(f(x, y), y)$ is a zero-operation. Otherwise, the operation $g(x, f(y, x))$ satisfies the conditions of Subcase B.2.

Subcase B.4. $g(2,0) = g(2,1) = 2$.
Since g is a $(1-2)$-operation, if $\{0,2\}$ is a subuniverse then we are done. Otherwise, by Lemma 6, one of the operations (1),(5),(7),(8),(9),(11),(13),(15) is a term operation of $\mathbb{A}$. Therefore, either, the conditions of Lemma 8 or Lemma 7 hold and there is a term zero-operation of $\mathbb{A}$ or $\mathbb{A}$ satisfies one of the following conditions: $\{0,1\}$- or $\{0,2\}$-extendibility, $(2-1)$-replacement property, $\{0,1\}$-rectangularity, or one of the operations (7),(8),(11),(13),(15),(16) is a term operation of $\mathbb{A}$. Denote this term operation by h. If h is one of (7),(13),(15), i.e. $h(0,2) = h(1,2) = 1$, then 2 is a zero-element with respect to the operation $g(h(x, y), y)$. If h is one of (8),(11),(16) then $g(h(x, y), y)$ satisfies the conditions of one of Cases E–H.

Case C.

g	0	1	2
0	0	0	0
1	0	1	2
2			2

.

If $g(2,0) \neq 2$ or $g(2,1) \neq 2$ then g is a zero-operation. So, suppose that $g(2,0) = g(2,1) = 2$. Since g is the operation (2), if $\{0,2\}$ is a subuniverse, then by Lemma 7, either a zero-operation is a term operation of $\mathbb{A}$, or $\mathbb{A}$ is $\{0,2\}$-rectangular, or $\mathbb{A}$ satisfies the $\{0,2\}$-extendibility property.

If $\{0,2\}$ is not a subuniverse then, as in Subcase B.4, either the conditions of Lemma 8 or Lemma 7 hold, or one of (7),(8),(11),(13),(15),(16) is a term operation of $\mathbb{A}$; denote this operation by h. If h is one of (7),(13),(15) then 2 is a zero-element with respect to the operation $g(h(x, y), y)$. If f is one of (8),(11),(16) then $g(h(x, y), y)$ satisfies the conditions of one of Cases E–H.

Case D.

g	0	1	2
0	0	1	0
1	1	1	2
2			2

.

If $g(2,1) = 1$ then g is a zero-operation. If $g(2,1) = 0$ then $g(g(x, y), y)$ is a zero-operation. Finally, let $g(2,1) = 2$. Then if $g(2,0) = 2$ then g is a zero operation, and if $g(2,0) = 1$ then $g(x, g(x,y))$ is a zero-operation. If $g(2,0) = 0$ then g is a conservative commutative operation.

CASE E.

g	0	1	2
0	0	1	2
1	0	1	0
2			2

In this case, either g itself, or the operation $g(x, g(x,y))$ is a zero-operation or a $(2-0)$-operation.

CASE F.

g	0	1	2
0	0	0	2
1	1	1	0
2			2

SUBCASE F.1. $g(2,0) = 0, g(2,1) = 0$.

Set $h(x,y) = g(x, g(x,y))$; its operation table is

h	0	1	2
0	0	0	2
1	1	1	1
2	0	0	2

. Then the $h(x, h(y,x))$ is a $(2-0)$-operation.

SUBCASE F.2. $g(2,1) = 2,\ g(2,0) \in \{1,2\}$.

The operation $g(g(x,y),y)$ is a zero-operation.

SUBCASE F.3. All other cases.

In this case, for the operation $h(x,y) = g(x, g(x,y))$ we have $1 \in \{h(2,0), h(2,1)\}$, and $h(2,1) = 1$. Therefore, h is a zero operation.

CASE G.

g	0	1	2
0	0	0	2
1	0	1	0
2			2

SUBCASE G.1. $g(2,0) = 0$. The operation g is a zero-operation.

SUBCASE G.2. $g(2,0) = 1$.

If $g(2,1) \in \{0,2\}$ then $g(x, g(x,y))$ is a zero-operation. If $g(2,1) = 1$ then $g(g(y,x),y)$ is a zero-operation.

SUBCASE G.3. $g(2,0) = 2$.

If $g(2,1) = 2$ then 2 is a zero-element with respect to $g(g(x,y),y)$. If $g(2,1) = 0$ then, for the operation $h(x,y) = g(x, g(x,y))$, we have $h(2,1) = 2$, and we get the previous case. Finally, if $g(2,1) = 1$ then $g(g(y,x),y)$ falls to the previous cases.

CASE H.

g	0	1	2
0	0	1	2
1	1	1	0
2			2

If $g(2,0) = 2, g(2,1) = 0$ then g is the operation (6), and, by Lemma 8, $\mathbb{A}$ satisfies the $\{1,2\}$-extendibility property. If $g(2,0) = g(2,1) = 2$ then g is a zero-operation, and if $g(2,1) = 2, g(2,0) \in \{0,1\}$ then $g(g(x,y),y)$ is a zero-operation. Further, if $g(2,0) = g(2,1) = 0$ then $g(x, g(y,x))$ is a zero-operation. In all other cases $g(x, g(x, y))$ is a zero-operation.

The proof in the case when $1 \in \{g(0, 2), g(1, 2)\}$ is quite similar. □

Lemma 11 *If $\mathbb{A}$ satisfies the conditions of Lemma 9 and has no binary operation destroying θ, then either a zero-operation, or* $(2-0)$*- or* $(2-1)$*-operation, or the operation (1) is a term operation of* $\mathbb{A}$.

Proof. Let f denote the operation (16), and g the ternary operation satisfying the conditions of Lemma 9. Consider the operation $g'(x, y) = g(x, x, y)$; its operation table is

g'	0	1	2
0	0		0
1		1	1
2			2

. The restriction of g' on $\{0, 1\}$ is either a projection or a semilattice operation. If $g'(2, 0) = g'(2, 1) \in \{0, 1\}$ and $g'|_{\{0,1\}}$ is the first projection, then g' is either $(2-0)$- or $(2-1)$-operation. If $g'|_{\{0,1\}}$ is the second projection, $g'(y, g'(y, x))$ is a zero operation. In the case when $g'|_{\{0,1\}}$ is a semilattice operation, g' is a zero-operation. Further, if $g'(2, 0) = 0, g'(2, 1) = 1$ then either g' itself or $g'(y, x)$ is the operation (1), or g' is a zero-operation. In the case $g'(2, 0) = 1, g'(2, 1) = 0$, the operation $g'(x, g(x, y))$ is either a zero-operation, or the operation (1), or the operation (1) with permuted variables. The only case remaining to consider is $g'(2, 0) = g'(2, 1) = 2$.

If the restriction of g' onto $\{0, 1\}$ is a semilattice operation, then $f(g(x, y), y)$ is one of the operations (8),(11). By Lemma 9, there exists a binary term operation of $\mathbb{A}$ that destroys the equivalence θ, a contradiction with the conditions of Lemma 11. If $g'|_{\{0,1\}}$ is the second projection then consider the operation $h(x, y, z) = g(x, y, g(z, z, x))$. It is not hard to check that h satisfied the conditions applied to g in Lemma 11, but the operation table of $h(x, x, y)$ is

	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

. Finally, if $g'|_{\{0,1\}}$ is the first projection then the operation table of $h(x, y) = g(x, f(x, y), y)$ is

	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

, and therefore, h is a zero-operation.

□

5.4 Conservative algebras

An algebra is called *conservative* if every its subset is a subuniverse.

If $\mathbb{B}$ is a 2-element subalgebra of $\mathbb{A}$ then $\mathbb{B}$ is tractable, and therefore, by Schaefer's theorem, there is a term operation f of $\mathbb{A}$ such that $f|_B$ is either a semilattice, or majority, or affine operation. We have 5 cases depending on operations of which kind provide the tractability of 2-element subalgebras.

5.4.1 $\mathbb{A}$ has a term operation f whose restriction on a 2-element subuniverse is a semilattice operation.

Suppose that f is a semilattice operation on $B = \{0, 1\}$, and $f(0, 1) = f(1, 0) = 1$. Then, by Lemma 5(1), the restriction of f onto any other 2-element subuniverse is either a semilattice operation, or a projection. Moreover, by Lemma 5(2) replacing f with $f(x, f(y, x))$, the projections may be assumed to be first projections. We consider 3 subcases.

CASE 1. f is a semilattice operation on all three 2-element subuniverses. In this case, f is a commutative conservative binary operation, and, by [4], $\mathbb{A}$ is of width 3.

CASE 2. f is a semilattice operation on two of the 2-element subuniverses, and it is the first projection on the third one.

SUBCASE 2.1. $f|_{\{0,2\}}$ is a semilattice operation.

Then the Cayley table of f is one of the following

f	0	1	2
0	0	1	0
1	1	1	1
2	0	2	2

,

f	0	1	2
0	0	1	2
1	1	1	1
2	2	2	2

.

In the first case, f is a zero-operation, and therefore, by Lemma 3, $\mathbb{A}$ satisfies the partial zero property. In the second case, f is the operation (2), hence, by Lemma 7, $\mathbb{A}$ satisfies the $\{1, 2\}$-rectangularity property.

SUBCASE 2.2. $f|_{\{1,2\}}$ is a semilattice operation.

Then the Cayley table of f is one of the following

f	0	1	2
0	0	1	0
1	1	1	1
2	2	1	2

,

.	0	1	2
0	0	1	0
1	1	1	2
2	2	2	2

In the first case, 1 is a zero-element with respect to f. In second case f is a zero-operation. Therefore, in both cases $\mathbb{A}$ satisfies the partial zero property.

CASE 3. f is a semilattice operation on one of the 2-element subuniverses, and is the first projection on the remaining ones.

In this case, the Cayley table of f is

	0	1	2
0	0	1	0
1	1	1	1
2	2	2	2

, and f is a $(0-1)$-operation . By Lemma 2, $\mathbb{A}$ satisfies the $(0-1)$-replacement property.

Before considering the remaining cases, we observe some properties of ternary term operations of $\mathbb{A}$ in the case when no 2-element subalgebras of $\mathbb{A}$ has a semilattice term operation. An operation $f(x, y, z)$ is said to be *minority* if the identities $f(x, x, y) = f(x, y, x) = f(y, x, x) = y$ hold. There is only one minority operation on a 2-element set, the affine operation $x - y + z$.

Lemma 12 *Let $\mathbb{A}$ be such that the restriction of every its binary term operation onto every 2-element subuniverse is a projection. Then there exists a term operation $h(x, y, z)$ of $\mathbb{A}$ such that, for any 2-element subuniverse B of $\mathbb{A}$, the restriction $h|_B$ is the majority operation if there is a term operation f of $\mathbb{A}$ such that $f|_B$ is the majority operation, and $h|_B$ is the minority operation otherwise.*

Proof. Let $B \subseteq A$, $|B| = 2$, $f' = f|_B$, and $f_1(x, y) = f'(x, y, y)$, $f_2(x, y) = f'(y, x, y)$, $f_3(x, y) = f'(y, y, x)$. If one of these operations is not a projection, then it must be a semilattice operation, a contradiction to the assumptions done.

Claim 1. If $f_i(x, y) = y$ for some i and f' is not a projection, then there is a term operation g of $\mathbb{A}$ such that $g|_B$ is the majority operation.
Let $f_1(x, y) = y$. Then there are 3 possibilities. (a) f' is a majority operation and we are done; (b) $f_2(x, y) = x, f_3(x, y) = y$ or $f_2(x, y) = y, f_3(x, y) = x$, and f' is a projection, which is impossible; (c) $f_2 = f_3 = x$. In this case set $g(x, y, z) = f(f(x, y, z), y, z)$; we have

$$\begin{aligned} g|_B(x, y, y) &= f'(f'(x, y, y), y, y) = f'(y, y, y) = y, \\ g|_B(y, x, y) &= f'(f'(y, x, y), x, y) = f'(x, x, y) = y, \\ g|_B(y, y, x) &= f'(f'(y, y, x), y, x) = f'(x, y, x) = y. \end{aligned}$$

Denote by B_1, B_2, B_3 the sets $\{0, 1\}, \{0, 2\}, \{1, 2\}$ respectively. By Claim 1, there are ternary term operations g_1, g_2, g_3 of $\mathbb{A}$ such that $g_i|_{B_i}$ is the majority operation if there are a term operation f of $\mathbb{A}$ such that $f|_{B_i}$ is the majority operation, $g_i|_{B_i}$ is the minority otherwise, and $g_i|_{B_j}$ is either the majority operation, or the minority operations, or the first projection for $j \neq i$.

Claim 2. g_1, g_2, g_3 can be chosen such that their restrictions on each 2-element subset is either the majority or the minority operation.
To prove the claim it is enough to notice that, for any $i, j \in \{1, 2, 3\}$, the operation

$$g_{ij}(x, y, z) = g_j(g_i(x, y, z), g_i(y, z, x), g_i(z, x, y))$$

is either the majority or minority operation on B_i, B_j, and $g_{ij}|_{B_i} = g_i|_{B_i}$. Then we may replace g_1, g_2, g_3 with

$$g_{23}(g_{12}(x, y, z), g_{12}(y, z, x), g_{12}(z, x, y)) \qquad (1)$$

$$g_{31}(g_{23}(x,y,z), g_{23}(y,z,x), g_{23}(z,x,y)) \quad (2)$$
$$g_{12}(g_{31}(x,y,z), g_{31}(y,z,x), g_{31}(z,x,y)) \quad (3)$$

Finally, if all g_1, g_2, g_3 are minority operations on each 2-element subset, then any of them suits as h. Suppose that $g_1|_{B_1}, g_2|_{B_2}$ are majority operations, but $g_1|_{B_2}$ is the minority operation. Then for $g'(x,y) = g_1(x,x,y)$, we have $g'|_{B_1}(x,y) = x, g'|_{B_2}(x,y) = y$; and the operation $g''(x,y,z) = g'(g_1(x,y,z), g_2(x,y,z))$ is the majority operation on both B_1, B_2. If $g''|_{B_3} \neq g_3|_{B_3}$ we repeat this procedure for B_1, B_3. □

5.4.2 All three 2-element subalgebras have the majority term operation, but no semilattice term operation.

By Lemma 12, there is a term operation f which is the majority operation on each 2-element subset. This means that f is a majority operation on A, and therefore, $\mathbb{A}$ has a majority term operation.

5.4.3 All three 2-element subalgebras have the minority term operation, but no semilattice or majority operation.

By the analogous reason, $\mathbb{A}$ has a Mal'tsev term operation.

5.4.4 Two of the 2-element subalgebras have the majority term operation, but no semilattice operation; and the third subalgebra has the minority term operation, but neither semilattice nor majority operation.

By Lemma 12, there is a term operation f of $\mathbb{A}$ such that $f|_{B_1}$ is the minority operation, and $f|_{B_2}, f|_{B_3}$ are majority operations. Then the operation $g(x,y) = f(x,y,y)$ is the first projection on B_1, and the second projection on B_2, B_3.

Lemma 13 *The algebra $\mathbb{A}$ satisfies the B_2-semisplitting property.*

Proof. For $I, J \subseteq \underline{n}$ $I \cap J = \varnothing$, and $\mathbf{a} \in R_I, \mathbf{b} \in R_J$, we write $(\mathbf{a}, \mathbf{b})$ for the $|I| + |J|$-tuple $\mathbf{c} = (c_I)_{i \in I \cup J}$ with $c_i = a_i$ if $i \in I$, and $c_i = b_i$ if $i \in J$.

Let $R \in \mathsf{Inv}\, F$ be an irreducible (n-ary) relation, $W = \{i \in \underline{n} \mid R_i = A\}$, and $W_j = \{i \in \underline{n} \mid R_i = B_i\}$, $j = 1, 2, 3$. We have to prove that

$$\varnothing \neq (R_W \cap B_2^{|W|}) \times R_{W_2 \cup W_3} \times R_{W_1} \subseteq R.$$

Claim 1. For any $\mathbf{a} \in R_{W_1}$, $i \in W \cup W_2 \cup W_3$, and any $a \in R_i$, the tuple $(\mathbf{a}, a)$ belongs to $R_{W_1 \cup \{i\}}$.

There is b such that $(\mathbf{a}, b) \in R_{W_1 \cup \{i\}}$, and $\mathbf{b} \in R_{W_1}$ such that $(\mathbf{b}, a) \in R_{W_1 \cup \{i\}}$. If $b \in \{0,1\}$ then take a tuple of the form $(\mathbf{c}, 2) \in R_{W_1 \cup \{i\}}$. The tuple $\begin{pmatrix} \mathbf{a} \\ 2 \end{pmatrix} = g\left(\begin{pmatrix} \mathbf{a} \\ b \end{pmatrix}, \begin{pmatrix} \mathbf{c} \\ 2 \end{pmatrix}\right)$ belongs to $R_{W_1 \cup \{i\}}$. If $a \in \{0,1\}$ then $\begin{pmatrix} \mathbf{a} \\ a \end{pmatrix} = g\left(\begin{pmatrix} \mathbf{a} \\ 2 \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ a \end{pmatrix}\right)$.

CLAIM 2. For any $i, j \in W$, any $(a, \mathbf{a}) \in R_{W_1 \cup \{i\}}$ with $a \in B_2$, there is $b \in B_2$ such that $(\mathbf{a}, a, b) \in R_{W_1 \cup \{i,j\}}$.

Let us denote $R_{W_1 \cup \{i,j\}}$ by R''. Since $(\mathbf{a}, a) \in R_{W_1 \cup \{i\}}$, there is c such that $(\mathbf{a}, a, c) \in R''$. If $c \in B_2$ then we are done, so, suppose that $c = 1$.

CASE 1. $a = 2$.

By Claim 1, there is $d \in A$ such that $(\mathbf{a}, d, 0) \in R''$. As is easily seen, the tuple $g\left(\begin{pmatrix} \mathbf{a} \\ d \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{a} \\ 2 \\ 1 \end{pmatrix}\right)$ is as required.

CASE 2. $a = 0$.

By Claim 1, there are $c, d \in A$ such that $(\mathbf{a}, c, 0), (\mathbf{a}, d, 2) \in R''$. If $c \in \{0,2\}$ or $d \in \{0,1\}$ then $g\left(\begin{pmatrix} \mathbf{a} \\ c \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{a} \\ 0 \\ 1 \end{pmatrix}\right)$ or $g\left(\begin{pmatrix} \mathbf{a} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \mathbf{a} \\ d \\ 2 \end{pmatrix}\right)$ is as required. Therefore, we may assume that $c = 1, d = 2$.

Since R is irreducible, there is $\mathbf{b} \in R_{W_1}$ such that $(\mathbf{b}, c, d) \in R''$ where $(c, d) \in \{(0/1, 2), (2, 0/1), (0,0), (1,1)\}$. In the first case, $g\left(\begin{pmatrix} \mathbf{a} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ 0/1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{a} \\ 0 \\ 2 \end{pmatrix}$ is as required. In the second case we have

$$g\left(\begin{pmatrix} \mathbf{a} \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ 2 \\ 0/1 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{a} \\ 2 \\ 0/1 \end{pmatrix}; \quad g\left(\begin{pmatrix} \mathbf{a} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{a} \\ 2 \\ 0/1 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{a} \\ 2 \\ 0 \end{pmatrix};$$

$$g\left(\begin{pmatrix} \mathbf{a} \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{a} \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{a} \\ 0 \\ 0 \end{pmatrix},$$

and we get the required tuple. In the third case, $g\left(\begin{pmatrix}\mathbf{a}\\2\\2\end{pmatrix},\begin{pmatrix}\mathbf{b}\\0\\0\end{pmatrix}\right)=$ $\begin{pmatrix}\mathbf{a}\\0\\0\end{pmatrix}$ is as required. Finally, in the last case,

$$g\left(\begin{pmatrix}\mathbf{a}\\2\\2\end{pmatrix},\begin{pmatrix}\mathbf{b}\\1\\1\end{pmatrix}\right)=\begin{pmatrix}\mathbf{a}\\1\\1\end{pmatrix},\text{ and } f\left(\begin{pmatrix}\mathbf{a}\\0\\1\end{pmatrix},\begin{pmatrix}\mathbf{a}\\1\\1\end{pmatrix},\begin{pmatrix}\mathbf{a}\\1\\0\end{pmatrix}\right)=\begin{pmatrix}\mathbf{a}\\0\\0\end{pmatrix},$$

and we again obtain the required tuple.

CLAIM 3. For any $i_1, i_2 \in W \cup W_2 \cup W_3$, any $(a_{i_1}, a_{i_2}) \in R_{\{i_1,i_2\}}$ such that $a_{i_j} \in B_2$ if $i_j \in W \cup W_2$, any $I \subseteq (W \cup W_1 \cup W_2) - \{i_1, i_2\}$, and any $\mathbf{a} \in R_{W_1}$, a tuple $\mathbf{b}$ with

$$\mathbf{b}[i] = \begin{cases} a_{i_j}, & \text{if } i = i_j,\ j = 1,2; \\ \mathbf{a}[i], & \text{if } i \in W_1; \end{cases}$$

and $\mathbf{b}[i] \in B_2$ for $i \in I \cap W$, belongs to $R_{W_1 \cup I \cup \{i_1,i_2\}}$.

The base case of induction, $I = \varnothing$, follows from Claim 2. Indeed, by Claim 2, there is c such that $(\mathbf{a}, a_{i_1}, c) \in R_{W_1 \cup \{i_1,i_2\}}$, and $c \in B_2$ whenever $i_1 \in W \cup W_2$; and there is $\mathbf{c}$ such that $(\mathbf{c}, a_{i_1}, a_{i_2}) \in R_{W_1 \cup \{i_1,i_2\}}$. Then $g\left(\begin{pmatrix}\mathbf{a}\\a_{i_1}\\c\end{pmatrix},\begin{pmatrix}\mathbf{c}\\a_{i_1}\\a_{i_2}\end{pmatrix}\right)$ is the required tuple.

Let $I \neq \varnothing$. Without loss of generality, let $i_1 = 1, i_2 = 2$, and $I = \{3, \dots, k\}$. Suppose that the claim holds for all $I \subseteq W \cup W_2 \cup W_3$ with $|I| \le k-3$. Hence, there is $b \in A$ such that $\mathbf{c} = (a_1, a_2, \dots, a_{k-1}, b, \mathbf{a}) \in R_{\{1,2\} \cup I \cup W_1}$ where $a_3, \dots, a_{k-1}$ satisfy the conditions of the claim. If $k \in W_2 \cup W_3$ or $b \in B_2$ then we are done; so let $k \in W$ and $b = 1$.

CASE 1. $a_1 = 2$.

By Claim 1, $(0, \mathbf{a}) \in R_{\{k\} \cup W_1}$, therefore $\mathbf{d} = (c, b_2, \dots, b_k - 1, 0, \mathbf{a}) \in R_{\{1,2\} \cup I \cup W_1}$ for certain $b_2, \dots, b_{k-1}$ satisfying the conditions of the claim. Set $\mathbf{e} = g(\mathbf{c}, \mathbf{b})$. Since $\{\mathbf{d}[i], \mathbf{c}[i]\} \neq \{0,1\}$ for all $i \in \{2, \dots, k-1\}$, we have $\mathbf{e}[i] = \mathbf{c}[i] = a_i$. Then $\mathbf{e}[1] = 2$ because $a_1 = 2$, and $\mathbf{e}[k] = g(0,1) = 0$.

CASE 2. $a_1 = 0$ or $a_1 = 1$ if $1 \in W_3$.

The tuple $(2, \mathbf{a})$ belongs to $R_{\{1\} \cup W_1}$, and by Case 1 can be extended to $\mathbf{d} = (2, c_2, \dots, c_{k-1}, 0, \mathbf{a}) \in R_{\{1,\dots,k\} \cup W_1}$. The tuple $\mathbf{e} = g(\mathbf{d}, \mathbf{c})$ is as required. Indeed, if $i \in \{2, \dots, k-1\}$ then $\{\mathbf{c}[i], \mathbf{d}[i]\} \neq \{0,1\}$, and therefore, $\mathbf{e}[i] = \mathbf{c}[i] = a_i$; $\mathbf{e}[1] = g(2,0) = 0$ (or $\mathbf{e}[1] = g(2,1) = 1$ if $1 \in W_3$); $\mathbf{e}[k] = g(0,1) = 0$.

CLAIM 4. For any $I \subseteq W \cup W_2 \cup W_3$, any $\mathbf{b} \in R_I$ such that $\mathbf{b}[i] \in B_2$ whenever $i \in W$, and any $\mathbf{a} \in R_{W_1}$ the tuple $(\mathbf{b}, \mathbf{a})$ is in $R_{I \cup W_1}$.

By Claim 3, there are tuples $(\mathbf{b},\mathbf{c}),(\mathbf{d},\mathbf{a}) \in R_{I\cup W_1}$ with $\mathbf{d}[i] \in B_2$ whenever $i \in W$. It is easy to see, that $g\left(\begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix}\right)=\begin{pmatrix}\mathbf{b}\\ \mathbf{a}\end{pmatrix}$.

To finish the proof of Lemma 13, we just should put $I = W \cup W_2 \cup W_3$ in Claim 4. □

By the assumptions made, f is the majority operation on B_2, and therefore, in this case, $\mathbb{A}$ satisfies the conditions of Theorem 6(5).

5.4.5 One of the 2-element subalgebras has the majority term operation, but no semilattice operation; and the two others have the minority term operation, but no semilattice or majority operation.

By Lemma 12, $\mathbb{A}$ has a term operation f which is the majority operation on B_1, and the minority operation on B_2, B_3. Then the operation $g(x,y) = f(x,x,y)$ is the first projection on B_1, and the second projection on B_2, B_3.

Let θ denote the equivalence relation whose classes are $\overline{0} = \{0,1\}$ and $\overline{2} = \{2\}$. The class containing an element $a \in A$ will be denoted by a^θ, and for an n-ary relation R, we set $R^\theta = \{(a_1^\theta,\ldots,a_n^\theta) \mid (a_1,\ldots,a_n) \in R\}$.

Lemma 14 *Let $R \in \mathsf{Inv}$ be a binary relation such that $R_1 = R_2 = A$. Then R is either the identity relation, or A^2, or the graph of the non-identity bijection with the fixed point 2, or $R^\theta \in \{\mu_1,\mu_2\}$ where*

$$\mu_1 = \begin{pmatrix}\overline{0} & \overline{2}\\ \overline{0} & \overline{2}\end{pmatrix},\quad \mu_1 = \begin{pmatrix}\overline{0} & \overline{0}\\ \overline{2} & \overline{2}\end{pmatrix}.$$

Proof. Suppose first that R is the graph of a mapping φ, and $\varphi(2) \neq 2$, say, $\varphi(2) = 0$. Then denoting $b = \varphi^{-1}(1)$, we have $g\left(\begin{pmatrix}b\\ 1\end{pmatrix},\begin{pmatrix}2\\ 0\end{pmatrix}\right) = \begin{pmatrix}2\\ 1\end{pmatrix} \in R$ which contradicts the assumptions made. Thus $\varphi(2) = 2$.

Suppose that R is neither the graph of a bijective mapping, nor $R^\theta \in \{\mu_1,\mu_2\}$. Then, R contains one of the tuples $\begin{pmatrix}0\\ 2\end{pmatrix},\begin{pmatrix}1\\ 2\end{pmatrix},\begin{pmatrix}2\\ 0\end{pmatrix},\begin{pmatrix}2\\ 1\end{pmatrix}$, and either the tuple $\begin{pmatrix}2\\ 2\end{pmatrix}$ or a tuple of the form $\begin{pmatrix}a\\ b\end{pmatrix}$ where $a,b \in \{0,1\}$.

If $\begin{pmatrix}2\\ 0\end{pmatrix} \in R$ then there is $a \in A$ such that $\begin{pmatrix}a\\ 1\end{pmatrix} \in R$; consequently, $\begin{pmatrix}2\\ 1\end{pmatrix} = g\left(\begin{pmatrix}a\\ 1\end{pmatrix},\begin{pmatrix}2\\ 0\end{pmatrix}\right) \in R$. Analogously, if $\begin{pmatrix}2\\ 1\end{pmatrix} \in R$

then $\begin{pmatrix} 2 \\ 0 \end{pmatrix} \in R$, and if one of the tuples $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ belongs to R, then the other also belongs to R. Since $R_1 = A$, we have two cases to consider.

CASE 1. $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \in R$.

Since $R_2 = A$, there is $\begin{pmatrix} a \\ b \end{pmatrix}$ in R where $b \in \{0,1\}$. Then $\begin{pmatrix} 2 \\ b \end{pmatrix} = f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) \in R$, and therefore, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in R$. Moreover, for any $c, d \in A$, we have $\begin{pmatrix} c \\ d \end{pmatrix} = f\left(\begin{pmatrix} c \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ d \end{pmatrix}\right) \in R$; hence, $R = A^2$.

CASE 2. $\begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in R$.

In this case there is $\begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \{0,1\}$, therefore, $\begin{pmatrix} 2 \\ 2 \end{pmatrix} = f\left(\begin{pmatrix} a \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} 2 \\ b \end{pmatrix}\right) \in R$, and we get the previous case. □

We consider two cases.

I. $\mathbb{A}$ is simple.

In this case θ does not belong to $\mathsf{Inv}\ F$, and moreover, any binary relation R, such that $R^\theta \in \{\mu_1, \mu_2\}$, and R is not the graph of a mapping, does not belong to $\mathsf{Inv}\ F$ as well. Indeed, every such relation R is not of the form $\begin{pmatrix} 0 & 1 & 2 & 2 \\ 2 & 2 & 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} a & a & b & 2 \\ c & d & d & 2 \end{pmatrix}$ where $\{a,b\} = \{c,d\} = \{0,1\}$. Then $\theta = R \circ R^{-1} \in \mathsf{Inv}\ F$, that contradicts the assumptions made

Lemma 15 *Let $R \in \mathsf{Inv}$ be an (n-ary) relation such that $R_1 = \ldots = R_n = A$, and $R_{i,j} = A^2$ for any $i, j \in \{1, \ldots, n\}$. Then $R = A^n$.*

Proof. We prove the lemma by induction. To prove the base case $n = 3$, take a ternary relation R from $\mathsf{Inv}\ F$, and, for $a \in A$, denote $R_a = \{(b,c) \in A^2 \mid (a,b,c) \in R\}$. Each R_a satisfies the conditions of Lemma 14, and therefore, is either a graph of a bijection with the fixed point 2, or A^2. (Notice that in both cases $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \in R_a$.) Since $R_0 \cup R_1 \cup R_2 = A^2$, one of R_0, R_1, R_2 is A^2.

CASE 1. $R_2 = A^2$.

For any $a, b, c \in \{0,1\}$, we have $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = f\left(\begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ b \\ c \end{pmatrix}\right) \in R$.

Therefore, $\{0,1\}^2 \subseteq R_a$, and by Lemma 14, $R_a = A^2$. Thus, $R = A^3$.

CASE 2. $R_a = A^2$, $a \in \{0,1\}$.

In this case, for any $b, c \in \{0,1\}$, we have $\begin{pmatrix} 2 \\ b \\ c \end{pmatrix} = f\left(\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \in R$. Therefore, $R_2 = A^2$, and we get the previous case.

To prove the induction step, suppose that the claim of the lemma holds for $n > 2$, and $R \in \mathsf{Inv}$ is an ($(n+1)$-ary) relation. As before, let $R_a = \{(a_2, \ldots, a_{n+1}) \mid (a, a_2, \ldots, a_{n+1}) \in R\}$. By the induction hypothesis, $R_a = A^n$ for any $a \in A$, and therefore, $R = A^{n+1}$. □

Lemma 16 *The algebra $\mathbb{A}$ satisfies the splitting property.*

Proof. Let $R \in \mathsf{Inv}\, F$ be an (n-ary) relation, and $W = \{i \mid R_i = A\}$, $W_j = \{i \mid R_i = B_j\}$, $i = 1,2,3$. We prove that $R = R_W \times R_{W_1} \times R_{W_2 \cup W_3}$, that is an even stronger condition.

Prove first that $R = R_{W \cup W_2 \cup W_3} \times R_{W_1}$. Take $\mathbf{a}' \in R_{W \cup W_2 \cup W_3}$, $\mathbf{b}' \in R_{W_1}$, and $\mathbf{a}, \mathbf{b} \in R$ such that $\mathbf{a}_{W \cup W_2 \cup W_3} = \mathbf{a}'$, $\mathbf{b}_{W_1} = \mathbf{b}'$. By Lemma 15, $R_W = A^{|W|}$, therefore, there is $\mathbf{c} \in R$ such that $\mathbf{c}[i] = 2$ for all $i \in W$. For $\mathbf{d} = g(g(\mathbf{b}, \mathbf{c}), \mathbf{a})$ we have

$\mathbf{d}[i] = g(g(\mathbf{b}[i], \mathbf{c}[i]), \mathbf{a}[i]) = g(g(\mathbf{b}[i], 2), \mathbf{a}[i]) = g(2, \mathbf{a}[i]) = \mathbf{a}[i]$ if $i \in W$;

$\mathbf{d}[i] = g(g(\mathbf{b}[i], \mathbf{c}[i]), \mathbf{a}[i]) = \mathbf{a}[i]$ if $i \in W_2 \cup W_3$;

$\mathbf{d}[i] = g(g(\mathbf{b}[i], \mathbf{c}[i]), \mathbf{a}[i]) = g(\mathbf{b}[i], \mathbf{c}[i]) = \mathbf{b}[i]$ if $i \in W_1$.

Then we prove that $R_{W \cup W_2 \cup W_3} = R_W \times R_{W_2 \cup W_3}$. Let $v \in W$, and $R' = R_{\{v\} \cup W_2 \cup W_3}$. Without loss of generality, we may assume that $v = 1$, $W_2 \cup W_3 = \{2, \ldots, k\}$. Notice first that if $\begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix} \in R'$ then $\begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix} \in R'$, and vice versa. Indeed, there is $\mathbf{b} \in R'_{W_2 \cup W_3}$ such that $\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} \in R'$; and $\begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix} = g\left(\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}\right) \in R'$.

Furthermore, if $(2, \mathbf{a}), (0, \mathbf{a}) \in R'$, for certain $\mathbf{a}$, then $(0, \mathbf{b}), (1, \mathbf{b}), (2, \mathbf{b}) \in R'$, for every $\mathbf{b} \in R_{W_2 \cup W_3}$. This follows from the equalities

$$\begin{aligned}
\begin{pmatrix} 2 \\ \mathbf{b} \end{pmatrix} &= f\left(\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix}, \begin{pmatrix} 2 \\ \mathbf{a} \end{pmatrix}\right) = f\left(\begin{pmatrix} 1 \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix}, \begin{pmatrix} 2 \\ \mathbf{a} \end{pmatrix}\right), \\
\begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} &= f\left(\begin{pmatrix} 2 \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} 2 \\ \mathbf{a} \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix}\right).
\end{aligned}$$

Hence, either $R' = A \times R_{W_2 \cup W_3}$ or $R_{W_2 \cup W_3} = R^1 \cup R^2$ with $R^1 \cap R^2 = \varnothing$, and $R' = (\{2\} \times R^1) \cup (\{0,1\} \times R^2)$.

In the former case, let $P(x, x_2, \ldots, x_k)$ be the predicate corresponding to R'. Then, set

$$P_Q(y, z) = \exists x_2, \ldots, x_k(P(y, x_2, \ldots, x_k) \wedge P(z, x_2, \ldots, x_k)).$$

The relation Q is the equivalence relation with the classes $\{0, 1\}, \{2\}$, that contradicts the simplicity of $\mathbb{A}$.

Finally, for $\mathbf{a} \in R_{W_2 \cup W_3}$, denote $R_{\mathbf{a}} = \{\mathbf{b} \in R_W \mid (\mathbf{b}, \mathbf{a}) \in R_{W \cup W_2 \cup W_3}\}$. By what was proved above $(R_{\mathbf{a}})_i = A$, for any $i \in W$ and $\mathbf{a} \in R_{W_2 \cup W_3}$. Hence, by Lemma 15, $R_{\mathbf{a}} = A^{|W|}$; therefore, $R_{W \cup W_2 \cup W_3} = R_W \times R_{W_2 \cup W_3}$. The lemma is proved. □

II. $\mathbb{A}$ is not simple.

Lemma 17 *The algebra* $\mathbb{A}$ *satisfies the* B_1*-semirectangular property.*

Proof. Let $R \in \mathsf{Inv}$ be an (n-ary) relation, $W = \{i \mid 0, 1 \in R_i\}$, and $W_1, \ldots, W_k$ the classes of $\delta(R)$. It will be convenient for us to denote $\overline{2}$ the tuple consisting of 2s; the length of this tuple will be clear from the context. Suppose that $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_k) \in R$ where $\mathbf{b}_0 \in R_{W'}$, $W' = \underline{n} - W$, $\mathbf{b}_i \in R_{W_i}$, $i \in \underline{k}$, and $\mathbf{b}_i = \overline{2}$ for $i \in \underline{k} - I$ and $\mathbf{b}_i \in R_{W_i} \cap \{0, 1\}^{|W_i|}$ for $i \in I$. We have to prove that, for any $\mathbf{a}_i \in R_{W_i} \cap \{0, 1\}^{|W_i|}$, $i \in I$, the tuple $(\mathbf{b}_0, \mathbf{d}_1, \ldots, \mathbf{d}_k)$ with

$$\mathbf{d}_i = \begin{cases} \mathbf{a}_i & \text{if } i \in I, \\ \overline{2} & \text{otherwise} \end{cases}$$

belongs to R.

We prove by induction that, for any $J = \{i_1, \ldots, i_l\} \subseteq I$, the tuple $\mathbf{c}_J = (\mathbf{b}_0, \mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_l})$ belongs to R_{W_J}, $W_J = W' \cup W_{i_1} \cup \ldots \cup W_{i_l}$. The base case of induction is obvious, $\mathbf{c}_\varnothing = \mathbf{b}$. The next case is $|J| = 1$. It will not be loss of generality if we assume $J = \{1\}$. There is $\mathbf{c}_0 \in R_{W'}$ and $\mathbf{c}_2 \in R_{W_2}$ such that $\mathbf{c} = (\mathbf{c}_0, \mathbf{a}_1, \mathbf{c}_2) \in R_{W' \cup W_2}$. The tuple $g(\mathbf{c}, \mathbf{b}) \in R_{W' \cup W_2}$ has the form $(\mathbf{b}_0, \mathbf{a}_1, \mathbf{c}'_2)$ as required.

Let us suppose now that the we proved what is required for all 1-element sets J. Take J with $|J| = 2$, as usual, it can be supposed to be $\{1, 2\}$. By what was proved, there are $\mathbf{d}_1 \in R_{W_1}$, $\mathbf{d}_2 \in R_{W_2}$ such that $(\mathbf{b}_0, \mathbf{d}_1, \mathbf{a}_2), (\mathbf{b}_0, \mathbf{a}_1, \mathbf{d}_2) \in R_{W' \cup W_1 \cup W_2}$. The tuples $\mathbf{d}_1, \mathbf{d}_2$ can be assumed to be from $\{0, 1\}^{|W_1|}, \{0, 1\}^{|W_2|}$ respectively. Indeed, if $\mathbf{d}_1 = \overline{2}$, then

$$g\left(\begin{pmatrix} \mathbf{b}_0 \\ \overline{2} \\ \mathbf{a}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}\right) = \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{a}_2 \end{pmatrix}.$$

Furthermore, since W_1, W_2 are different classes of $\delta(R)$, there are $\mathbf{c}_0 \in R_{W'}$ and $\mathbf{c}_1 \in R_{W_1}$ (or $\mathbf{c}_2 \in R_{W_2}$) such that $(\mathbf{c}_0, \mathbf{c}_1, \overline{2}) \in R_{W' \cup W_1 \cup W_2}$ $((\mathbf{c}_0, \overline{2}, \mathbf{c}_2) \in$

$R_{W'\cup W_1\cup W_2}$), and $\mathbf{c}_1 \in \{0,1\}^{|W_1|}$ ($\mathbf{c}_2 \in \{0,1\}^{|W_2|}$). The tuple $\mathbf{c}_1$ can be chosen to be $\mathbf{a}_1$. Indeed,

$$\begin{pmatrix} \mathbf{c}_0 \\ \mathbf{a}_1 \\ \overline{2} \end{pmatrix} = g\left(\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{a}_1 \\ \mathbf{d}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \overline{2} \end{pmatrix}\right) \in R_{W'\cup W_1\cup W_2}.$$

Then we have

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = g\left(\begin{pmatrix} \mathbf{c}_0 \\ \mathbf{a}_1 \\ \overline{2} \end{pmatrix}, \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{d}_1 \\ \mathbf{a}_2 \end{pmatrix}\right) \in R_{W'\cup W_1\cup W_2},$$

as required.

Then suppose that the inclusion $\mathbf{c}_J \in R_{W_J}$ is already proved for all J with $|J| < m$, and $K \subseteq \{1,\dots,k\}$ is such that $|K| = m$. Again, without loss of generality, assume that $K = \{1,\dots,m\}$.

By the induction hypothesis, there are tuples of the form $\mathbf{b}^1 = (\mathbf{b}_0, \mathbf{a}_1, \dots, \mathbf{a}_{m-1}, \mathbf{d}^1_m)$, $\mathbf{b}^2 = (\mathbf{b}_0, \mathbf{a}_1, \dots, \mathbf{a}_{m-2}, \mathbf{d}^2_{m-1}, \mathbf{a}_m)$, $\mathbf{b}^3 = (\mathbf{b}_0, \mathbf{a}_1, \dots, \mathbf{a}_{m-3}, \mathbf{d}^3_{m-2}, \mathbf{a}_{m-1}, \mathbf{a}_m)$ in R. The tuples $\mathbf{d}^1_m, \mathbf{d}^2_{m-1}, \mathbf{d}^3_{m-2}$ can be chosen to be distinct from $\overline{2}$. Indeed, if, say, $\mathbf{d}^1_m = \overline{2}$ then, since $\mathbf{a}_i, \mathbf{b}_i \in \{0,1\}^{|W_i|}$ for $i \in \{1,\dots,m-1\}$, $g(\mathbf{b}^1, \mathbf{b})$ is a tuple of the form $(\mathbf{b}_0, \mathbf{a}_1, \dots, \mathbf{a}_{m-1}, \mathbf{b}_m)$ where $\mathbf{b}_m \in \{0,1\}^{|W_m|}$. Finally, we have

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{m-3} \\ \mathbf{a}_{m-2} \\ \mathbf{a}_{m-1} \\ \mathbf{a}_m \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{m-3} \\ \mathbf{a}_{m-2} \\ \mathbf{a}_{m-1} \\ \mathbf{d}^1_m \end{pmatrix}, \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{m-3} \\ \mathbf{a}_{m-2} \\ \mathbf{d}^2_{m-1} \\ \mathbf{a}_m \end{pmatrix}, \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_{m-3} \\ \mathbf{d}^3_{m-2} \\ \mathbf{a}_{m-1} \\ \mathbf{a}_m \end{pmatrix}\right) \in R.$$

Finally, let $\mathbf{c} \in R$ be a tuple with $\mathbf{c}_{W_I} = \mathbf{c}_I$. As is easily seen, the tuple $(\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_k) = g(\mathbf{c}, \mathbf{b})$ satisfies the conditions: $\mathbf{d}_0 = \mathbf{b}_0$, $\mathbf{d}_i = g(\mathbf{a}_i, \mathbf{b}_i) = \mathbf{a}_i$ if $i \in I$, and $\mathbf{d}_i = g(\mathbf{c}_i, \overline{2}) = \overline{2}$ if $i \notin I$. The lemma is proved. □

5.5 Non-simple algebras

In this subsection we assume that $\mathbb{A}$ has a proper congruence. Without loss of generality, let us suppose that the equivalence relation θ, whose classes are $\{0,1\}, \{2\}$, is a congruence of $\mathbb{A}$. Then, for any term operation $f(x_1,\dots,x_n)$, and any $\mathbf{a} \in \{0,1\}^n$, we have

$$\begin{pmatrix} f(0,\dots,0) \\ f(\mathbf{a}[1],\dots,\mathbf{a}[n]) \end{pmatrix} = \begin{pmatrix} 0 \\ f(\mathbf{a}[1],\dots,\mathbf{a}[n]) \end{pmatrix} \in \theta.$$

Therefore, $f(\mathbf{a}[1], \ldots, \mathbf{a}[n]) \in \{0,1\}$, that is $B = \{0,1\}$ is a subuniverse. Denote $\mathbb{B}$ the subalgebra with the universe B. The condition (No-G-Set) implies that there are term operations f, g of $\mathbb{A}$ such that f is a semilattice, majority, or minority operation on B, and g^θ is a semilattice, majority, or minority operation.

5.5.1 $f|_B$ is a semilattice operation.

The operation f^θ is idempotent, hence, it is either a projection, or a semilattice operation. In the latter case we get Case 2. In the former case, we may assume that f^θ is the first projection, that is $f^\theta(x,y) = x$. Consider first the case $f(1,0) = f(0,1) = 1$. We prove that $\mathbb{A}$ or certain its reduct satisfies $(0-1)$-replacement property.

Since f preserve θ, the operation table of $f'(x,y) = f(f(x,y),y)$ is

	0	1	2
0	0	1	0/1
1	1	1	1
2	2	2	2

, and f' is a $(0-1)$-operation. If $\{1,2\}$ is a subuniverse of $\mathbb{A}$, then, by Lemma 2, $\mathbb{A}$ satisfies the $(0-1)$-replacement property. Otherwise, if g^θ is a semilattice operation we get Case 2, so, suppose that g^θ is a minority or a majority operation, and consider the operation

$$g'(x,y,z) = f'(f'(f'(g(x,y,z),x),y),z).$$

If $g(x,y,z) \in \{1,2\}$ then $g'(x,y,z) = g(x,y,z)$. If $g(x,y,z) = 0$ and $x,y,z \in \{1,2\}$, then, as is easily seen, $g'(x,y,z) = 1$. Therefore, $\{1,2\}$ is a subuniverse of the algebra $\mathbb{A}' = (A; f', g')$, and g'^θ is a minority or a majority operation. Hence, $\mathbb{A}'$ satisfies the condition (No-G-Set), and the $(0-1)$-replacement property. The case when $f(0,1) = f(1,0) = 0$ is quite analogous.

5.5.2 g^θ is a semilattice operation.

If $g^\theta(2^\theta, 0^\theta) = g^\theta(0^\theta, 2^\theta) = 2^\theta$ then 2 is a zero-element with respect to g. So suppose that $g^\theta(2^\theta, 0^\theta) = g^\theta(0^\theta, 2^\theta) = 0^\theta$.

Case 1. $g|_{\{0,1\}}$ is a projection.

Without loss of generality we may assume that $g|_{\{0,1\}}$ is the first projection. As can be straightforwardly verified, the operation table of $h(x,y) = g(x, g(x,y))$ is one of the following:

	0	1	2
0	0	0	0
1	1	1	1
2	0	0	2

,

	0	1	2
0	0	0	0
1	1	1	1
2	1	1	2

,

	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

.

In the first 2 cases h is a $(2-0)$- or $(2-1)$-operation; in the third case h is the operation (1). By Lemmas 2,7, $\mathbb{A}$ satisfies one on the conditions listed in Theorem 6.

CASE 2. $g|_{\{0,1\}}$ is a semilattice operation.
Suppose that $g(0,1) = g(1,0) = 1$. Set $h(x,y) = g(x, g(x,y))$, and $h'(x,y) = h(h(x,y),y)$, then, 1 is a zero-element with respect to h'.

In the following 3 cases we assume that there are term operations f, g such that $f|_{\{0,1\}}, g^\theta$ are minority or majority operations.

5.5.3 $\mathbb{A}$ is conservative.

In this case we are in the conditions of Subsection 5.2.

5.5.4 One of $\{0,2\}, \{1,2\}$ is a subuniverse.

Without loss of generality, suppose that $\{0,2\}$ is a subuniverse, but $\{1,2\}$ is not. Then $\mathbb{A}$ has one of the operations listed in Lemma 6. By Lemmas 8,7, some of them lead to a good property of $\mathbb{A}$. Since $\{0,2\}$ is a subuniverse, $\mathbb{B}$ has no semilattice term operation, and θ is a congruence, only the operation (14) remains. This operation is a $(1-0)$-operation, and since $\{0,2\}$ is a subuniverse, by Lemma 2, $\mathbb{A}$ satisfies the $(1-0)$-replacement property.

5.5.5 $\{0,1\}$ is the only subuniverse of $\mathbb{A}$.

As above, by making use of Lemmas 6,8,7, we have to consider only the cases when (16), or both (14), (15) are a term operation of $\mathbb{A}$.

CASE 1. (14),(15) are term operations of $\mathbb{A}$.
Let $r(x,y)$ denote the operation (14). For any term operation $h'(x_1,\dots,x_n)$ of $\mathbb{A}$, the operation

$$\hat{h}(x_1,\dots,x_n) = r(\dots r(h(x_1,\dots,x_n),x_1)\dots x_n)$$

satisfies the conditions: $\hat{h}(x_1,\dots,x_n) = 2$ if and only if $h(x_1,\dots,x_n) = 2$, otherwise if $2 \in \{x_1,\dots,x_n\}$ then $\hat{h}(x_1,\dots,x_n) = 0$, and $\hat{h}(x_1,\dots,x_n) = h(x_1,\dots,x_n)$ whenever $\{x_1,\dots,x_n\} \subseteq \{0,1\}$. As is easily seen, the operation $\hat{f}|_{\{0,1\}}$ ($\hat{g}^\theta$) is a minority or a majority operation if $f|_{\{0,1\}}$ (g^θ) is a minority or a majority operation.

Let us consider the reduct of $\mathbb{A}$, $\mathbb{A}' = (A; \hat{f}, \hat{g})$. The algebra $\mathbb{A}'$ satisfies (NO-G-SET), and $\{0,2\}$ is a subuniverse of $\mathbb{A}'$; We are in the conditions of Case 5.5.4.

CASE 2. (16) is a term operation of $\mathbb{A}$.

Lemma 18 *Let g be a minority operation on $\mathbb{A}/_\theta$, and h the operation (16). Then either $\mathbb{A}$ satisfies the conditions of one of the previous cases, or has a* *term operation g' which is minority on $\mathbb{A}/_\theta$, $g'|_{\{0,1\}} = g|_{\{0,1\}}$, and g' preserves $\{0,2\}, \{1,2\}$.*

Proof. Since, for any $x, y \in \{0,1\}$, $g(2,x,y) = g(x,2,y) = g(x,y,2) = 2$, we just are to show that there is a term operation g' which is a minority operation on $\mathbb{A}/_\theta$, and $g'(2,2,x) = g'(2,x,2) = g'(x,2,2) = x$. Suppose first, that $g(2,2,0) = g(2,2,1) = a \in \{0,1\}$. Then the operation table of $g(y,y,x)$ is

	0	1	2
0	0		a
1		1	a
2	2	2	2

, and we get Case 1. Then suppose that $g(2,2,1) = 0, g(2,2,0) = 1$. In this case, for the operation $g'(x,y,z) = g(x,y,h(z,x))$, we have $g'(x,y,2) = g(x,y,2)$, $g'|_{\{0,1\}} = g|_{\{0,1\}}$, and $g'(2,2,x) = x$. Repeating the same procedure for all three variables we get the required operation. □

Lemma 19 *Let g be a majority operation on $\mathbb{A}/_\theta$, and h the operation (16). Then either $\mathbb{A}$ satisfies the conditions of one of the previous cases, or has a term operation g' which is majority on $\mathbb{A}/_\theta$, $g'|_{\{0,1\}} = g|_{\{0,1\}}$, and g' preserves $\{0,2\}, \{1,2\}$.*

Proof. The proof is quite similar to that of Lemma 18, but the required operation g' must satisfy the conditions $g'(2,x,x) = g'(x,2,x) = g'(x,x,2) = x$. Suppose first, that $g(2,0,0) = g(2,1,1) = a \in \{0,1\}$. Then the operation table of $g(y,x,x)$ is

	0	1	2
0	0		a
1		1	a
2	2	2	2

, and we get Case 1. Then suppose that $g(2,1,1) = 0, g(2,0,0) = 1$. In this case, for the operation $g'(x,y,z) = h(g(x,y,z),x)$, we have $g'(x,y,z) = 2$ if and only if $g(x,y,z) = 2$, that is g'^θ is a majority operation; $g'|_{\{0,1\}} = g|_{\{0,1\}}$, $g'(x,2,x) = g(x,2,x)$, $g'(x,x,2) = g(x,x,2)$, and $g'(2,x,x) = x$. Repeating the same procedure for all pairs of variables we get the required operation. □

Since $g|_{\{0,1\}}$ is an idempotent operation, and the subalgebra of $\mathbb{A}$ with the universe $\{0,1\}$ has no semilattice term operation, $g|_{\{0,1\}}(x,x,y), g|_{\{0,1\}}(x,y,x)$, $g|_{\{0,1\}}(y,x,x) \in \{x,y\}$. If g^θ and $g|_{\{0,1\}}$ are minority (majority) operations, then applying one Lemmas 18 (Lemma 19) to g, we get a minority (majority) operation on A. Noticing that a minority operation is Mal'tsev operation, the algebra $\mathbb{A}$ has Mal'tsev (majority) term operation. Otherwise, an operation h' such that $h'^\theta(x,y) = x$, $h'|_{\{0,1\}}(x,y) = y$ can be derived from g.

Lemma 20 *Let f, g be term operations of $\mathbb{A}$ such that $f|_{\{0,1\}}$, g^θ are majority or minority operations, Then there is a term operation g' of $\mathbb{A}$ such that $g'^\theta = g^\theta$, $g'|_{\{0,1\}} = f|_{\{0,1\}}$.*

Proof. By the observation before the lemma, there is a term operation h' of $\mathbb{A}$ such that $h'^\theta(x,y) = x$, $h'|_{\{0,1\}}(x,y) = y$. Set $g'(x,y,z) = h'(g(x,y,z), f(x,y,z))$. Since $f(x,y,z), g(x,y,z) \in \{0,1\}$, for any $x,y,z \in \{0,1\}$, we have $g'|_{\{0,1\}}(x,y,z) = f|_{\{0,1\}}(x,y,z)$. Further, the equality $h^\theta(x,y) = x$, implies $g'^\theta(x,y,z) = g^\theta(x,y,z)$. The lemma is proved. □

Finally, applying Lemmas 18,19 we get an operation g'' which preserves $\{0,2\}$ and $\{1,2\}$, and such that each of g''^θ, $g''|_{\{0,1\}}$ is a majority or a minority operation. If either both of these operations are minority, or both are majority, then $\mathbb{A}$ has Mal'tsev or a majority term operation. Otherwise, the algebra $\mathbb{A}' = (A; g'')$ is a reduct of $\mathbb{A}$, and is conservative. We are going to show that θ is the only proper congruence of $\mathbb{A}'$. Indeed, as was observed above, an operation $h'(x,y)$ such that $h'^\theta(x,y) = x, h'|_{\{0,1\}}(x,y) = y$ is derivable from g''. Since g'' preserves all the 2-element subsets of A, so does h'; therefore, its operation table is

	0	1	2
0	0	1	0
1	0	1	1
2	2	2	2

. The pair $\begin{pmatrix} h'(0,1) \\ h'(2,1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ witnesses that h' destroys the equivalence relation with classes $\{0,2\},\{1\}$; while the pair $\begin{pmatrix} h'(1,0) \\ h'(2,0) \end{pmatrix}$ witnesses that h' destroys the equivalence relation with classes $\{1,2\},\{0\}$. Thus, $\mathbb{A}'$ satisfies the condition (NO-G-SET), and we get Case 5.5.3.

6 Conclusion

In fact, Theorem 6 implies a stronger result than that claimed in Theorem 4. The difference appears when considering infinite constraint languages satisfying the conditions of Conjecture 1. Theorem 4 claims that, for any finite subset Γ of such a language, there is its own polynomial time algorithm Alg(Γ) solving CSP(Γ), and for different subsets the corresponding algorithms can be quite different. Theorem 6 yields a uniform polynomial time algorithm that solves any problem from the class risen from the constraint language. Moreover, from the proof of Theorem 6 a general algorithm can be derived, which solves any problem instance $\mathcal{P}$ on a 3-element set provided that $\mathcal{P} \in$ CSP(Γ), for some tractable Γ.

As a matter of fact, Theorem 6 is proved by the 'rough force' method, that is by analysing a large number of operations which provide the condition

(No-G-Set). We believe that development of algebraic tools and more subtle usage of results from universal algebra will make it possible to obtain dichotomy results for larger domains, and eventually, for an arbitrary finite domain.

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