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Inapproximability results for bounded variants of optimization problems

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Abstract

We study small degree graph problems such as MAXIMUM INDEPENDENT SET and MINIMUM NODE COVER and improve approximation lower bounds for them and for a number of related problems, like MAX-B-SET PACKING, MIN-B-SET COVER, MAX-MATCHING in B-uniform 2-regular hypergraphs. For example, we prove NP-hardness factor of $\frac{95}{94}$ for MAX-3DM, and factor of $\frac{48}{47}$ for MAX-4DM; in both cases the hardness result applies even to instances with exactly two occurrences of each element.

1 Introduction

This paper deals with combinatorial optimization problems related to bounded variants of MAXIMUM INDEPENDENT SET (MAX-IS) and MINIMUM NODE COVER (MIN-NC) in graphs. We improve approximation lower bounds for small degree variants of them and apply our results to even highly restricted versions of set covering, packing and matching problems, including MAXIMUM-3-DIMENSIONAL-MATCHING (MAX-3DM).

It has been well known that MAX-3DM is MAX SNP-complete (or APX-complete) even when restricted to instances with the number of occurrences of any element bounded by 3. To the best of our knowledge, the first inapproximability result for bounded MAX-3DM with the bound 2 on the number of occurrences of any elements in triples, appeared in our paper [4], where the first explicit approximation lower bound for MAX-3DM problem is given. (For less restricted matching problem, MAX 3-SET PACKING, the similar inapproximability result for instances with 2 occurrences follows directly from hardness

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results for MAX-IS problem on 3-regular graphs [2], [3]). For B-DIMENSIONAL MATCHING problem with $B \geq 4$ the lower bounds on approximability were recently proven by Hazan, Safra and Schwartz [10]. A limitation of their method, as their explicitly state, is that it does not provide an inapproximability factor for 3-DIMENSIONAL MATCHING. But just inapproximability factor for 3-dimensional case is of major interest, as it allows the improvement of hardness of approximation factors for several problems of practical interest, e.g. scheduling problems, some (even highly restricted) cases of Generalized Assignment problem, and other packing problems.

This fact, and an important role of small degree variants of MAX-IS (MIN-NC) problem as intermediate steps in reductions to many other problems of interest, are good reasons for trying to push our technique to its limits. We build our reductions on a restricted version of MAXIMUM LINEAR EQUATIONS over $\mathbb{Z}_2$ with 3 variables per equation and with the (large) constant number of occurrences of each variable. Recall that this method, based on the deep Hastad's version of PCP theorem, was also used to prove $(\frac{117}{116} - \varepsilon)$-approximability lower bound for TRAVELING SALESMAN problem by Papadimitriou and Vempala [12], and for our lower bound of $\frac{96}{95}$ for STEINER TREE problem in graphs [5].

In this paper we optimize our equation gadgets and their coupling via a *consistency amplifier*. The notion of consistency amplifier varies slightly from problem to problem. Generally, they are graphs with suitable expanding (or mixing) properties. Interesting quantities, in which our lower bounds can be expressed, are parameters μ and λ of consistency amplifiers that provably exist.

Let us explain how our inapproximability results for bounded variants of MAX-IS and MIN-NC, namely B-MAX-IS and B-MIN-NC, imply the same bounds for some set packing, set covering and hypergraph matching problems. MAX SET PACKING (resp. MIN SET COVER) is the following: Given a collection $\mathcal{C}$ of sets draw from a finite set S, find a maximum cardinality collection $\mathcal{C}' \subseteq \mathcal{C}$ such that each element in S is contained in at most one (resp., in at least one) set in $\mathcal{C}'$. If each set in $\mathcal{C}$ is of size at most B, we speak about B-SET PACKING (res. B-SET COVER).

It may be phrased also in hypergraph notation; the set of nodes is S and elements of $\mathcal{C}$ are hyperedges. In this notation a set packing is just a matching in the corresponding hypergraph. For a graph $G = (V, E)$ we define its dual hypergraph $\widetilde{G} = (E, \widetilde{V})$ whose node set is just E, $\widetilde{V} = \{\widetilde{v} : v \in V\}$, and for each $v \in V$ hyperedge $\widetilde{v}$ consists of all $e \in E$ such that $v \in e$ in G. Hypergraph $\widetilde{G}$ defined by this duality is clearly 2-regular, each node of $\widetilde{G}$ is contained exactly in two hyperedges. G is of maximum degree B iff $\widetilde{G}$ is of dimension B, in particular G is B-regular iff $\widetilde{G}$ is B-uniform. Independent sets in G are in one-to-one correspondence with matchings in $\widetilde{G}$ (hence with set packings, in set-system notation), and node covers in G with set covers for $\widetilde{G}$. Hence any approximation hardness result for B-MAX-IS translates via this duality to the one for MAX-B-SET PACKING (with exact 2 occurrences), or to MAX MATCHING in 2-regular B-dimensional hypergraph. Similar is the relation of results on B-MIN-NC to MIN-B-SET COVER problem.

If G is B-regular edge B-colored graph, then $\widetilde{G}$ is, moreover, B-partite with balanced B-partition determined by corresponding color classes. Hence independent sets in such graphs correspond to B-dimensional matchings in natural way. Hence any inapproximability result for B-MAX-IS problem restricted to B-regular edge-B-colored graphs translates directly to inapproximability result for MAX-B-DIMENSIONAL MATCHING (MAX-B-DM), even on instances with exact two occurrences of each element.

Our results for MAX-3DM and MAX-4DM nicely complement recent results of [10] on MAX-B-DM given for $B \geq 4$. To compare our results with their for $B = 4$, we have better lower bound ($\frac{48}{47}$ vs. $\frac{54}{53} - \varepsilon$) and our result applies even to highly restricted version with two occurrences. On the other hand, their hard gap result has *almost perfect completeness.*

The main new explicit NP-hardess factors of this contribution are summarized in the following theorem. In more precise parametric way they are expressed in Theorems 3, 5, 6. Better upper estimates on parameters μ_B and λ_B immediately improve lower bounds given bellow.

Theorem. It is NP-hard to approximate:

- MAX-3DM and MAX-4DM to within $\frac{95}{94}$ and $\frac{48}{47}$ respectively, both results apply to instances with exactly two occurrences of each element;
- 3-MAX-IS (even on 3-regular graphs) and MAX TRIANGLE PACKING (even on 4-regular line graphs) to within $\frac{95}{94}$;
- 3-MIN-NC (even on 3-regular graphs) and MIN-3-SET COVER (with exactly two occurrences of each element) to within $\frac{100}{99}$;
- 4-MAX-IS (even on 4-regular graphs) to within $\frac{48}{47}$;
- 4-MIN-NC (even on 4-regular graphs) and MIN-4-SET COVER (with exactly two occurrences) to within $\frac{53}{52}$;
- B-MIN-NC to within $\frac{7}{6} - \frac{12 \log B}{B}$.

Preliminaries

Definition 1 MAX-E3-LIN-2 *is the following optimization problem: Given a system I of linear equation over $\mathbb{Z}_2$, with exactly* 3 *(distinct) variables in each equation. The goal is to maximize, over all assignments φ to the variables, the ratio $\frac{sat(\varphi)}{|I|}$, where $sat(\varphi)$ is the number of equations of I satisfied by φ.*

We use the notation Ek-MAX-E3-LIN-2 for the same maximization problem, where each variable occurs exactly k times. The following theorem follows from Håstad results [9] and the proof can be found in [4]

Theorem 1 *For every $\varepsilon \in \left(0, \frac{1}{4}\right)$ there is a constant $k(\varepsilon)$ such that for every $k \geq k(\varepsilon)$ the following problem is* NP*-hard: given an instance of* Ek-MAX-E3-LIN-2*, decide whether the fraction of more than $(1-\varepsilon)$ or less than $(\frac{1}{2}+\varepsilon)$ of all equations is satisfied by the optimal (i.e. maximizing) assignment.*

To use all properties of our equation gadgets, the order of variables in equations will play a role. We denote by E$[k,k,k]$-MAX-E3-LIN-2 those instances of E$3k$-MAX-E3-LIN-2 for which each variable occurs exactly k times as the first variable, k times as the second variable and k times as the third variable in equations. Given an instance I_0 of Ek-MAX-E3-LIN-2 we can easily transform it into an instance I of E$[k,k,k]$-MAX-E3-LIN-2 with the same optimum, as follows: for any equation $x+y+z=j$ of I_0 we put in I the triple of equations $x+y+z=j$, $y+z+x=j$, and $z+x+y=j$. Hence the same NP-hard gap as in Theorem 1 applies for E$[k,k,k]$-MAX-E3-LIN-2 as well. We describe several reductions from E$[k,k,k]$-MAX-E3-LIN-2 to bounded occurrence instances of NP-hard problems that preserve the hard gap of E$[k,k,k]$-MAX-E3-LIN-2.

2 Consistency Amplifiers

As a parameter of our reduction for B-MAX-IS (or B-MIN-NC) ($B \geq 3$), and MAX-3DM, we will use a graph H, so called *consistency $3k$-amplifier*, with the following structure:

(i) The degree of each node is at most B.

(ii) There are $3k$ pairs of *contact nodes* $\{(c_0^i, c_1^i) : i = 1, 2, \ldots, 3k\}$.

(iii) The degree of any contact node is at most $B-1$.

(iv) The first $2k$ pairs of contact nodes $\{(c_0^i, c_1^i) : i = 1, 2, \ldots, 2k\}$ are *implicitly linked* in the following sense: whenever J is an independent set in H, there is an independent set J' in H such that $|J'| \geq |J|$, a contact node c can belong to J' only if $c \in J$, and for any $i = 1, 2, \ldots, 2k$ at most one node of the pair (c_0^i, c_1^i) belongs to J'.

(v) *The consistency property:* Let us denote $C_j := \{c_j^1, c_j^2, \ldots, c_j^{3k}\}$ for $j \in \{0,1\}$, and $M_j := \max\{|J| : J$ is an independent set in H such that $J \cap C_{1-j} = \emptyset\}$. Then $M_1 = M_2$ $(:= M(H))$, and for every $\psi : \{1, 2, \ldots, 3k\} \rightarrow \{0,1\}$ and for every independent set J in $H \setminus \{c_{1-\psi(i)}^i : i = 1, 2, \ldots, 3k\}$ we have $|J| \leq M(H) - \min\{|\{i : \psi(i) = 0\}|, |\{i : \psi(i) = 1\}|\}$.

Remark 1 Let $j \in \{0,1\}$ and J be any independent set in $H \setminus C_{1-j}$ such that $|J| = M(H)$, then $J \supseteq C_j$. To show that, assume that for some $l \in \{1, 2, \ldots, 3k\}$ $c_j^l \notin J$. Define $\psi : \{1, 2, \ldots, 3k\} \rightarrow \{0,1\}$ by $\psi(l) = 1-j$, and $\psi(i) = j$ for

$i \neq l$. Now (v) above says $|J| < M(H)$, a contradiction. Hence, in particular, C_j is an independent set in H.

To obtain better inapproximability results we use equation gadgets that require some further restrictions on degrees of contact nodes of a consistency $3k$-amplifier: (iii-1) For B-MAX-IS, $B \geq 6$, the degree of any contact node is at most $B-2$. (iii-2) For B-MAX-IS, $B \in \{4,5\}$, the degree of any contact node c_j^i with $i \in \{1,\dots,k\}$ is at most $B-1$, the degree of c_j^i with $i \in \{k+1,\dots,3k\}$ is at most $B-2$, where $j = 1,2$.

For integers $B \geq 3$ and $k \geq 1$ let $\mathcal{G}_{B,k}$ stand for the set of corresponding consistency $3k$-amplifiers. Let $\mu_{B,k} := \min\{\frac{M(H)}{k} : H \in \mathcal{G}_{B,k}\}$, $\lambda_{B,k} := \min\{\frac{|V(H)|-M(H)}{k} : H \in \mathcal{G}_{B,k}\}$ (if $\mathcal{G}_{B,k} = \emptyset$, let $\lambda_{B,k} = \mu_{B,k} = \infty$), $\mu_B = \underline{\lim}_{k\to\infty}\mu_{B,k}$, and $\lambda_B = \underline{\lim}_{k\to\infty}\lambda_{B,k}$. The parameters μ_B and λ_B play a role of quantities in which our inapproximability results for B-MAX-IS and B-MIN-NC can be expressed. To obtain explicit lower bounds on approximability requires to find upper bounds on those parameters.

In what follows we describe some methods how consistency $3k$-amplifiers can be constructed. We will confine ourselves to highly regular amplifiers. This ensures that our inapproximability results apply to B-regular graphs for small values of B. We will look for a consistency $3k$-amplifier H as a bipartite graph with bipartition (D_0, D_1), where $C_0 \subseteq D_0$, $C_1 \subseteq D_1$ and $|D_0| = |D_1|$. The idea is that if D_j ($j = 0,1$) is significantly larger than $3k$ ($= |C_j|$) then suitable probabilistic model of constructing bipartite graphs with bipartition (D_0, D_1) and prescribed degrees, will produce with high probability a graph H with good "mixing properties" that ensures the consistency property with $M(H) = |D_j|$. We will not develop probabilistic model here, rather we will rely on what has already been proved (using similar methods) for amplifiers. The starting point to our construction of consistency $3k$-amplifiers will be amplifiers, which were studied by Berman & Karpinski [3] and Chlebík & Chlebíková [4].

Definition 2 *A graph $G = (V,E)$ is a $(2,3)$-graph if G contains only the nodes of degree 2 (contacts) and 3 (checkers). We denote Contacts $= \{v \in V : \deg_G(v) = 2\}$, and Checkers $= \{v \in V : \deg_G(v) = 3\}$. Furthermore, a $(2,3)$-graph G is an* amplifier *if for every $A \subseteq V$: $|\mathrm{Cut}\, A| \geq |Contacts \cap A|$, or $|\mathrm{Cut}\, A| \geq |Contacts \setminus A|$, where $\mathrm{Cut}\, A = \{\{u,v\} \in E$: exactly one of nodes u and v is in $A\}$. An amplifier G is called a (k,τ)-amplifier if $|Contacts| = k$ and $|V| = \tau k$.*

To simplify proofs we will use in our constructions only such (k,τ)-amplifiers which contain no edge between contact nodes. Recall, that the infinite families of amplifiers with $\tau = 7$ [3], and even with $\tau \leq 6.9$ constructed in [4], are of this kind.

The consistency $3k$-amplifier for $B = 3$

Let a $(3k,\tau)$-amplifier $G = (V(G), E(G))$ from Definition 2 be fixed, and x^1, ..., x^{3k} be its contact nodes. We assume, moreover, that there is a matching in G consisting of nodes $V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. Let us point out that both, the wheel-amplifiers with $\tau = 7$ [3], and also their generalization given in [4] with $\tau \leq 6.9$, clearly contain such matchings.

Let one such matching $\mathcal{M} \subseteq E(G)$ be fixed from now on. Each node $x \in V(G)$ is replaced with a small gadget A_x. The gadget of $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$ is a path of 4 nodes x_0, X_1, X_0, x_1 (in this order). For $x \in \{x^{2k+1}, \dots, x^{3k}\}$ we take as A_x a pair of nodes x_0, x_1 without an edge. Denote $E_x := \{x_0, x_1\}$ for each $x \in V(G)$, and $F_x := \{X_0, X_1\}$ for $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. The union of gadgets A_x (over all $x \in V(G)$) contains already all nodes of our consistency $3k$-amplifier H, and some of its edges. Now we identify the remaining edges of H. For each edge $\{x, y\}$ of G we connect corresponding gadgets A_x, A_y with a pair of edges in H, as follows: if $\{x, y\} \in \mathcal{M}$, we connect X_0 with Y_1 and X_1 with Y_0; if $\{x, y\} \in E(G) \setminus \mathcal{M}$, we connect x_0 with y_1, and x_1 with y_0.

Having this done, one after another for each edge $\{x, y\} \in E(G)$, we obtain the consistency $3k$-amplifier $H = (V(H), E(H))$ with contact nodes x^i_j determined by contact nodes x^i of G, for $j \in \{0, 1\}$, $i \in \{1, 2, \dots, 3k\}$. The proof of all conditions from the definition of a consistency $3k$-amplifier follows.

Clearly H is a bipartite graph with the bipartition (D_0, D_1) where D_j is the set of nodes of H with a lower index j, $j \in \{0, 1\}$. Further, $|D_0| = |D_1| = (6\tau - 1)k =: M(H)$. Moreover, degree of each contact node in H is 2, and degree of any other node is 3. First we prove that pairs $\{(x^i_0, x^i_1) : i = 1, \dots, 2k\}$ are implicitly linked. In fact, we will prove the following stronger result:

Claim 1 *Whenever J is an independent set in H, there is an independent set J' in H such that $|J'| \geq |J|$ and the following holds: if $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$ with $|E_x \cap J| = 2$, then $|E_x \cap J'| = 1$; in all other cases $E_x \cap J' = E_x \cap J$.*

Proof. Consider $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$ with $|E_x \cap J| = 2$ and make the following modification of J. Take $y \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$ such that $\{x, y\} \in \mathcal{M}$. As $\{Y_0, Y_1\} \in E(H)$ there is $j \in \{0, 1\}$ such that $Y_j \notin J$. Take one such j and replace x_j in J by X_{1-j}. Having the above modification of J done, one after another for each $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$, we obtain J' as required. □

Hence J' obtained from J using Claim 1 is an independent set even in the graph $\widetilde{H}$ obtained from H adding an edge $\{x_0, x_1\}$ connecting the pair E_x, for each $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. We denote further by $\widetilde{\widetilde{H}}$ the graph obtained from H adding an edge $\{x_0, x_1\}$ for all pairs E_x, $x \in V(G)$.

Now our aim is to prove that H satisfies the consistency property. For this purpose we keep fixed one (arbitrary) assignment $\psi : \{1, 2, \dots, 3k\} \to \{0, 1\}$, and denote by $\mathcal{J}$ the set of all independent sets J in H such that $J \cap \{x^i_{1-\psi(i)} : i = 1, 2, \dots, 3k\} = \emptyset$. If $\psi \equiv 0$ (respectively, $\psi \equiv 1$) clearly there is $J \in \mathcal{J}$ with $|J| = M(H)$, namely $J := D_0$ (respectively, $J := D_1$). To complete the proof of consistency of H we have to show that

$$|J| \leq M(H) - \min\big\{|\{i : \psi(i) = 0\}|, |\{i : \psi(i) = 1\}|\big\} \qquad (1)$$

for every $J \in \mathcal{J}$. For this purpose we need to introduce some notations: Given an assignment $\sigma : V(G) \to \{0,1\}$, then $N(\sigma)$ contains for each $x \in V(G)$ exactly those nodes from A_x which have lower index $\sigma(x)$. Clearly $|N(\sigma)| = M(H)$. In general, $N(\sigma)$ is not an independent set in H. But the structure of violating edges of $N(\sigma)$, i.e. edges of H with both endpoints in $N(\sigma)$, can be described as follows: for each $\{x,y\} \in E(G)$ with $\sigma(x) \neq \sigma(y)$ there is exactly one violating edge in H, namely $\{x_{\sigma(x)}, y_{\sigma(y)}\}$ if $\{x,y\} \in E(G) \setminus \mathcal{M}$; and $\{X_{\sigma(x)}, Y_{\sigma(y)}\}$ if $\{x,y\} \in \mathcal{M}$.

An assignment $\sigma : V(G) \to \{0,1\}$ is said to be *admissible*, if the set of violating edges of $N(\sigma)$ forms a matching in H. Clearly, σ is admissible iff for each $x \in V(G)$ there is at most one $y \in V(G)$ such that $\{x,y\} \in E(G) \setminus \mathcal{M}$ and $\sigma(y) \neq \sigma(x)$.

We will call an independent set J in H (in fact, even in $\tilde{\tilde{H}}$) *σ-regular*, if $J \subseteq N(\sigma)$. To obtain a σ-regular set from $N(\sigma)$ we have to remove at least one endpoint for every violating edge if the set of violating edges forms a matching. The cardinality of the set of violating edges is the same as of Cut (in G) of the set $\{x \in V(G) : \sigma(x) = 0\}$. As G is an amplifier, this cardinality is at least $\min\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\}$. It means, for any admissible assignment $\sigma : V(G) \to \{0,1\}$ any σ-regular independent set J in H satisfies

$$|J| \le M(H) - \min\{|\{i : \sigma(x^i) = 0|\}, |\{i : \sigma(x^i) = 1\}|\}. \tag{2}$$

Our strategy to prove (1) is to relate it to (2).

Now we are back to our fixed ψ and $\mathcal{J}$ as above. Denote further by $\tilde{\mathcal{J}}$ the set of $J \in \mathcal{J}$ for which J is also independent set in $\tilde{H}$ (in fact, J is then an independent set also in $\tilde{\tilde{H}}$). Let $\tilde{\mathcal{J}}_{\max}$ be the set of all independent sets from $\tilde{\mathcal{J}}$ of the maximum size, i.e. of size $\max\{|J| : J \in \tilde{\mathcal{J}}\}$. Using Claim 1 we easily get that this maximum is the same as $\max\{|J| : J \in \mathcal{J}\}$. Hence it is sufficient to prove (1) for an element $J \in \tilde{\mathcal{J}}_{\max}$.

Clearly, for any $J \in \tilde{\mathcal{J}}$ all nodes of $A_x \cap J$ have the same index for each $x \in V(G)$. For $J \in \tilde{\mathcal{J}}_{\max}$ we have, moreover, that $A_x \cap J \neq \emptyset$ for each $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. Keep, for a moment one $J \in \tilde{\mathcal{J}}_{\max}$ fixed. It determines an assignment σ $(= \sigma_J)$: $V(G) \to \{0,1\}$ in the following way:

- (i) For $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$, $\sigma(x) \in \{0,1\}$ is uniquely determined by $(\emptyset \neq)$ $A_x \cap J \subseteq \{x_{\sigma(x)}, X_{\sigma(x)}\}$.
- (ii) For $x = x^i$ with $i \in \{2k+1, \dots, 3k\}$ we take $\sigma(x^i) = \psi(i)$, unless $A_{x^i} \cap J = \emptyset$ and σ assigns (by the rule (i)) $1 - \psi(i)$ to the both neighbors of x in G; in that case we put $\sigma(x^i) = 1 - \psi(i)$.

Clearly, J is σ-regular. We will show that one can take J in such way that σ is, moreover, an admissible assignment. For this purpose we introduce the following notation for elements $J \in \tilde{\mathcal{J}}_{\max}$:

$$n_1(J) = |\{\{x,y\} \in E(G) : \sigma(x) = \sigma(y)\}|,$$
$$n_2(J) = |\{i \in \{1,2,\dots,2k\} : X^i_{1-\psi(i)} \notin J\}|.$$

For $J_1, J_2 \in \widetilde{\mathcal{J}}_{\max}$ we write $J_1 \prec J_2$ whenever $(n_1(J_1), n_2(J_1)) < (n_1(J_2), n_2(J_2))$ in the lexicographic order.

Let us keep fixed from now on one maximal element J of $(\widetilde{\mathcal{J}}_{\max}, \prec)$. Due to extremality of J in the order $\prec$ we will be able to prove that σ, determined by J as above, is admissible, and to derive (1) from that. We will proceed in several steps.

Claim 2 *Assume that $x \in V(G)$ is a checker node, y, z, $w \in V(G)$ are (pairwise distinct) neighbors of x in G, such that $\{x, w\} \in \mathcal{M}$. Suppose $\sigma(x) = j$, and $\sigma(y) = \sigma(z) = 1 - j$. Then $\sigma(w) = j$, and W_j, X_j, $x_j \in J$.*

Proof. Clearly $\sigma(w) = j$, because otherwise one could find larger $J' \in \widetilde{\mathcal{J}}_{\max}$ replacing in J the set $J \cap (A_x \cup \{W_0, W_1\})$ of cardinality at most 2 by $\{x_{1-j}, X_{1-j}, W_{1-j}\}$, a contradiction. Then also $W_j, X_j \in J$ easily follows. Assuming $x_j \notin J$ one could obtain contradiction replacing X_j in J by x_{1-j} that leads to $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$. Hence $x_j \in J$ as well. □

Now we strengthen Claim 2 showing that its assumptions are never satisfied for our extremal J.

Claim 3 *Assume that $x \in V(G)$ is a checker node, $y, z \in V(G)$ are distinct neighbors of x in G such that both edges $\{x, y\}$ and $\{x, z\}$ are from $E(G) \setminus \mathcal{M}$. Then either $\sigma(y) = \sigma(x)$ or $\sigma(z) = \sigma(x)$.*

Proof. Put $j := \sigma(x)$, and assume for contradiction that $\sigma(y) = \sigma(z) = 1 - j$. Using Claim 2 we conclude that X_j, $x_j \in J$, and consequently $y_{1-j}, z_{1-j} \notin J$. We will discus several possibilities for node y separately; in all of them we get a contradiction.

(a) Let y be a contact, i.e. $y = x^i$ for some $i \in \{1, 2, \ldots, 3k\}$. Assume first that $i \in \{1, 2, \ldots, 2k\}$. As $A_{x^i} \cap J \neq \emptyset$ but $x^i_{1-j} \notin J$, clearly $A_{x^i} \cap J = \{X^i_{1-j}\}$. Assuming $\psi(i) = j$ one could replace X^i_{1-j} in J by x^i_j. Otherwise $\psi(i) = 1 - j$ and one could replace x_j and X_j in J by x_{1-j} and y_{1-j}. In both cases it results in $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction.

Assume now that $i \in \{2k+1, \ldots, 3k\}$. As $\sigma(y) = 1 - j$ but $y_{1-j} \notin J$, it is only possible if $\psi(i) = 1 - j$ and σ assigns $1 - j$ to the second neighbor of y in G. But then replacing x_j and X_j in J by x_{1-j} and y_{1-j} we get $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction.

(b) Let y be a checker. Take $u \in V(G) \setminus \{x\}$ such that $\{y, u\} \in E(G) \setminus \mathcal{M}$. Assuming $\sigma(u) = j$ leads to a contradiction with Claim 2 when applied to the checker y with $1 - j := \sigma(y)$ in place of x with $j := \sigma(x)$. Namely, by the Claim 2, $y_{1-j} \in J$, a contradiction. Hence $\sigma(u) = 1 - j$, and in particular $u_j \notin J$. Consequently, one can replace x_j and X_j in J by x_{1-j} and y_{1-j} to obtain $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction. □

Claim 4 *σ is an admissible assignment.*

Proof. Assume, for contradiction, that σ is not admissible. That means, for some $x \in V(G)$ there are two distinct $y, z \in V(G)$ such that $\{x, y\} \in E(G) \setminus \mathcal{M}$, $\{x, z\} \in E(G) \setminus \mathcal{M}$, and $\sigma(y) = \sigma(z) = 1 - \sigma(x)$. Due to Claim 3, x must be a contact $x = x^i$. Clearly $i \in \{2k+1, \ldots, 3k\}$, because otherwise one of two edges of G adjacent to x belongs to $\mathcal{M}$. Due to our definition of $\sigma(x^i)$ in that case we conclude that necessarily $\sigma(x^i) = \psi(i)$ and $x^i_{\psi(i)} \in J$. Now, y being a checker, take $u \in V(G) \setminus \{x\}$ such that $\{y, u\} \in E(G) \setminus \mathcal{M}$. Assuming $\sigma(u) = \sigma(x^i)$ $(= \psi(i) = 1 - \sigma(y))$ we get a contradiction with Claim 3 (applied to the checker y in place of x). Hence $\sigma(u) = 1 - \psi(i)$. But now we can replace $x^i_{\psi(i)}$ in J by $y_{1-\psi(i)}$ to obtain $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction. That completes the proof. $\square$

Claim 5 *Let $x = x^i \in V(G)$ be a contact node with $\sigma(x^i) = 1 - \psi(i)$, and y, z be its neighbors in G. Then $\sigma(y) = \sigma(z) = 1 - \psi(i)$.*

Proof. For $i \in \{2k+1, \ldots, 3k\}$ it is clear from our definition of σ. Thus assume $i \in \{1, 2, \ldots, 2k\}$. Clearly $X^i_{1-\psi(i)} \in J$. One of neighbors of x, say y, satisfies $\{x, y\} \in \mathcal{M}$. If $\sigma(y) = \psi(i)$, we can replace $X^i_{1-\psi(i)}$ in J by $X^i_{\psi(i)}$; if $\sigma(z) = \psi(i)$ we can replace $X^i_{1-\psi(i)}$ in J by $x^i_{\psi(i)}$. In both cases we would obtain $J' \in \widetilde{\mathcal{J}}$ with $J \prec J'$, a contradiction. Hence necessarily $\sigma(y) = \sigma(z) = 1 - \psi(i)$. $\square$

Denote $Z := \{x^i_{1-\psi(i)} : \sigma(x^i) = 1 - \psi(i)\}$. From Claims above it easily follows that σ is an admissible assignment and that even $J \cup Z$ is a σ-regular independent set in H. So we can apply (2) to $J \cup Z$ in place of J to get

$$|J| + |\{i : \sigma(x^i) \neq \psi(i)\}| \leq M(H) - \min\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\},$$

from which (1) easily follows verifying that always

$$\begin{aligned} &\min\{|\{i : \psi(i) = 0\}|, |\{i : \psi(i) = 1\}|\} \leq \\ &\quad \min\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\} + |\{i : \sigma(x^i) \neq \psi(i)\}|. \end{aligned}$$

We have proved that H is a consistency $3k$-amplifier, as claimed. As $M(H) = (6\tau - 1)k$, $|V(H)| = 2M(H)$ and τ can be taken ≤ 6.9 (see [4]), it easily follows $\mu_3 \leq 40.4$, $\lambda_3 \leq 40.4$.

The consistency $3k$-amplifier for $B = 4$

The construction will be similar as in the case $B = 3$. Given k, we will look for a consistency $3k$-amplifier $H = (V(H), E(H))$ with the following properties:

(A) The first $2k$ pairs $\{(c^i_0, c^i_1), i = 1, 2, \ldots, 2k\}$ are connected by edges.

(B) The nodes c^i_0, c^i_1, $i \in \{1, 2, \ldots, k\}$ are of degree 3, the nodes c^i_0, c^i_1, $i \in \{k+1, \ldots, 3k\}$ are of degree 2. All other nodes of H are of degree 4.

(C) H is a bipartite graph with the bipartition (D_0, D_1), where $C_0 \subseteq D_0$, $C_1 \subseteq D_1$ and $|D_0| = |D_1| = M(H)$. (Here $M(H)$ is the one from the consistency property.)

Let a $(3k, \tau)$-amplifier $G = (V(G), E(G))$ be fixed, and $x^1, \ldots, x^{3k}$ be its contact nodes. Each node $x \in V(G)$ is replaced with a small gadget A_x. The gadget of a checker x is a pair of nodes x_0, x_1 connected by an edge. The same kind of gadget we take for any of the first k-contacts, i.e. for each $x \in \{x^1, x^2, \ldots, x^k\}$. For $x \in \{x^{2k+1}, x^{2k+2}, \ldots, x^{3k}\}$ we take as A_x a pair of nodes x_0, x_1 (i.e. x_0^i and x_1^i, if $x = x^i$ for $i \in \{2k+1, \ldots, 3k\}$ without an edge. For $x \in \{x^{k+1}, x^{2k+2}, \ldots, x^{2k}\}$ we take as A_x a 4-cycle (x_0, x_1, X_0, X_1) (with nodes in this order). Denote further $E_x = \{x_0, x_1\}$ for each $x \in V(G)$. The union of gadgets A_x (over all $x \in V(G)$) already contains all nodes of our consistency $3k$-amplifier H, and some of its edges. Now we identify the remaining edges of H. If two nodes $x, y \in V(G)$ are connected by an edge in G, we connect their pairs E_x and E_y with a pair of edges in such way that the node of E_x with an index j ($j \in \{0, 1\}$) is connected with the node of E_y indexed by $1 - j$. Having this done, one after another, for each edge $\{x, y\}$ of G, we obtain a graph $H = (V(H), E(H))$. The contact nodes are $c_0^i := X_0^i$, $c_1^i := X_1^i$ for $i \in \{k+1, k+2, \ldots, 2k\}$, otherwise $c_0^i := x_0^i$ and $c_1^i := x_1^i$. Clearly H is a bipartite graph with the bipartition (D_0, D_1), where D_j is the set of nodes with a lower index j, $j \in \{0, 1\}$. Further, $|D_0| = |D_1| = (3\tau + 1)k =: M(H)$. One can easily check that the above requirement (B) on H concerning degrees of nodes is satisfied as well as (A).

Our aim now is to prove the consistency property. For this purpose we keep fixed one (arbitrary) assignment $\psi : \{1, 2, \ldots, 3k\} \to \{0, 1\}$ and denote by $\mathcal{J}$ the set of all independent sets J in H such that $J \cap \{c^i_{1-\psi(i)} : i = 1, 2, \ldots, 3k\} = \emptyset$. We have to show that (1) holds for every $J \in \mathcal{J}$. It is clear that $|J| \leq M(H)$, as for each $x \in V(G)$ at most one of x_0 and x_1 can belong to J, and if $x \in \{x^{k+1}, \ldots, x^{2k}\}$ at most one of X_0, X_1 as well. Moreover, in the case $\psi \equiv 0$, or $\psi \equiv 1$, one has in fact $\max\{|J| : J \in \mathcal{J}\} = M(H)$, as $|D_0| = |D_1| = M(H)$. Hence the first part of the consistency property is obviously satisfied.

Let us describe our strategy for the proof of (1). We need to introduce some notions: An assignment $\sigma : V(G) \to \{0, 1\}$ to the nodes of G is said to be *nice*, if for each $x \in V(G)$ there is at most one neighbor y of x in G such that $\sigma(y) \neq \sigma(x)$. Given an assignment $\sigma : V(G) \to \{0, 1\}$, consider the set $N(\sigma) \subseteq V(H)$ which for each $x \in V(G)$ contains exactly the nodes from A_x with the lower index of $\sigma(x)$. Clearly, $|N(\sigma)| = M(H)$. In most cases $N(\sigma)$ is not an independent set. But the structure of the set of violating edges of H, i.e. those with both endpoints in $N(\sigma)$, is simple, assuming that σ is nice. In that case they are exactly the edges $\{x_{\sigma(x)}, y_{\sigma(y)}\} \in E(H)$ such that $\{x, y\} \in E(G)$ and $\sigma(x) \neq \sigma(y)$. In particular, they form a matching in H, and the cardinality of this matching is the same as the cardinality of Cut (in G) of the set $\{x \in V(G) : \sigma(x) = 0\}$. As G is an amplifier, this is at least $\min\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\}$.

Further, any independent set J in H that is subset of $N(\sigma)$ is said to be σ-regular (in fact, J is an independent set also in a graph obtained from H adding an edge $\{x_0, x_1\}$ connecting the pair E_x, for each contact node x). We have just proved that for any nice $\sigma : V(G) \to \{0, 1\}$ any σ-regular independent set J in H satisfies (2). This is because to obtain an independent set J from $N(\sigma)$ (of cardinality $M(H)$) we have to remove at least one endpoint for every violating edge. So, our strategy to prove (1) is to relate it to (2).

Now we are back to our fixed ψ and $\mathcal{J}$ as above. We want to prove that $\max |J|$ over $J \in \mathcal{J}$ is achieved even on "very regular" independent sets from $\mathcal{J}$. Let us introduce the following notation for $J \in \mathcal{J}$:

$$\begin{aligned} n_1(J) &= |J|, \\ n_2(J) &= |J \cap (\cup_{x \in V(G)} E_x)|, \\ n_3(J) &= \max\{|J \cap D_0|, |J \cap D_1|\}. \end{aligned}$$

For J_1, $J_2 \in \mathcal{J}$ we write $J_1 \prec J_2$ whenever $(n_1(J_1), n_2(J_1), n_3(J_1)) < (n_1(J_2), n_2(J_2), n_3(J_2))$ in the lexicographic order. Let us keep fixed any maximal element J of $(\mathcal{J}, \prec)$. Clearly, $|J| = \max\{|J'| : J' \in \mathcal{J}\}$, hence it is sufficient to prove (1) for this single J. Due to extremality of J in our order $\prec$ we will be able to relate J to σ-regular independent set of H for some nice assignment $\sigma : V(G) \to \{0, 1\}$. Preliminarily, let σ be defined only on those $x \in V(G)$ for which $E_x \cap J \neq \emptyset$: let $\sigma(x)$ be the index of a (unique) node of $E_x \cap J$ (i.e. $E_x \cap J = \{x_{\sigma(x)}\}$).

Claim 6 *Let $x \in V(G)$ be a checker node with $E_x \cap J = \emptyset$. Then for each y such that $\{x, y\} \in E(G)$ the set $E_y \cap J$ is nonempty. In another words, σ is already defined for all three neighbors of x in G. Moreover, σ attains both 0 and 1 as value on neighbors of x.*

Proof. Let y, z, w be all three neighbors of x in G. Assume, for example, that $E_y \cap J = \emptyset$. As neither $J \cup \{x_0\}$ nor $J \cup \{x_1\}$ is an independent set in H, necessarily for some $j \in \{0, 1\}$ $E_z \cap J = \{z_j\}$ and $E_w \cap J = \{w_{1-j}\}$. But then replacing in J either z_j by x_{1-j}, or w_{1-j} by x_j, will result in $J' \in \mathcal{J}$ with $J \prec J'$ (namely, $n_3(J) < n_3(J')$), a contradiction. Hence σ is already defined for y, and in the same way for z and w too.

Assume now that $\sigma(y) = \sigma(z) = \sigma(w) =: j \in \{0, 1\}$. But then adding x_j to J will produce larger $J' \in \mathcal{J}$, a contradiction. □

Claim 7 *Let $x = x^i \in V(G)$ be a contact node with $E_x \cap J = \emptyset$, and y, z be its neighbors in G. By our assumption about G, y and z have to be checker nodes with $\sigma(y)$ and $\sigma(z)$ already defined (due to Claim 6).*

(a) If $i \in \{k + 1, \dots, 2k\}$ then $X^i_{\psi(i)} \in J$ and $\sigma(y) \neq \sigma(z)$.

(b) If $i \in \{1, \dots, k\} \cup \{2k + 1, \dots, 3k\}$ then either $\sigma(y) \neq \sigma(z)$, or $\sigma(y) = \sigma(z) = 1 - \psi(i)$. In the latter case $J \cup \{x^i_{1-\psi(i)}\}$ is an independent set in H (and also in a graph obtained from H adding an edge $\{x_0, x_1\}$ connecting the pair E_x for each contact node x) too.

Proof. (a) In this case clearly $X^i_{\psi(i)} \in J$ due to maximality of J. Further, neither $J' := J \cup \{x^i_{\psi(i)}\}$ nor $J' := J \setminus \{X^i_{\psi(i)}\} \cup \{x^i_{1-\psi(i)}\}$ is an independent set in H (it would imply $J \prec J'$). Hence for some $j \in \{0,1\}$ $E_y \cap J = \{y_j\}$ and $E_z \cap J = \{z_{1-j}\}$.

(b) In this case $J' := J \cup \{x^i_{\psi(i)}\}$ is not an independent set in J (it would imply $J \prec J'$). Hence at least for one $u \in \{y, z\}$ we have $\sigma(u) = 1 - \psi(i)$. Moreover, if $\sigma(y) = \sigma(z) = 1 - \psi(i)$, it follows that $y_{\psi(i)} \notin J$ and $z_{\psi(i)} \notin J$, hence $J \cup \{x^i_{1-\psi(i)}\}$ is an independent set as well. □

Now we are ready to extend σ to a nice assignment for which J is σ-regular.

(i) If $x \in V(G)$ is a checker node with $E_x \in J = \emptyset$, by Claim 6 both 0 and 1 are attained by σ on neighbors of x. Necessarily one $j \in \{0,1\}$ is attained twice there, and we let $\sigma(x) := j$.

(ii) If $x = x^i \in V(G)$ is a contact node with $E_x \cap J = \emptyset$, by Claim 7 either both 0 and 1 are attained by σ on neighbors of x, or both neighbors of x have assigned $1 - \psi(i)$ by σ. In the former case we let $\sigma(x^i) = \psi(i)$, in the latter one $\sigma(x^i) = 1 - \psi(i)$.

Denote further $Z := \{x^i_{1-\psi(i)} : \sigma(x^i) = 1 - \psi(i)\}$. Clearly J is σ-regular. Using Claim 7(b), even $J \cup Z$ is σ-regular independent set in H.

Now we want to prove that σ is a nice assignment. Clearly, by our extension of σ based on Claims 6 and 7, for each $x \in V(G)$ with $E_x \cap J = \emptyset$ at most one neighbor y of x in G has $\sigma(y) \neq \sigma(x)$. We have to prove that this is also true for each $x \in V(G)$ with $E_x \cap J \neq \emptyset$.

Claim 8 *Let $x \in V(G)$ be either checker or contact node with $E_x \cap J \neq \emptyset$. Then there exists at most one neighbor u of x in G with $\sigma(u) \neq \sigma(x)$.*

Proof. Consider a neighbor u of x in G with $\sigma(u) \neq \sigma(x)$. Clearly $E_x \cap J = \{x_{\sigma(x)}\}$ which implies $u_{\sigma(u)} = u_{1-\sigma(x)} \notin J$, hence $E_u \cap J = \emptyset$. If u is a checker node, then due to Claim 6 the other two neighbors of u have assigned $1 - \sigma(x)$ already in the first stage, in particular $(J \setminus \{x_{\sigma(x)}\}) \cup \{u_{1-\sigma(x)}\}$ is an independent set as well. If u is a contact node, then due to Claim 7 the other neighbor of u has assigned $1 - \sigma(x)$ already in the first stage, in particular $(J \setminus \{x_{\sigma(x)}\}) \cup \{u_{1-\sigma(x)}\}$ is an independent set in this case as well.

Assume now, that two distinct neighbors y and z of x in G have $\sigma(y) = \sigma(z) = 1 - \sigma(x)$. The analysis above shows that then $J' := (J \setminus \{x_{\sigma(x)}\}) \cup \{y_{1-\sigma(x)}, z_{1-\sigma(x)}\}$ will be a larger independent set, a contradiction. □

From above Claims we know that σ is a nice assignment and that even $J \cup Z$ is a σ-regular independent set in H. So, we can apply (2) to $J \cup Z$,

$$|J| + |\{i : \sigma(x^i) \neq \psi(i)\}| \leq M(H) - \min\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\},$$

from which we easily obtain (1) as in case $B = 3$. Hence H is really a consistency $3k$-amplifier, as claimed. As $M(H) = (3\tau + 1)k$, and $|V(H)| = 2M(H)$, this gives us basic estimates $\mu_4 \leq 21.7$, $\lambda_4 \leq 21.7$.

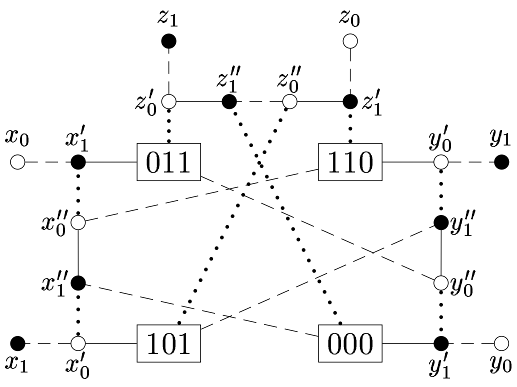

Figure 1: The equation gadget $G_0 := G_0[3]$ for 3-MAX-IS and MAX-3DM.

We do not try to optimize our estimates for $B \geq 5$ in this paper, we are mainly focused on cases $B = 3, 4$. For larger B we provide our inapproximability results based on small degree amplifiers constructed above. Of course, one can expect that amplifiers with much better parameters can be found for these cases by suitable constructions. We only slightly change the consistency $3k$-amplifier H constructed for case $B = 4$ to get some (very small) improvement for $B \geq 5$ case. Namely, also for $x \in \{x^{k+1}, x^{k+2}, \ldots, x^{2k}\}$ we take as A_x a pair of nodes connected by an edge. The corresponding c_0^i, c_1^i nodes of H will have degree 3 in H, but we will have now $M(H) = 3\tau k$. The same proof of consistency for H will work. This consistency amplifier H will be clearly simultaneously a consistency $3k$-amplifier for any $B \geq 5$. In this way we get the upper bound $\mu_B \leq 20.7$, $\lambda_B \leq 20.7$ for any $B \geq 5$.

3 The Equation Gadgets

In the reduction to our problems we use the equation gadgets G_j for equations $x + y + z = j$, $j = 0, 1$. To obtain better inapproximability results, we use slightly modified equation gadgets for distinct value of B in B-MAX-IS problem (or B-MIN-NC problem). We define equation gadgets $G_j[3]$ for 3-MAX-IS problem (Fig. 1), $G_j[4]$ for 4(5)-MAX-IS (Fig. 2(i)), $G_j[6]$ for B-MAX-IS $B \geq 6$ (Fig. 2(ii)). In each case the gadget $G_1[*]$ can be obtained from $G_0[*]$ replacing each $i \in \{0, 1\}$ in indices and labels by $1 - i$.

For each $u \in \{x, y, z\}$ we denote by F_u the set of all accented u-nodes from G_j (hence F_u is a subset of $\{u_0', u_1', u_0'', u_1''\}$), and $F_u := \emptyset$ if G_j does not contain any accented u-node; $T_u := F_u \cup \{u_0, u_1\}$. For a subset A of nodes of G_j and any independent set J in G_j we will say that J *is pure in* A if all nodes of $A \cap J$ have the same lower index (0 or 1). If moreover, $A \cap J$ consists exactly of all nodes of A of one index, we say that J *is full in* A.

The following theorem describes basic properties of equation gadgets.

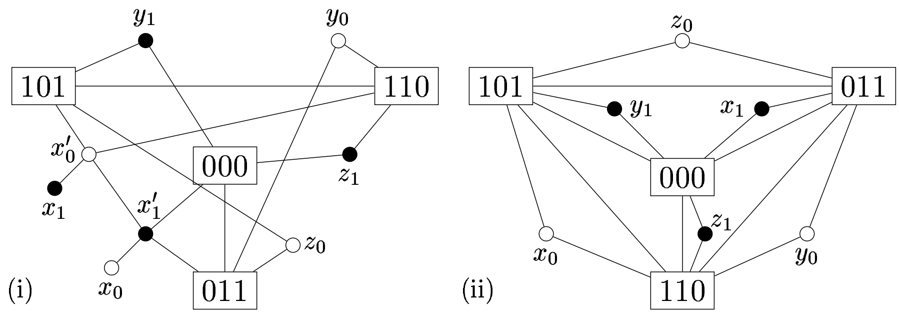

Figure 2: The equation gadget (i) $G_0 := G_0[4]$ for B-Max-IS, $B \in \{4, 5\}$, (ii) $G_0 := G_0[6]$ for B-Max-IS ($B \geq 6$).

Theorem 2 *Let G_j ($j \in \{0,1\}$) be one of the following gadgets: $G_j[3]$, $G_j[4]$, or $G_j[6]$, corresponding to an equation $x+y+z=j$. Let J be an independent set in G_j such that for each $u \in \{x, y\}$ at most one of two nodes u_0 and u_1 belongs to J. Then there is an independent set J' in G_j with the following properties:*

(I) $|J'| \geq |J|$,

(II) for each $u \in \{x, y\}$ it holds $J' \cap \{u_0, u_1\} = J \cap \{u_0, u_1\}$,

(III) $J' \cap \{z_0, z_1\} \subseteq J \cap \{z_0, z_1\}$ *and* $|J' \cap \{z_0, z_1\}| \leq 1$,

(IV) J' contains (exactly) one special node, say $\psi(x)\psi(y)\psi(z)$. Furthermore, J' is pure in T_u and full in F_u.

Proof. We write proofs for the gadget G_0, the modifications for G_1 are obvious. Denote by S the set of four special nodes: $\boxed{000}$, $\boxed{110}$, $\boxed{101}$, $\boxed{011}$.

A: The equation gadget for 3-Max-IS (Figure 1)

(a) First we show that we can always modify J to J' satisfying (I), (II), and (III) such that $|J' \cap \{z_0, z_1\}| \leq 1$. For this purpose let J as above be fixed with both $z_0 \in J$ and $z_1 \in J$. Then clearly $z'_0 \notin J$ and $z'_1 \notin J$. We can assume that either z''_0 or z''_1 is in J because otherwise we could either add z''_1 to J (if a special node $\boxed{000} \notin J$), or to replace $\boxed{000}$ in J by z''_1, to ensure this property. Hence we will assume in what follows that $z''_1 \in J$ (the discussion for the case $z''_0 \in J$ is, due to symmetry, analogous).

So, we are in the situation $\{z_0, z_1, z''_1\} \subseteq J$, implying $z''_0 \notin J$, $\boxed{000} \notin J$. We can further assume that $\boxed{110} \in J$ (because otherwise replacing z_0 in J by z'_1 we are done with this part of the proof).

(i) Assume first that $\boxed{101} \notin J$. Replacing z''_1 in J by z''_0 we reduce this to the case $\{z_0, z_1, z''_0, \boxed{110}\} \subseteq J$, $\boxed{101} \notin J$, $\boxed{000} \notin J$. We can further assume that $\boxed{011} \in J$ (because otherwise replacing z_1 in J by z'_0 we are done). As both $\boxed{110}$ and $\boxed{011}$ belong to J, clearly $|F_x \cap J| \leq 1$. So we can modify J inside F_x, T_z and S to J' with $|J'| \geq |J|$ as follows. Let $j \in \{0,1\}$ be fixed such that $x_{1-j} \notin J$. We take J' with $F_x \cap J' = \{x'_j, x''_j\}$, $T_z \cap J' = \{z_{1-j}, z'_{1-j}, z''_{1-j}\}$ and $S \cap J' = \{\boxed{j1(1-j)}\}$.

(ii) Assume now that $\boxed{101} \in J$. We can also assume that $\boxed{011} \notin J$ (because otherwise one could replacing $\boxed{101}$ in J by y''_1 get the situation already discussed in (i)). So recall that we have $\{z_0, z_1, z''_1, \boxed{110}, \boxed{101}\} \subseteq J$, $\boxed{011} \notin J$, $\boxed{000} \notin J$. Clearly $|F_y \cap J| \leq 1$. Now we can modify J inside F_y, T_z and S to J' with $|J'| \geq |J|$ as follows. Let $j \in \{0,1\}$ be fixed such that $y_{1-j} \notin J$. We take J' with $F_y \cap J' = \{y'_j, y''_j\}$, $T_z \cap J' = \{z_{1-j}, z'_{1-j}, z''_{1-j}\}$ and $S \cap J' = \{\boxed{1j(1-j)}\}$.

The proof of the part (a) is complete.

(b) After reduction from part (a) we can assume that J is an independent set in G_0 such that for each $u \in \{x,y,z\}$ $|J \cap \{u_0, u_1\}| \leq 1$. Keep one such J fixed and denote by $\mathcal{J}$ the set of all independent sets J' in G_0 satisfying (II) and (III). Our aim is to prove that some of sets from $\mathcal{J}$ have to satisfy (I) and (IV) as well. In the following part we will prove that there is J' in $\mathcal{J}$ satisfying (I) and (IV′), where (IV′) is a slight relaxation of (IV), namely

(IV′) J' contains at most one special node and for each $u \in \{x,y,z\}$ the set J' is pure and full in F_u.

To prove that such J' exists, we will show that some extremal elements of $\mathcal{J}$ have this property. Consider $J' \in \mathcal{J}$ with the maximum cardinality, from those with maximum cardinality the one with the least number of special nodes, and from such sets the one which is pure in as many of T_x, T_y, T_z, as possible. Let us keep one such extremal $J' \in \mathcal{J}$ fixed. We will show that J' satisfies (IV′) ((I) being trivial). We will proceed in several steps.

Observation 1. If $u \in \{x,y,z\}$ and J' is pure in T_u then it is full in F_u.

Proof. Take $j \in \{0,1\}$ such that $T_u \cap J'$ contains nodes with the index j only. Fix a node $v \in F_u$ with index j. Our aim is to show that $v \in J'$. Assume, on the contrary, that $v \notin J'$. As $J' \cup \{v\}$ is not an independent set due to the maximality of J', a neighbor of v (one of special nodes) belongs to J'. Replacing this special node in J' by v we obtain $J'' \in \mathcal{J}$ with $|J''| = |J'|$ but with less special nodes, a contradiction. □

Observation 2. If $u \in \{x,y,z\}$ and J' is not pure in T_u then one of the following possibilities occurs:

(i) $T_u \cap J' = \{u'_0, u'_1\}$ and both special nodes adjacent to u''_0 and u''_1 belong to J';

(ii) for some $j \in \{0,1\}$: $T_u \cap J' = \{u_j, u''_{1-j}\}$ and both special nodes adjacent to u'_j and u''_j belong to J'.

Proof. Assume first that $T_u \cap J' = \{u'_0, u'_1\}$. If for some $j \in \{0,1\}$ the special node adjacent to u''_j does not belong to J', then replacing u'_{1-j} in J' by u''_j results in $J'' \in \mathcal{J}$ which is more pure than J', a contradiction.

Now it is clear that if J' is not pure in T_u and the case (ii) does not occur, then for some $j \in \{0,1\}$ $T_u \cap J' = \{u_j, u''_{1-j}\}$. If the special node adjacent to u'_j (respectively, to u''_j) does not belong to J', then replacing u''_{1-j} in J' by u'_j (respectively, by u''_j) will result in $J'' \in \mathcal{J}$ which is more pure than J', a contradiction. □

Observation 3. $|S \cap J'| \leq 2$.

Proof. If for $p = 0$ or $p = 1$ we have $|S \cap J'| = 4 - p$, clearly for each $u \in \{x,y,z\}$ $|F_u \cap J'| \leq p$. We can clearly find $J'' \in \mathcal{J}$ pure in T_x, T_y and T_z such that $|T_u \cap J''| = 2$ for each $u \in \{x,y,z\}$, and $S \cap J'' = \emptyset$. Clearly $|J''| \geq |J| + 2 - 2p \geq |J|$, and J'' has less special nodes than J', a contradiction. □

Now we are ready to complete the proof of part (b) showing that J' is, in fact, pure in each T_u, $u \in \{x,y,z\}$. By Observation 1 it is then even full in each F_u and clearly $|S \cap J'| \leq 1$ will follow.

Assume, on the contrary that J' is not pure in some (at least one) T_u, $u \in \{x,y,z\}$. Using Observations 2 and 3 we obtain that $|S \cap J'| = 2$. Let $S \cap J' = \{s_1, s_2\}$. There are 6 theoretical possibilities how this pair $\{s_1, s_2\}$ from S is chosen. But *each* pair $\{s_1, s_2\}$ of nodes from S has the following property that can be easily verified. There is $v \in \{x,y\}$ for which two nodes of F_v adjacent to $\{s_1, s_2\}$ have distinct indices and at least one of them belongs to $\{u'_0, u'_1\}$. This fact (together with $S \cap J' = \{s_1, s_2\}$) easily leads to a contradiction.

Thus J' is not full in F_v and, due to Observation 1, it cannot be pure in T_v. Hence Observation 2 applies. But neither (i) nor (ii) case of this observation is consistent with what we know about nodes of F_v adjacent to $\{s_1, s_2\}$. This contradiction completes the proof of part (b).

(c) We have already seen that an independent set J' satisfying (I), (II), (III), (IV′) exists. Let for $u \in \{x,y,z\}$ $\psi(u) \in \{0,1\}$ be such that $F_u \cap J'$ contains exactly all nodes of lower index $\psi(u)$. If $\psi(x) + \psi(y) + \psi(z) = 0$, $J' \cup \{\boxed{\psi(x)\psi(y)\psi(z)}\}$ is an independent set as required.

Otherwise one can add $\{\boxed{\psi(x)\psi(y)(1-\psi(z))}\}$ to J', remove $z_{\psi(z)}$ from J' if it belongs to it, and modify J' in F_z to obtain J'' such that $F_z \cap J'' = \{z'_{1-\psi(z)}, z''_{1-\psi(z)}\}$. Now J'' is as required.

B: For the equation gadget for B-Max-IS, $B \in \{4,5\}$ (Figure 2(i))

(a) Assume first that J contains no special node. One can choose $\psi(x) \in \{0,1\}$ such that $x_{1-\psi(x)} \notin J$ and $x'_{1-\psi(x)} \notin J$, and $\psi(y) \in \{0,1\}$ such that $y_{1-\psi(y)} \notin J$. Let s be the special node labeled by $\psi(x)\psi(y)\psi(z)$, where $\psi(z) = (\psi(x) + \psi(y)) \bmod 2$. If $z_{1-\psi(z)} \notin J$ then clearly one can take $J' = J \cup \{s, x'_{\psi(x)}\}$, otherwise $J' = (J \setminus \{z_{1-\psi(z)}\}) \cup \{s, x'_{\psi(x)}\}$.

(b) Assume now that J contains exactly one special node, say s, and let its label starts with $\psi(x) \in \{0,1\}$. Then clearly $|J \cap \{z_0, z_1\}| \leq 1$. If $x_{1-\psi(x)} \notin J$

one can clearly take $J' = J \cup \{x'_{\psi(x)}\}$. Otherwise modify J replacing s by $x'_{1-\psi(x)}$ to contain no special nodes, and continue as in the case (a).

(c) If J contains 2 special nodes then the label of one of them, say s_0, starts with 0, and the label of the other one, say s_1, starts with 1. From the structure of G_0 we can see that then $J \cap \{x'_0, x'_1\} = \emptyset$. Let further $\psi(x) \in \{0,1\}$ be chosen such that $x_{1-\psi(x)} \notin J$. Now replacing $s_{1-\psi(x)}$ in J by $x'_{\psi(x)}$ will produce J' as required.

C: The equation gadget for B-Max-IS, $B \geq 6$, (Figure 2(ii)).
If J contains a special node then clearly $|J \cap \{z_0, z_1\}| \leq 1$ and one can take $J' = J$. Assume now that J contains no special node. Let $\psi(x), \psi(y) \in \{0,1\}$ be chosen in such way that $x_{1-\psi(x)} \notin J$ and $y_{1-\psi(y)} \notin J$. Let s be the special node in G_0 labeled by $\psi(x)\psi(y)\psi(z)$, where $\psi(z) = (\psi(x) + \psi(y)) \bmod 2$. If $z_{1-\psi(z)} \notin J$ then clearly one can take $J' = J \cup \{s\}$, otherwise one can obtain J' from J replacing $z_{1-\psi(z)}$ by s. □

4 Reduction for B-Max-IS and B-Min-NC

For arbitrarily small fixed $\varepsilon > 0$ consider k large enough such that conclusion of Theorem 1 for E$[k,k,k]$-Max-E3-Lin-2 is satisfied. Further, let a consistency $3k$-amplifier H have $\frac{M(H)}{k}$ (resp. $\frac{|V(H)|-M(H)}{k}$) as close to μ_B (resp. λ_B) as we need. Keeping one consistency $3k$-amplifier H fixed, our reduction f $(= f_H)$ from E$[k,k,k]$-Max-E3-Lin-2 to B-Max-IS (resp., B-Min-NC) is as follows: Let I be an instance of E$[k,k,k]$-Max-E3-Lin-2, $\mathcal{V}(I)$ be the set of variables of I, $m := |\mathcal{V}(I)|$. Hence I has mk equations, each variable $u \in \mathcal{V}(I)$ occurs exactly in $3k$ of them: k times as the first variable, k times as the second one, and k times as the third variable in the equation. Assume, for convenience, that equations are numbered by $1, 2, \ldots, mk$. Given variable $u \in \mathcal{V}(I)$ and $s \in \{1,2,3\}$ let $r_s^1(u) < r_s^2(u) < \cdots < r_s^k(u)$ be the numbers of equations in which variable u occurs as the s-th variable. On the other hand, if for fixed $r \in \{1, 2, \ldots, mk\}$ the r-th equation is $x + y + z = j$ $(j \in \{0,1\})$, there are uniquely determined numbers $i(x,r)$, $i(y,r)$, $i(z,r) \in \{1, 2, \ldots, k\}$ such that $r_1^{i(x,r)}(x) = r_2^{i(y,r)}(y) = r_3^{i(z,r)}(z) = r$.

Take m disjoint copies of H, one for each variable. Let H_u denote a copy of H that correspondents to a variable $u \in \mathcal{V}(I)$. The corresponding contacts are in H_u denoted by $C_j(u) = \{u_j^i : i = 1, 2, \ldots, 3k\}$, $j = 0, 1$. Now we take mk disjoint copies of equation gadgets G^r, $r \in \{1, 2, \ldots, mk\}$. More precisely, if the r-th equation reads as $x + y + z = j$ $(j \in \{0,1\})$ we take as G^r a copy of $G_j[3]$ for 3-Max-IS (or $G_j[4]$ for 4(5)-Max-IS or $G_j[6]$ for B-Max-IS, $B \geq 6$). Then the nodes x_0, x_1, y_0, y_1, z_0, z_1 of G^r are identified with nodes $x_0^{i(x,r)}$, $x_1^{i(x,r)}$ (of H_x), $y_0^{k+i(y,r)}$, $y_1^{k+i(y,r)}$ (of H_y), $z_0^{2k+i(z,r)}$, $z_1^{2k+i(z,r)}$ (of H_z), respectively. It means that in each H_u the first k-tuple of pairs of contacts corresponds to the occurrences of u as the first variable, the second k-tuple corresponds to the occurrences as the second variable, and the third one occurrences as the last variable. Making the above identification for all equations, one after another, we get a graph of degree at most B, denoted by $f(I)$. Clearly, the above reduction f (using the fixed H as a parameter) to special instances of B-MAX-IS is polynomial. Now we show how the NP-hard gap of E[k,k,k]-MAX-E3-LIN-2 is preserved.

We look at $f(I)$ as at an instance of the MAX-IS problem. An independent set J in $f(I)$ is called *standard* if for each $u \in \mathcal{V}(I)$ there is (necessarily unique) $\varphi(u) \in \{0,1\}$ such that $J \cap C_{1-\varphi(u)}(u) = \emptyset$ and $|J \cap V(H_u)| = M(H)$. It implies, in particular, that $J \supseteq C_{\varphi(u)}(u)$ (see Remark 1). Clearly, any standard independent set J in $f(I)$ determines an assignment $\varphi : \mathcal{V}(I) \to \{0,1\}$; an independent set J is called, more specifically, φ-standard. Further, it is clear that a φ-standard independent set J can contain one special node for each equation satisfied by the assignment φ. More precisely, if r-th equation of I reads as $x + y + z = j$ then J can contain a special node from the equation gadget G^r iff $\varphi(x) + \varphi(y) + \varphi(z) = j \bmod 2$, namely the special node labeled by $\varphi(x)\varphi(y)\varphi(z)$.

Hence if $\mathrm{sat}(\varphi)$ means the number of equations of I satisfied by φ, one can express easily the maximum cardinality of a φ-standard independent set as $M(H)m + \mathrm{sat}(\varphi)$, for B-MAX-IS, $B \geq 6$; $M(H)m + mk + \mathrm{sat}(\varphi)$, for 4-MAX-IS and 5-MAX-IS, and $M(H)m + 6mk + \mathrm{sat}(\varphi)$, for 3-MAX-IS.

Taking φ optimal, i.e. such that $\mathrm{sat}(\varphi) = \mathrm{OPT}(I)|I| = \mathrm{OPT}(I)mk$, allows to express simply $\mathrm{OPT}_{\mathrm{std}}(f(I)) := \max\{|J|\colon J$ is a standard independent set in $f(I)\}$ using $\mathrm{OPT}(I)$. Namely $\mathrm{OPT}_{\mathrm{std}}(f(I)) = mk(M(H)/k + 6 + \mathrm{OPT}(I))$ for 3-MAX-IS, $\mathrm{OPT}_{\mathrm{std}}(f(I)) = mk(M(H)/k + 1 + \mathrm{OPT}(I))$ for 4(5)-MAX-IS and $\mathrm{OPT}_{\mathrm{std}}(f(I)) = mk(M(H)/k + \mathrm{OPT}(I))$ for B-MIS, $B \geq 6$.

The key point now is that the properties of our consistency gadget H imply that there can never be advantageous to use independent set which is not standard, to achieve the maximum cardinality. In other words, $\mathrm{OPT}(f(I))$ is achieved on standard independent set.

Let us prove now the fact that $\mathrm{OPT}(f(I)) = \mathrm{OPT}_{\mathrm{std}}(f(I))$. For this purpose consider one independent set J of $f(I)$ such that $|J| = \mathrm{OPT}(f(I))$. The aim is to show, in several steps, that one can modify J to another independent set J' in $f(I)$ such that $|J'| \geq |J|$ and J' is standard.

First, for each $u \in \mathcal{V}(I)$, one after another, modify J inside H_u to obtain another optimal independent set J_0 containing no pair of implicitly linked nodes. In another words, for each $u \in \mathcal{V}(I)$ an independent set $J_0 \cap V(H_u)$ contains at most one node from each of first $2k$ pairs of contact nodes.

Now, for each equation of I, one after another, modify J_0 inside the corresponding equation gadget G^r according to Theorem 2, to obtain another optimal independent set J_1 with the following properties: For each $u \in \mathcal{V}(I)$ an independent set $J_1 \cap V(H_u)$ contains from each pairs of contact nodes at most one node and for each $r = 1, 2, \ldots, mk$ the graph $J_1 \cap V(G^r)$ contains exactly one special node. If this special node for the r-th equation $x + y + z = j$ is labeled by $\psi(x)\psi(y)\psi(z)$, those bits can be viewed as a local satisfying assignment for occurrences of variables x, y and z in this equation. Moreover, for each $u \in \{x,y,z\}$ the set J_1 in this equation gadget is pure and full in F_u

(with nodes of label $\psi(u)$ there), in particular $u_{1-\psi(u)} \notin J_1$. In this way the set J_1 uniquely determines local assignment ψ to all occurrences of each variable. More precisely, as $\psi(u)$ can vary from occurrence to occurrence of u, we should write more precisely $\psi(u^i)$ for particular occurrences of u. For fixed u we will also write $\psi_u(i) := \psi(u^i)$.

Now for each variable $u \in \mathcal{V}(I)$ we can take the prevailing value from $\{0,1\}$ of local assignment to occurrences of u as the definition of the value $\varphi(u)$, assigned to this variable (by an independent set J_1). In the case of equal number of 0's and 1's of local assignment to occurrences of u the choice of $\varphi(u) \in \{0,1\}$ can be arbitrary.

Keeping $u \in \mathcal{V}(I)$ fixed, denote by $S(u)$ the set of special nodes in J_1 that determine for u the local assignment ψ inconsistent with $\varphi(u)$. Clearly,

$$|S(u)| = |\{i : \psi_u(i) \neq \varphi(u)\}| = \min\{|\{i : \psi_u(i) = 0\}|, |\{i : \psi_u(i) = 1\}|\},$$

hence $|J_1 \cap V(H)| \leq M(H) - |S(u)|$ as follows from the consistency property of H. If some $u \in \mathcal{V}(I)$ are inconsistent, i.e. $S(u) \neq \emptyset$, we will further modify J_1 in the following way:

(i) Remove first from J_1 special nodes that caused the inconsistency, i.e. $\cup_{u \in \mathcal{V}(I)} S(u)$, of cardinality $|\cup_u S(u)| \leq \sum_u |S(u)|$.

For each inconsistent occurrence of u we further modify J_1 inside the corresponding equation gadget: the node $u'_{1-\varphi(u)}$, resp. $u''_{1-\varphi(u)}$ is replaced by $u'_{\varphi(u)}$, resp. $u''_{\varphi(u)}$, if such nodes exist in the equation gadget.

(ii) Then for each u replace $J_1 \cap V(H_u)$ (of cardinality $\leq M(H) - |S(u)|$) by an independent set in $H_u \setminus C_{1-\varphi(u)}(u)$ of cardinality $M(H)$.

The result of (i) and (ii) will be new independent set J' with $|J'| \geq |J|$. Moreover J' is φ-standard. This completes the proof that OPT is achieved on standard independent set of $f(I)$. Hence we have an affine dependence of $\mathrm{OPT}(f(I))$ on $\mathrm{OPT}(I)$ as described earlier.

Let us now check how the NP-hard gap of E$[k,k,k]$-Max-E3-Lin-2 is preserved. If an instance I of E$[k,k,k]$-Max-E3-Lin-2 has m variables as above, then $f(I)$ has $n := m|V(H)| + 16mk$ nodes, and $\mathrm{OPT}(f(I)) = mk(M(H)/k + 6 + \mathrm{OPT}(I))$ for 3-Max-IS; $n := m|V(H)| + 6mk$ nodes, and $\mathrm{OPT}(f(I)) = mk(M(H)/k + 1 + \mathrm{OPT}(I))$ for 4(5)-Max-IS; $n := m|V(H)| + 4mk$ nodes, and $\mathrm{OPT}(f(I)) = mk(M(H)/k + \mathrm{OPT}(I))$ for B-Max-IS, $B \geq 6$.

Hence the NP-hard question of whether $\mathrm{OPT}(I)$ is greater than $(1-\varepsilon)$, or less than $\left(\frac{1}{2}+\varepsilon\right)$ (but surely at least $\frac{1}{2}$) is transformed to NP-hard partial decision problem of whether

• for 3-Max-IS:

$$n\frac{2M(H)/k + 13 + 2\varepsilon}{2|V(H)|/k + 32} > \mathrm{OPT}(f(I)) \quad \text{or} \quad \mathrm{OPT}(f(I)) > n\frac{2M(H)/k + 14 - 2\varepsilon}{2|V(H)|/k + 32};$$

• for 4(5)-Max-IS:

$$n\frac{2M(H)/k + 3 + 2\varepsilon}{2|V(H)|/k + 12} > \mathrm{OPT}(f(I)) \quad \text{or} \quad \mathrm{OPT}(f(I)) > n\frac{2M(H)/k + 4 - 2\varepsilon}{2|V(H)|/k + 12};$$

• for B-Max-IS, $B \geq 6$:

$$n\frac{2M(H)/k+1+\varepsilon}{2|V(H)|/k+8} > \mathrm{OPT}(f(I)) \quad \text{or} \quad \mathrm{OPT}(f(I) > n\frac{2M(H)/k+2-2\varepsilon}{2|V(H)|/k+8}.$$

Consequently, it is NP-hard to approximate the solution of 3-Max-IS within $1+(1-4\varepsilon)/(2M(H)/k+13+2\varepsilon)$, 4(5)-Max-IS within $1+(1-4\varepsilon)/(2M(H)/k+3+2\varepsilon)$, B-Max-IS, $B \geq 6$ within $1+(1-4\varepsilon)/(2M(H)/k+1+2\varepsilon)$.

Passing to the complements one can state similar results for the Minimum Node Cover problem. Clearly, $\mathrm{OPT}_{\mathrm{nc}}(f(I)) = mk((|V(H)|-M(H))/k+10-\mathrm{OPT}(I))$ for 3-Min-NC; $\mathrm{OPT}_{\mathrm{nc}}(f(I)) = mk((|V(H)|-M(H))/k+5-\mathrm{OPT}(I))$ for 4(5)-Min-NC; $\mathrm{OPT}_{\mathrm{nc}}(f(I)) = mk((|V(H)|-M(H))/k+4-\mathrm{OPT}(I))$ for B-Min-NC, $B \geq 6$. So we get the NP-hardness of partial decision problem

• for 3-Min-NC:

$$n^{\frac{2[|V(H)|-M(H)]/k+18+2\varepsilon}{2|V(H)|/k+32}} > \mathrm{OPT} \quad \text{or} \quad \mathrm{OPT} > n^{\frac{2[|V(H)|-M(H)]/k+19-2\varepsilon}{2|V(H)|/k+32}},$$

• for 4(5)-Min-NC:

$$n^{\frac{2[|V(H)|-M(H)]/k+8+2\varepsilon}{2|V(H)|/k+12}} > \mathrm{OPT} \quad \text{or} \quad \mathrm{OPT} > n^{\frac{2[|V(H)|-M(H)]/k+9-2\varepsilon}{2|V(H)|/k+12}},$$

B-Min-NC, $B \geq 6$:

$$n^{\frac{2[|V(H)|-M(H)]/k+6+2\varepsilon}{2|V(H)|/k+8}} > \mathrm{OPT} \quad \text{or} \quad \mathrm{OPT} > n^{\frac{2[|V(H)|-M(H)]/k+7-2\varepsilon}{2|V(H)|/k+8}},$$

is NP-hard. Consequently, it is NP-hard to approximate the solution of 3-Min-NC within $1+(1-4\varepsilon)/[2(|V(H)|-M(H))/k+18+2\varepsilon]$; 4(5)-Min-NC within $1+(1-4\varepsilon)/[2(|V(H)|-M(H))/k+8+2\varepsilon]$; B-Min-NC ($B \geq 6$) within $1+(1-4\varepsilon)/[2(|V(H)|-M(H))/k+6+2\varepsilon]$.

The following main theorem summarizes the results

Theorem 3 *It is* NP-*hard to approximate: the solution of* 3-Max-IS *to within any constant smaller than* $1+1/(2\mu_3+13)$*; for* $B \in \{4,5\}$ *the solution of* B-Max-IS *to within any constant smaller than* $1+1/(2\mu_B+3)$*, the solution of* B-Max-IS*,* $B \geq 6$*, to within any constant smaller than* $1+1/(2\mu_B+1)$*. Similarly, it is* NP-*hard to approximate the solution of* 3-Min-NC *to within any constant smaller than* $1+1/(2\lambda_3+18)$*, for* $B \in \{4,5\}$ *the solution of* B-Min-NC *to within any constant smaller than* $1+1/(2\lambda_B+8)$*, the solution of* B-Min-NC*,* $B \geq 6$*, to within any constant smaller than* $1+1/(2\lambda_B+6)$.

Using our upper bounds given for μ_B, λ_B for distinct value of B we obtain the following corollary

Corollary 1 *It is* NP-*hard to approximate the solution of* 3-Max-IS *to within* 1.010661 ($> \frac{95}{94}$)*; the solution of* 4-Max-IS *to within* 1.0215517 ($> \frac{48}{47}$)*, the solution of* 5-Max-IS *to within* 1.0225225 ($> \frac{46}{45}$) *and the solution of* B-Max-IS*,* $B \geq 6$ *to within* 1.0235849 ($> \frac{44}{43}$)*. Similarly, it is* NP-*hard to approximate the solution of* 3-Min-NC *to within* 1.0101215 ($> \frac{100}{99}$)*; the solution of* 4-Min-NC *to within* 1.0194553 ($> \frac{53}{52}$)*; the solution of* 5-Min-NC *to within* 1.0202429 ($> \frac{51}{50}$) *and* B-Min-NC*,* $B \geq 6$*, to within* 1.021097 ($> \frac{49}{48}$)*. For each* $B \geq 3$*, the corresponding result applies to* B*-regular graphs as well.*

5 Asymptotic Approximability Bounds

This paper is focused mainly on graphs of very small degree. In this section we discuss also the asymptotic relation between hardness of approximation and degree for INDEPENDENT SET and NODE COVER problem in bounded degree graphs.

For the INDEPENDENT SET problem in the class of graphs of maximum degree B the problem is known to be approximable with performance ratio arbitrarily close to $\frac{B+3}{5}$ (Berman & Fujito, [2]). But asymptotically better ratios can be achieved by polynomial algorithms, currently the best one approximates to within a factor of $O(B \log\log B/\log B)$, as follows from [1], [11]. On the other hand, Trevisan [13] has proved NP-hardness to approximate the solution to within $B/2^{O(\sqrt{\log B})}$.

For the NODE COVER problem the situation is more challenging, even in general graphs. A recent result of Dinur and Safra [8] shows that for any $\delta > 0$ the MINIMUM NODE COVER problem is NP-hard to approximate to within $10\sqrt{5} - 21 - \delta$. One can observe that their proof can give hardness result also for graphs with (very large) bounded degree $B(\delta)$. This follows from the fact that after their use of Raz's parallel repetition (where each variable appears in only a constant number of tests), the degree of produced instances is bounded by a function of δ. But the dependence of $B(\delta)$ on δ in their proof is really very complicated. The earlier $\frac{7}{6} - \delta$ lower bound proved by Håstad [9] was extended by Clementi & Trevisan [7] to graphs with bounded degree $B(\delta)$.

Our next result improve on their; it has better trade-off between non-approximability and the degree bound. There are no hidden constants in our asymptotic formula, and it provides good explicit inapproximability results for degree bound B starting from few hundreds. First we need to introduce some notation.

Notation. Denote $F(x) := -x\log x - (1-x)\log(1-x)$, $x \in (0,1)$, where log means the natural logarithm. Further, $G(c,t) := (F(t) + F(ct))/(F(t) - ctF(\frac{1}{c}))$ for $0 < t < \frac{1}{c} < 1$, $g(t) := G(\frac{1-t}{t}, t)$ for $t \in (0, \frac{1}{2})$. More explicitly, $g(t) = 2[-t\log t - (1-t)\log(1-t)]/[-2(1-t)\log(1-t) + (1-2t)\log(1-2t)]$. Using Taylor series of the logarithm near 1 we see that the denominator here is $t^2 \cdot \sum_{k=0}^{\infty} \frac{2^{k+2}-2}{(k+1)(k+2)} t^k > t^2$, and $-(1-t)\log(1-t) = t - t^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} t^k < t$, consequently $g(t) < \frac{2}{t}(1 + \log\frac{1}{t})$.

For large enough B we look for $\delta \in (0, \frac{1}{6})$ such that $3\lfloor g(\frac{\delta}{2})\rfloor + 3 \leq B$. As $g(\frac{1}{12}) \approx 75.62$ and g is decreasing in $(0, \frac{1}{12})$, we can see that for $B \geq 228$ any $\delta > \delta_B := 2g^{-1}(\lfloor\frac{B}{3}\rfloor)$ will do. Trivial estimates on δ_B (using $g(t) < \frac{2}{t}(1 + \log\frac{1}{t})$) are $\delta_B < \frac{12}{B-3}(\log(B-3) + 1 - \log 6) < \frac{12\log B}{B}$.

We will need the following lemma about regular bipartite expanders to prove the Theorem 4.

Lemma 1 *Let $t \in (0, \frac{1}{2})$ and d be an integer for which $d > g(t)$. For every sufficiently large positive integer n there is a d-regular n by n bipartite graph H with bipartition (V_0, V_1), such that for each independent set J in H either $|J \cap V_0| \leq tn$, or $|J \cap V_1| \leq tn$.*

Sketch. In the standard model of random d-regular bipartite graphs it is well known (and easy to prove) that the conditions $0 < t < \frac{1}{c} < 1$ and $d > G(c,t)$ are sufficient for existence, for every sufficiently large n, of d-regular bipartite graph with n by n bipartition (V_0, V_1), which is (c,t,d)-expander (i.e., $U \subseteq V_0$ or $U \subseteq V_1$, and $|U| \le tn$ imply $|\Gamma(U)| \ge c|U|$; here $\Gamma(U) := \{y\colon\ y$ is a node adjacent to some $x \in U\}$) (see e.g. Theorem 6.6 in [6] for this result). If $d > g(t)$ $(= G(\frac{1-t}{t}, t))$, by continuity of G also $d > G(c,t)$ for some $c > \frac{1-t}{t}$. So with these parameters (c,t,d)-expanders exist for n sufficiently large, and they clearly have the required property. $\square$

Theorem 4 *For every* $\delta \in (0, \frac{1}{6})$ *it is* NP*-hard to approximate* MINIMUM NODE COVER *to within* $\frac{7}{6} - \delta$ *even in graphs of maximum degree* $\le 3\lfloor g(\frac{\delta}{2})\rfloor + 3 \le 3\lceil \frac{4}{\delta}(1 + \log \frac{2}{\delta})\rceil$. *Consequently, for any* $B \ge 228$ *it is* NP*-hard to approximate* B-MIN-NC *to within any constant smaller than* $\frac{7}{6} - \delta_B$*, where* $\delta_B := 2g^{-1}(\lfloor \frac{B}{3} \rfloor) < \frac{12}{B-3}(\log(B-3) + 1 - \log 6) < \frac{12 \log B}{B}$.

Proof. Let $\delta \in (0, \frac{1}{6})$ be given, put $d := \lfloor g(\frac{\delta}{2})\rfloor + 1$. Then we choose $t \in (0, \frac{\delta}{2})$ so close to $\frac{\delta}{2}$ that $d > g(t)$. Further we choose $\varepsilon \in (0, \frac{1}{4})$ such that $(\frac{7}{2} - \varepsilon - 6t)/(3 + \varepsilon) > \frac{7}{6} - \delta$. Then a positive integer k is chosen so large that

- (i) NP-hard gap $\langle \frac{1}{2} + \varepsilon, 1 - \varepsilon \rangle$ of Theorem 1 applies to the problem Ek-MAX-E3-LIN-2, and
- (ii) there is d-regular $2k$ by $2k$ bipartite graph H with bipartition (V_0, V_1), such that for each independent set J in H either $|J \cap V_0| \le 2kt$, or $|J \cap V_1| \le 2kt$ (see Lemma 1). Keep one such H fixed from now on.

We will describe reduction f from Ek-MAX-E3-LIN-2 to graphs and will check how the NP-hard gap of (i) is preserved for MIN-NC problem.

Let I be an instance of Ek-MAX-E3-LIN-2, $\mathcal{V}(I)$ be the set of variables of I, and $m := |\mathcal{V}(I)|$. Clearly, the system I has $\frac{mk}{3}$ equations. For each equation of I we take a quadruple of labeled nodes. More precisely, if the equation reads as $x + y + z = j$ $(j \in \{0,1\})$ we take 4 nodes with labels $\boxed{xyz = 00j}$, $\boxed{xyz = 01(1-j)}$, $\boxed{xyz = 10(1-j)}$ and $\boxed{xyz = 11j}$. Notice, that nodes correspond to all partial assignments to variables making the equation satisfied. Denote by G_I the graph whose node set consists of the union of nodes of those $\frac{mk}{3}$ quadruples, with an edge added for each pair of inconsistently labeled nodes. The pair of nodes is inconsistent if a variable $u \in \mathcal{V}(I)$ exists that is assigned differently in their labels. It is clear that independent sets in G_I correspond to subsets of I satisfied by an assignment to variables. Consequently, $\alpha(G_I) = \frac{mk}{3}\mathrm{OPT}(I)$. (Here $\alpha(\cdot)$ stands for the cardinality of maximum independent set.) Clearly, the hard gap of (i) is preserved for MAX-IS problem and translates to another one for the problem MIN-NC for graphs G_I.

Using our fixed expander H we can enforce similar preserving of that NP-hard gap even in graphs of maximum degree $\le 3d$.

Consider a variable $u \in \mathcal{V}(I)$. Let $V_j(u)$ ($j \in \{0,1\}$) be the set of all $2k$ nodes in which u has assigned bit j. Choose any bijection between $V_0(u)$ and V_0 (of H), and between $V_1(u)$ and V_1 (of H). Now take edges between $V_0(u)$, $V_1(u)$ exactly as prescribed by our expander H. Having this done, one after another, for each $u \in \mathcal{V}(I)$, we get the graph $G_I^H =: f(I)$. Clearly, the transformation f is polynomial, and the maximum degree of G_I^H is at most $3d$.

Any independent set in G_I is an independent set also in G_I^H, hence $\alpha(G_I^H) \geq \alpha(G_I) = \frac{mk}{3}\mathrm{OPT}(I)$ and $nc(G_I^H) \leq nc(G_I) = \frac{mk}{3}(4 - \mathrm{OPT}(I))$.

On the other hand, one can show that $\alpha(G_I^H) \leq \alpha(G_I) + 2kmt$ as follows: Consider an independent set J of G_I^H with $|J| = \alpha(G_I^H)$. For each $u \in \mathcal{V}(I)$, one after another, remove exactly one of sets $J \cap V_0(u)$, $J \cap V_1(u)$ from J, namely the one with cardinality $\leq 2kt$. (The existence is ensured by properties of our expander H, and the way how G_I^H was created.) Having this done for all $u \in \mathcal{V}(I)$ we get an independent set of G_I (hence of size $\leq \alpha(G_I)$), removing no more than $2kmt$ nodes. Hence $\alpha(G_I^H) \leq \alpha(G_I) + 2kmt = \frac{mk}{3}(\mathrm{OPT}(I) + 6t)$, and $nc(G_I^H) \geq \frac{mk}{3}(4 - \mathrm{OPT}(I) - 6t)$. Hence NP-hard question of whether $\mathrm{OPT}(I)$ is greater than $(1-\varepsilon)$, or less than $(\frac{1}{2}+\varepsilon)$, is transformed to the one of whether $nc(G_I^H)$ is less than $\frac{mk}{3}(3+\varepsilon)$, or greater than $\frac{mk}{3}(\frac{7}{2} - \varepsilon - 6t)$. Consequently, it is NP-hard to approximate MIN-NC to within $(\frac{7}{2} - \varepsilon - 6t)/(3+\varepsilon) > \frac{7}{6} - \delta$ on instances G_I^H of maximum degree $\leq 3d$.

The consequence about inapproximability of B-MIN-NC is straightforward. □

Typically, the methods used for asymptotic results cannot be used for small values of B to achieve interesting lower bounds. Therefore we work on new techniques that improve the results of Berman & Karpinski [3] and Chlebík & Chlebíková [4].

6 MAX-3DM and Other Problems

Clearly, the restriction of B-MAX-IS problem to edge-B-colored B-regular graphs is a subproblem of MAXIMUM B-DIMENSIONAL MATCHING (see [4] for more details). Hence we want to prove that our reduction to B-MAX-IS problem can produce as instances edge-B-colored B-regular graphs. In this contribution we present results for $B = 3, 4$. For the equation $x + y + z = j$ ($j \in \{0,1\}$) of E$[k,k,k]$-MAX-E3-LIN-2 we will use an equation gadget $G_j[B]$, see Fig. 1 and Fig. 2(i). The basic properties of these gadgets are described in Theorem 2.

Maximum 3-Dimensional Matching

As follows from Fig. 1 a gadget $G_0[3]$ can be edge-3-colored by colors a, b, c in such way that all edges adjacent to nodes of degree one (contacts) are colored by one fixed color, say a (for $G_1[3]$ we take the corresponding analogy). As an amplifier of our reduction $f = f_H$ from E$[k,k,k]$-MAX-E3-LIN-2 to MAX-3DM we use a consistency $3k$-amplifier $H \in \mathcal{G}_{3,k}$ with some additional properties: degree of any contact node is exactly 2, degree of any other node is

3 and moreover, a graph H is an edge-3-colorable by colors a, b, c in such way that all edges adjacent to contact nodes are colored by two colors b and c. Let $\mathcal{G}_{3\mathrm{DM},k} \subseteq \mathcal{G}_{3,k}$ be the class of all such amplifiers. Denote $\mu_{3\mathrm{DM},k} = \min\{\frac{M(H)}{k} : H \in \mathcal{G}_{3\mathrm{DM},k}\}$ and $\mu_{3\mathrm{DM}} := \underline{lim}_{k\to\infty}\mu_{3\mathrm{DM},k}$.

We use the same construction for consistency $3k$-amplifiers as was presented for 3-Max-IS, but now we have to show that produced graph H fulfills conditions about coloring of edges. For fixed $(3k, \tau)$-amplifier G and the matching $\mathcal{M} \subseteq E(G)$ of nodes $V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$ we define edge coloring in two steps: (i) Take preliminary the following edge coloring: for each $\{x, y\} \in \mathcal{M}$ we color the corresponding edges in H as depicted on Fig. 3(i). The remaining edges of H are easily 2-colored by colors b and c, as the rest of the graph is bipartite and of degree at most 2. So, we have a proper edge-3-coloring but some edges adjacent to contacts are colored by color a. It will happen exactly if $x \in \{x^1, x^2, \ldots, x^{2k}\}$, $\{x, y\} \in \mathcal{M}$. (We assume that no two contacts of G are adjacent, hence y is a checker node of G.) Clearly, one can ensure that in the above extension of coloring of edges by colors c and b both other edges adjacent to x_0 and x_1 have the same color. (ii) Now we modify our edge coloring in all these violating cases as follows. Fix $x \in \{x^1, \ldots, x^{2k}\}$, $\{x, y\} \in \mathcal{M}$, and let both other edges adjacent to x_0 and x_1 have assigned color b. Then change coloring according Fig. 3(ii). The case when both edges have assigned color c, can be solved analogously (see Fig. 3(iii)). From the construction follows $\mu_{3\mathrm{DM}} \leq 40.4$.

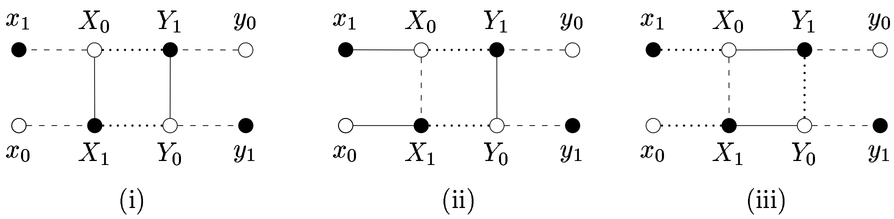

Figure 3: a color: dashed line, b color: dotted line, c color: full line

Keeping one such consistency $3k$-gadget H fixed, our reduction f $(= f_H)$ reduction from E$[k, k, k]$-Max-E3-Lin-2 is exactly the same as for B-Max-IS described in Section 3. Let us fix an instance I of E$[k, k, k]$-Max-E3-Lin-2 and consider an instance $f(I)$ of 3-Max-IS. As $f(I)$ is edge 3-colored 3-regular graph, it is at the same time an instance of 3DM with the same objective function. We can show how the NP-hard gap of E$[k, k, k]$-Max-E3-Lin-2 is preserved exactly in the same way as for 3-Max-IS. Consequently it is NP-hard to approximate the solution of Max-3DM to within $1+(1-4\varepsilon)(\frac{2M(H)}{k}+13+2\varepsilon)$, even on instances with each element occurring in exactly two triples.

Maximum 4-Dimensional Matching

We will use the following edge-4-coloring of our gadget $G_0[4]$ in Fig. 2(i) (analogously for $G_1[4]$): a-colored edges $\{x'_0, \boxed{101}\}$, $\{x'_1, \boxed{011}\}$, $\{y_1, \boxed{000}\}$, $\{y_0, \boxed{110}\}$;

b-colored edges $\{x'_0, \boxed{110}\}$, $\{x'_1, \boxed{000}\}$, $\{y_1, \boxed{101}\}$, $\{y_0, \boxed{011}\}$; c-colored edges $\{x_1, x'_0\}$, $\{x_0, x'_1\}$, $\{\boxed{101}, \boxed{110}\}$, $\{z_0, \boxed{011}\}$, $\{z_1, \boxed{000}\}$; d-colored edges $\{x'_0, x'_1\}$, $\{\boxed{000}, \boxed{011}\}$, $\{z_0, \boxed{101}\}$, $\{z_1, \boxed{110}\}$. Now we will show that an edge-4-coloring of a consistency $3k$-amplifier H exists that fit well with the above coloring of equation gadgets. We suppose that the $(3k, \tau)$-amplifier G from which H was constructed has a matching $\mathcal{M}$ of all checkers. (This is true for amplifiers of [3] and [4]). The color d will be used for edges $\{x_0, x_1\}$, $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$. Also, for any $x \in \{x^{k+1}, \ldots, x^{2k}\}$, the corresponding $\{X_0, X_1\}$ edge will have color d too. The color c will be reserved for coloring edges of H "along the matching $\mathcal{M}$", i.e. if $\{x, y\} \in \mathcal{M}$, edges $\{x_0, y_1\}$ and $\{x_1, y_0\}$ have color c. Furthermore, for $x \in \{x^{k+1}, \ldots, x^{2k}\}$ the corresponding edges $\{x_0, X_1\}$ and $\{x_1, X_0\}$ will be of color c too. The edges that are not colored by c and d form a 2-regular bipartite graph, hence they can be edge 2-colored by colors a and b. The above edge 4-coloring of H and $G_j[4]$ ($j \in \{0, 1\}$) ensures that instances produced in our reduction to 4-MAX-IS are edge-4-colored 4-regular graphs.

The following theorem summarizes both results for MAX-3DM and MAX-4DM:

Theorem 5 *It is* NP*-hard to approximate the solution of* MAX-3DM *to within any constant smaller than* $1 + 1/(2\mu_{3\mathrm{DM}} + 13) > 1.010661 > \frac{95}{94}$*, and the solution of* MAX-4-DM *to within* 1.0215517 $(> \frac{48}{47})$*. The both inapproximability results hold also on instances with each element occurring in exactly two triples, resp. quadruples.*

Lower bound for MIN-B-SET COVER follows from that of B-MIN-NC, as explained in Introduction. It is also easy to see that instances obtained by our reduction for 3-MAX-IS are 3-regular triangle-free graphs. Hence, we get the same lower bound for MAXIMUM TRIANGLE PACKING by simple reduction (see [4] for more details).

Theorem 6 *It is* NP*-hard to approximate the solution of the problems* MAXIMUM TRIANGLE PACKING *(even on 4-regular line graphs) to within any constant smaller than* $1 + \frac{1}{2\mu_3 + 13} > 1.010661 > \frac{95}{94}$*,* MIN-3-SET COVER *with exactly two occurrences of each elements to within any constant smaller than* $1 + \frac{1}{2\lambda_3 + 13} > 1.0101215 > \frac{100}{99}$*; and* MIN-4-SET COVER *with exactly two occurrences of each elements to within any constant smaller than* $1 + \frac{1}{2\lambda_4 + 8} > 1.0194553 > \frac{53}{52}$*.*

Conclusion remarks

A plausible direction to improve further our inapproximability results is to give better upper bounds on parameters λ_B, μ_B. We think that there is still a potential for improvement here, using a suitable probabilistic model for the construction of amplifiers.

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