

# RP is Small in SUBEXP else ZPP equals PSPACE and NP equals EXP

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## Abstract

We use recent results from [ABK<sup>+</sup>02] on the hardness of resource-bounded Kolmogorov random strings, to prove that RP is small in SUBEXP else ZPP = PSPACE and NP = EXP. Along the way we improve a result from [IM03] by reducing the time bounds from exponential to subexponential. More precisely we show that if NP is not small in SUBEXP, then NP = AM.

## 1 Introduction

An interesting question in Lutz's resource-bounded measure theory is whether the zero-one law holds within EXP, i.e. does every complexity class contained in EXP have either measure zero or one. Although the answer to this question is currently not known, it has been shown in [Mel00] and [IM03] that the zero-one law holds for all three standard probabilistic classes ZPP, RP and BPP, for Lutz's resource-bounded measure in E. These results can be interpreted as "it is likely that RP has measure zero in E, otherwise the unlikely ZPP = EXP is true".

What happens if the resource time bound of the martingales are reduced, by considering for instance the measure of [AS94] in the smaller class SUBEXP instead of E. Unfortunately the proofs of the zero-one laws in [Mel00] and [IM03] do not translate to smaller time bounds. For instance the proof in [Mel00] relies heavily on the following result from [IW98]: if BPP  $\neq$  EXP, then every BPP algorithm can be approximated on infinitely many input lengths, by a subexponential time algorithm, which errs only on a small fraction of inputs. Using this result, a martingale succeeding on every BPP language can be constructed. The point is that this martingale needs to bet on *every* string, to ensure that the small fraction of strings on which the subexponential time approximation algorithm errs is too small to prevent the martingale from winning. Since martingales with small time bounds cannot bet on all strings, the proof of [Mel00] does not carry over to measure on small complexity classes.

In this paper we take another path to give some evidences that RP has measure zero in SUBEXP. We use recent results from [ABK<sup>+</sup>02] on the power of Kolmogorov random strings to prove that RP has measure zero in SUBEXP, otherwise ZPP = PSPACE and NP = EXP.

We also improve a result from [IM03], by reducing the power of the martingales from exponential to subexponential time. More precisely we prove that if NP is not small in SUBEXP, then Arthur-Merlin can be fully derandomized, thus gaining more evidence that NP = AM.

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## 2 Preliminaries

We use standard notation for traditional complexity classes; see for instance [BDG95] and [BDG90], or [Pap94]. For  $\epsilon > 0$ , denote by  $\mathbf{E}_\epsilon$  the class  $\mathbf{E}_\epsilon = \bigcup_{\delta < \epsilon} \text{DTIME}(2^{n^\delta})$ .  $\text{SUBEXP}$  is the class  $\bigcap_{\epsilon > 0} \mathbf{E}_\epsilon$ . Let us fix some notations for strings and languages. Let  $s_0, s_1, \dots$  be the standard enumeration of the strings in  $\{0, 1\}^*$  in lexicographical order, where  $s_0 = \lambda$  denotes the empty string. Denote by  $s_0^{(n)}, s_1^{(n)}, \dots, s_{2^n-1}^{(n)}$  all strings of size  $n$  ordered lexicographically. A sequence is an element of  $\{0, 1\}^\infty$ . If  $w$  is a string or a sequence and  $0 \leq i < |w|$  then  $w[i]$  and  $w[s_i]$  denotes the  $i$ th bit of  $w$ . Similarly  $w[i \dots j]$  and  $w[s_i \dots s_j]$  denote the  $i$ th through  $j$ th bits. We identify language  $L$  with its characteristic function  $\chi_L$ , where  $\chi_L$  is the sequence such that  $\chi_L[i] = 1$  iff  $s_i \in L$ . If  $w_1$  is a string and  $w_2$  is a string or a sequence extending  $w_1$ , we write  $w_1 \sqsubseteq w_2$ . Denote by  $L_{=n}^{(k)} = L \cap \{s_0^{(n)}, \dots, s_{k-1}^{(n)}\}$ . A language  $L$  is said polynomially dense if there exists a polynomial  $p$ , such that  $|L_{=n}| \geq 2^n/p(n)$ , where  $L_{=n}$  denotes the set of strings of size  $n$  contained in  $L$ .

### 2.1 Pseudorandom Generators

The hardness of a generator is the size of the smallest circuit which can distinguish the output of the generator from truly random bits. More precisely,

**Definition 1** *Let  $A$  be any language. The hardness  $H^A(G_{m,n})$  of a random generator  $G_{m,n} : \{0, 1\}^m \rightarrow \{0, 1\}^n$ , is defined as the minimal  $s$  such that there exists an  $n$ -input circuit  $C$  with oracle gates to  $A$ , of size at most  $s$ , for which:*

$$\left| \Pr_{x \in \{0,1\}^m} [C(G_m(x)) = 1] - \Pr_{y \in \{0,1\}^n} [C(y) = 1] \right| \geq \frac{1}{s}.$$

The  $A$ -oracle circuit complexity of a Boolean function  $f$ , denoted  $\text{Size}^A(f)$  is defined as the size of the smallest circuit computing  $f$ . It was discovered in [KvM99] that the construction of pseudorandom generator from high circuit complexity Boolean functions does relativize.

**Theorem 1 (Klivans-Melkebeek)** *Let  $A$  be any language. There is a polynomial-time computable function  $F : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$ , with the following properties. For every  $\epsilon > 0$ , there exists  $a, b \in \mathbb{N}$  such that*

$$F : \{0, 1\}^{n^a} \times \{0, 1\}^{b \log n} \rightarrow \{0, 1\}^n,$$

*and if  $r$  is the truth table of a  $(a \log n)$ -variables Boolean function of  $A$ -oracle circuit complexity at least  $n^{\epsilon a}$ , then the function  $G_r(s) = F(r, s)$  is a generator, mapping  $\{0, 1\}^{b \log n}$  into  $\{0, 1\}^n$ , which has hardness  $H^A(G_r) > n$ .*

### 2.2 Measure on SUBEXP

In this section we describe the fragment of Lutz's measure theory that we will need. For a more detailed presentation of this theory we refer the reader to the survey by Lutz [Lut97].

The measure on resource-bounded complexity classes is obtained by imposing appropriate resource-bound on a game theoretical characterization of the classical Lebesgue measure.

A martingale is a function  $d : \{0, 1\}^* \rightarrow [0, \infty[$  such that,

$$d(w) = \frac{d(w0) + d(w1)}{2} \tag{1}$$

for every  $w \in \{0, 1\}^*$ .

This definition can be motivated by the following betting game in which a gambler puts bets on the successive membership bits of a hidden language  $A$ . The game proceeds in infinitely many rounds where at the end of round  $n$ , it is revealed to the gambler whether  $s_n \in A$  or not. The game starts with capital 1. Then, in round  $n$ , depending on the first  $n - 1$  outcomes  $w = \chi_A[0 \dots n - 1]$ , the gambler bets a certain fraction  $\epsilon_w d(w)$  of his current capital  $d(w)$ , that the  $n$ th word  $s_n \in A$ , and bets the remaining capital  $(1 - \epsilon_w)d(w)$  on the complementary event  $s_n \notin A$ . The game is fair, i.e. the amount put on the correct event is doubled, the one put on the wrong guess is lost. The value of  $d(w)$ , where  $w = \chi_A[0 \dots n]$  equals the capital of the gambler after round  $n$  on language  $A$ . The player wins on a language  $A$  if he manages to make his capital arbitrarily large during the game. We say that a martingale  $d$  succeeds on a language  $A$ , if  $d(A) := \limsup_{w \sqsubseteq A, w \rightarrow A} d(w) = \infty$ , where we identify language  $A$  with its characteristic sequence  $\chi_A$ . The success set  $S^\infty[d]$  of a martingale  $d$  is the class of all languages on which  $d$  succeeds. Strategies satisfying Equation 1 with the equality sign replaced by " $\geq$ " are called supermartingales.

In [AS94], Lutz's resource-bounded measure theory [Lut92] was generalized to small complexity classes. For the sake of completeness, let us recall the definitions of [AS94] adapted to the class SUBEXP. To define a measure on  $E_\alpha$ , with  $\alpha > 0$ , supermartingales computed by Turing machines with random access to their inputs were considered in [AS94], that is on input  $w$ , the machine can query any bit of  $w$  to its oracle. In order to allow such Turing machines to compute the lengths of their inputs  $w$  without querying their oracles, they also have access to  $s_{|w|}$ ; this convention is denoted by  $M^w(s_{|w|})$ .

It is widely believed that random access subexponential time Turing machines are too strong to define a measure on subexponential time classes, because it is not clear whether the whole class has not measure zero relatively to subexponential-time computed supermartingales. Therefore the concept was weakened in [AS94] by bounding the number of recursive queries such a Turing machine is allowed to make. Let  $M$  be a random access Turing machine and  $n \in \mathbb{N}$ . Define the query set  $G_{M,n} \subseteq \{1, 2, \dots, 2^{n+1} - 1\}$  such that for every string  $w \in \{0, 1\}^*$  coding for words of size up to  $n$ ,  $M$  can compute  $M^w(s_{|w|})$  querying only input bits in  $G_{M,n}$ .

A measure notion on SUBEXP was introduced in [AS94] by considering subexponential-time random access Turing machines with subexponential size query sets.

**Definition 2** *Let  $\alpha > 0$ . A class  $A$  of languages is said to have  $E_\alpha$ -measure zero if there exists a supermartingale  $d$ , such that  $d(w)$  is computable in time  $2^{n^\delta}$ , with  $0 < \delta < \alpha$  and  $n = |s_{|w|}|$ , by a random access Turing machine with query set of size  $2^{n^\delta}$ , and such that  $A \subseteq S^\infty[d]$ .*

Such a supermartingale is called an  $E_\alpha$ -computable supermartingale.

It is implicit in [AS94] that this measure notion satisfies all three measure axioms; i.e. first every single language  $L \in E_\alpha$  has  $E_\alpha$ -measure zero, second  $E_\alpha$  has not  $E_\alpha$ -measure zero, and "easy" infinite unions of sets of  $E_\alpha$ -measure zero have  $E_\alpha$ -measure zero. More precisely,

**Definition 3** *Let  $\alpha > 0$ .  $X = \bigcup_{i \geq 1} X_i$  is an  $E_\alpha$ -union of  $E_\alpha$ -measure zero sets if there exists an indexed supermartingale  $d$  such that  $X_i \subseteq S^\infty[d_i]$ , where  $d_i(w) = d(i, w)$  is computable in time  $2^{n^\delta} + q(i)$ , for some polynomial  $q$ , with  $0 < \delta < \alpha$  and  $n = |s_{|w|}|$ , by a random access Turing machine with query set of size  $2^{n^\delta}$ .*

It is implicit in [AS94] that the third axiom holds for the measure notion on  $E_\alpha$ .

**Theorem 2** [AS94] *Let  $X = \bigcup_{i \geq 1} X_i$  be an  $E_\alpha$ -union of  $E_\alpha$ -measure zero sets. Then  $X$  has  $E_\alpha$ -measure zero.*

### 2.3 Resource-Bounded Kolmogorov Complexity

Let us give the basic notions on resource-bounded Kolmogorov complexity that we will need.

**Definition 4** (Levin) *Let  $U$  be a universal Turing machine. Define  $K_t(x)$  to be  $\min\{|d| + \log t : U(d) = x \text{ in at most } t \text{ steps}\}$ .*

There is a similar definition of space-bounded Kolmogorov complexity.

**Definition 5** *Let  $U$  be a universal Turing machine. Define  $KS(x)$  as  $\min\{|d| + s : U(d) = x \text{ in space at most } s\}$ .*

It was shown in [ABK<sup>+</sup>02], that any polynomially dense set of strings with high time-bounded Kolmogorov complexity is complete for EXP under nondeterministic Turing reductions.

**Theorem 3** *Let  $R \in \text{EXP}$  be a polynomially dense set and  $0 < \delta < 1$  such that  $r \in R$  implies  $K_t(r) \geq |r|^\delta$ . Then  $\text{NP}^R = \text{EXP}$ .*

A similar result for space-bounded Kolmogorov complexity was also shown in [ABK<sup>+</sup>02].

**Theorem 4** *Let  $R \in \text{PSPACE}$  be a polynomially dense set and  $0 < \delta < 1$  such that  $r \in R$  implies  $KS(r) \geq |r|^\delta$ . Then  $\text{ZPP}^R = \text{PSPACE}$ .*

## 3 RP is small else $\text{RP} = \text{PSPACE}$ and $\text{NP} = \text{EXP}$

To prove our main result we will need the following theorem which essentially says that if RP is not small in  $E_\epsilon$ , then P contains a dense set of strings with great Kolmogorov complexity.

**Theorem 5** *Suppose RP has  $E_\epsilon$ -measure non zero, for some  $\epsilon > 0$ , then there exists a polynomially dense set  $R \in \text{P}$  and  $0 < \delta < 1$ , such that for every  $r \in R$ ,  $K_t(r) > |r|^\delta$  and  $KS(r) > |r|^\delta$ .*

*Proof.* To prove Theorem 5, we will need the following lemma.

**Lemma 1** *If RP has  $E_\epsilon$ -measure non-zero, then there exists  $0 < \gamma < 1$ , and a language  $L$  in RP, and a probabilistic polynomial time Turing machine  $M$  deciding  $L$  such that: For all but finitely many  $m \in \mathbb{N}$ ,  $L_{=m}^{(m)}$  is non-empty and for all but finitely many  $m \in \mathbb{N}$ , for every  $x \in L_{=m}^{(m)}$ , every random string  $t$  such that  $M(x, t) = 1$  has great Kolmogorov complexity, i.e.  $K_t(t) > |x|^\gamma$  and  $KS(t) > |x|^\gamma$ .*

*Proof.* Consider the sum of the following two martingales  $d_1$  and  $d_2$  (enforcing the two conditions above). For every string's size  $n$ ,  $d_1$  only bets on the  $n$  strings  $s_0^{(n)}$  to  $s_{n-1}^{(n)}$ . On input  $(w, s_{|w|})$  where  $s_{|w|} = s_i^{(n)}$  with  $i < n$ ,  $d_1$  checks that there is no element of  $L$  in the

range  $s_0^{(n)}$  to  $s_{i-1}^{(n)}$ , and if so  $d_1$  wagers  $2^i/2^n$  that  $s_i^{(n)} \notin L$ , else  $d_1$  stops betting on  $n$ -sized strings. If  $s_i^{(n)}$  is the first element of  $L$  in the range,  $d_1$  loses

$$\left| \sum_{j=0}^{i-1} \frac{2^j}{2^n} - \frac{2^i}{2^n} \right| = 1/2^n.$$

Since  $\sum_{n \geq 0} 1/2^n$  converges, by starting with a finite amount of money we ensure that  $d_1$  never goes broke, and its total losses are finite. On the other hand, each time there is no  $x \in L$  between  $s_0^{(n)}$  and  $s_{n-1}^{(n)}$ ,  $d_1$  wins  $2^n/2^n = 1$ . Thus, if this happens infinitely often,  $d_1$ 's wins grow arbitrarily large. It is easy to see that  $d_1$  can be computed in  $2^{n^{\epsilon/4}}$  steps, and that the query set is polynomial-sized.

The second martingale  $d_2$  only bets on the  $n$  first  $n$ -sized strings  $s_0^{(n)}$  to  $s_{n-1}^{(n)}$ , for every length  $n$ . Fix an enumeration of polytime probabilistic Turing machines so that  $M_i$  always halts within  $n^j$  steps where  $j \leq \log i$ , and a deterministic universal Turing machine  $U$ . On input  $(w, s_{|w|})$  where  $s_{|w|} \in \{s_0^{(n)}, \dots, s_{n-1}^{(n)}\}$ ,  $d_2$  simulates the first  $\log n$  probabilistic machines on all random strings for inputs of length at most  $\log n$ . Let  $M_i$  be the first such machine that agrees with  $L$  on all such inputs, making errors only when  $x \in L$ , and then on less than  $1/4$  of its random strings, i.e. decides  $L$  correctly (RP-wise) on all inputs smaller than  $\log n$ .  $d_2$  simulates  $U$  on all programs  $d$  of size smaller than  $n^\gamma$ , during  $2^{n^\gamma}$  steps, where  $0 < \gamma < 1$  will be determined later. For every string  $t$  output during  $U$ 's simulation, check whether  $M_i(s_{|w|}, t) = 1$ . If there is such a  $t$ ,  $d_2$  bets half its current capital that  $s_{|w|} \in L$ . Otherwise,  $d_2$  does not bet.

**Claim 1** *Let  $L \in \text{RP}$ . Let  $M_{i_0}$  be the first probabilistic poly-time machine deciding  $L$ . Then if there are infinitely many lengths  $m$ , such that there exists  $x \in L_{=m}^{(m)}$  and a random string  $t$  with  $K_t(t) < |x|^\gamma$ , such that  $M_{i_0}(x, t) = 1$ , then  $L \in S^\infty[d_2]$ .*

After a finite amount of time  $d_2$  will always pick  $i = i_0$ . Thereafter for each length  $m$  such that there exists  $x \in L_{=m}^{(m)}$  and a random string  $t$  with low Kolmogorov complexity, such that  $M_{i_0}(x, t) = 1$ ,  $d_2$  will always bet half its current capital that  $x \in L$ . Since  $M_{i_0}$  has one-sided error, this bet will be correct, and  $d_2$ 's total stake will be multiplied by  $3/2$ . Otherwise,  $d_2$  does not bet, so will never lose (once  $i = i_0$ ). So if there are infinitely many such lengths  $m$ ,  $d_2$ 's capital grows unbounded.

The running time of  $d_2$  is less than  $2^{n^{\epsilon/4}}$ , since on input  $(w, s_{|w|})$  as above, there are  $\log n$  machines to simulate on all inputs of size  $\log n$ , where each machine runs in time smaller than  $(\log n)^{\log \log n}$ , i.e. takes time  $2^{(\log n)^{\log \log n}}$  to be simulated by trying all random seeds and taking a majority vote. Next  $d_2$  needs to simulate  $U$  on  $2^{n^\gamma}$  programs during  $2^{n^\gamma}$  steps. Thus  $d_2(w)$  can be computed in time  $2^{n^{\epsilon/4}}$ , by an appropriate choice of  $\gamma$ , with  $0 < \gamma < 1$ . The query set of  $d_2$  is equal to  $Q_{d_2}(n) = \{s_0^{(n)}, \dots, s_{n-1}^{(n)}\} \cup \bigcup_{j=1}^{\log n} L_{=j}$ , hence polynomial-sized.

This proves Lemma 1, since if RP does not have  $E_\epsilon$ -measure zero, there must be a language  $L \in \text{RP}$  such that  $d_1 + d_2$ 's capital does not grow unbounded on  $L$ .

The proof of Theorem 5 then follows. Let  $L$  and  $M$  be as in Lemma 1, and suppose  $M$  runs in time  $n^k$ . Consider the set  $R$  of random strings  $t$  such that  $M(x, t) = 1$  for some  $x \in L_{=n}^{(n)}$ , where  $n^k = |t|$ . Because  $L \in \text{RP}$ ,  $R$  is polynomially dense. Moreover if  $t \in R$ , then  $M(x, t) = 1$  for some  $x \in L_{=n}^{(n)}$ , therefore  $K_t(t) > |x|^\gamma > |t|^{\gamma/k}$ . Putting  $\delta := \gamma/k$  ends the proof.  $\square$

We are now ready to prove our main result stating that if  $\text{RP}$  is not meager in  $\mathbf{E}_\epsilon$ , then  $\text{ZPP} = \text{PSPACE}$  and  $\text{NP} = \text{EXP}$ .

**Theorem 6** *For every  $\epsilon > 0$ ,  $\text{RP}$  has  $\mathbf{E}_\epsilon$ -measure zero, else  $\text{ZPP} = \text{PSPACE}$  and  $\text{NP} = \text{EXP}$ .*

*Proof.* Suppose  $\text{RP}$  has  $\mathbf{E}_\epsilon$ -measure non-zero, for some  $\epsilon > 0$ , then by Theorem 5,  $\mathbf{P}$  contains a polynomially dense set  $R$  of strings with high  $K_t$  and  $KS$  complexity. Since  $\text{NP} = \text{NP}^R$  and  $\text{ZPP} = \text{ZPP}^R$ , Theorem 3 and 4 end the proof.  $\square$

## 4 Derandomization of $\text{AM}$ if $\text{NP}$ is not small

The following result says that if  $\text{NP}$  is not small in  $\mathbf{E}_\alpha$ , then  $\text{NP} = \text{AM}$ .

**Theorem 7** *Suppose  $\text{NP}$  has  $\mathbf{E}_\alpha$ -measure non zero, for some  $\alpha > 0$ . Then  $\text{NP} = \text{AM}$ .*

*Proof.* In order to prove Theorem 7, we will need the following lemma.

**Lemma 2** *If  $\text{NP}$  has  $\mathbf{E}_\alpha$ -measure non-zero then there is a language  $L$  in  $\text{NP}$  and a nondeterministic polynomial time Turing machine  $M$  deciding  $L$  such that: For all but finitely many  $m \in \mathbb{N}$ ,  $L_{=m}^{(m)}$  is non-empty and for almost every  $m \in \mathbb{N}$ , for every  $x \in L^{(m)=m}$ , every non-deterministic witness  $t$  such that  $M(x, t) = 1$ , has great SAT-oracle circuit complexity, i.e.  $\text{Size}^{\text{SAT}}(t) > |x|^\gamma$ .*

*Proof.* The first property is easily verified by constructing the appropriate martingale  $d_1$  similarly to Lemma with  $K_t(t) < |x|^\gamma$  or  $KS(t) < |x|^\gamma, 1$ . For the second property consider the following martingale  $d$ , which only bets on the  $n$  first  $n$ -sized strings  $s_0^{(n)}$  to  $s_{n-1}^{(n)}$ , for every length  $n$ . Fix an enumeration of nondeterministic Turing machines so that  $M_i$  always halts within  $n^j$  steps where  $j \leq \log i$ . On input  $(w, s_{|w|})$  where  $s_{|w|} \in \{s_0^{(n)}, \dots, s_{n-1}^{(n)}\}$ ,  $d$  simulates the first  $\log n$  nondeterministic machines on all nondeterministic witnesses for inputs of length at most  $\log n$ . Let  $M_i$  be the first such machine that agrees with  $L$   $\text{NP}$ -wise on all such inputs, i.e. decides  $L$  correctly ( $\text{NP}$ -wise) on inputs smaller than  $\log n$ . Next  $d$  constructs all circuits with oracle gates for SAT of size smaller than  $|x|^\gamma$  on  $\log i \log |x|$  inputs, where  $0 < \gamma < 1$  will be determined later, and computes the truth table  $t$  for each. If  $M_i(x, t) = 1$  for any such  $t$ ,  $d$  bets half its current capital that  $x \in L$ . Otherwise,  $d$  does not bet.

**Claim 2** *Let  $L \in \text{NP}$ . Let  $M_{i_0}$  be the first nondeterministic poly-time machine deciding  $L$ . Then if there are infinitely many lengths  $m$ , such that there exists  $x \in L_{=m}^{(m)}$  and a random string  $t$ , such that  $M_{i_0}(x, t) = 1$ , and  $t$  viewed as a function of  $\log |t|$  inputs has circuit complexity less than  $|x|^\gamma$ , then  $L \in S^\infty[d]$ .*

After a finite amount of time  $d$  will always pick  $i = i_0$ . Thereafter we have that for each length  $m$  such that there exists  $x \in L_{=m}^{(m)}$  and a nondeterministic witness  $t$  with small circuit complexity, such that  $M_{i_0}(x, t) = 1$ ,  $d_2$  will always bet half its current capital that  $x \in L$ . Since  $M_{i_0}$  decides  $\text{NP}$ -wise, this bet will be correct, and  $d$ 's total stake will be multiplied by  $3/2$ . Otherwise,  $d$  does not bet, so will never lose (once  $i = i_0$ ). So if there are infinitely many such lengths  $m$ ,  $d$ 's capital grows unbounded. The running time of  $d_2$  is less than  $2^{n^{\alpha/4}}$ , since on input  $(w, s_{|w|})$  as above, there are  $\log n$  machines to simulate on all inputs of size  $\log n$ , where each machine runs in time smaller than  $(\log n)^{\log \log n}$ , i.e. takes time  $2^{(\log n)^{\log \log n}}$  to be simulated by trying all nondeterministic witnesses and taking a majority

vote. Next  $d$  needs to construct  $2^{n^{2\gamma}}$  circuits of size at most  $n^\gamma$  and construct their truth table. Since evaluating a SAT gate of size at most  $n^\gamma$  takes time  $2^{O(n^\gamma)}$ ,  $d(w)$  can be computed in time  $2^{n^{\alpha/4}}$ , by an appropriate choice of  $\gamma$ , with  $0 < \gamma < 1$ . The query set of  $d$  is equal to  $Q_{d_2}(n) = \{s_0^{(n)}, \dots, s_{n-1}^{(n)}\} \cup \bigcup_{j=1}^{\log n} L=j$ , hence polynomial-sized.  $\square$

This proves Lemma 2, since if NP does not have  $E_\alpha$ -measure zero, there must be a language  $L \in \text{NP}$  such that  $d_1 + d$ 's capital does not grow unboundedly on  $L$ .  $\square$

The proof of Theorem 7 then follows, let  $L$  and  $M$  be as in Lemma 2, where  $M$  runs in time  $n^d$ . Let  $L' \in \text{AM}$  be any language and  $N$  a probabilistic nondeterministic Turing machine deciding it, and assume that on input of size  $n$ ,  $N$  runs in time  $n^c$ . For  $\epsilon = \gamma/d$  let  $a$  and  $b$  be as in Theorem 1 and pick  $m$  so that  $m^\gamma > n^e$ , where  $e = \epsilon ac$ . For every  $x$  among  $s_0^{(m)}, \dots, s_{m-1}^{(m)}$  nondeterministically guess a witness  $t$  for machine  $M$  on input  $x$ . Check whether  $M(x, t) = 1$  – we know that there is at least one such witness for at least one such  $x$ , and every such witness has circuit complexity at least  $m^\gamma = n^{\epsilon ac}$  – if so use Theorem 1 to construct from  $t$  a pseudorandom generator from  $O(\log n)$  bits to  $n^c$  bits secure against circuits of size  $n^c$  with oracle gates to SAT. Use the outputs of this pseudorandom generator to simulate the nondeterministic Turing machine  $N$  for  $L'$ . This has expected nondeterministic polynomial time, and never errs for sufficiently large inputs, so  $L' \in \text{NP}$ .  $\square$

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