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On Chvátal Rank and Cutting Planes Proofs

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Abstract

We study the Chvátal rank of polytopes as a complexity measure of unsatisfiable sets of clauses. Our first result establishes a connection between the Chvátal rank and the minimum refutation length in the cutting planes proof system. The result implies that length lower bounds for cutting planes, or even for tree-like cutting planes, imply rank lower bounds. We also show that the converse implication is false. Rank lower bounds don't imply size lower bounds. In fact we give an example of a class of formulas that have high rank and small size. A corollary of the previous results is that cutting planes proofs cannot be balanced.

We also introduce a general technique for deriving Chvátal rank lower bounds directly from the syntactical form of the inequalities. We apply this technique to show that the polytope of the Pigeonhole Principle requires logarithmic Chvátal rank. The bound is tight since we also prove a logarithmic upper bound.

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1 Introduction

Let $P \subseteq \mathbb{R}^n$ be a polyhedron, that is, the set of solutions of a system of linear inequalities, and let $a_1x_1 + \cdots + a_nx_n \leq a_0$ be an inequality that is satisfied by all points in P. Clearly, if $a_1, \ldots, a_n$ are all integers, then $a_1x_1 + \cdots + a_nx_n \leq \lfloor a_0 \rfloor$ is also satisfied by all integer points of P, where $\lfloor a_0 \rfloor$ denotes the rounding of a_0 to the largest smaller integer. Such a *derived* inequality is known in the integer linear programming literature as a *Chvátal-Gomory cut*. The idea that integer points of a polyhedron are preserved by such cuts was used by Chvátal to study the integer hull of certain important polyhedra arising from combinatorial optimization theory. In a remarkably beautiful article, Chvátal [Chv73] introduced the rank of a bounded polyhedron (polytope), namely, the minimum number of rounds of cut operations that are required to reach its integer hull. With the aim of understanding the combinatorics of 0/1 linear programming problems with polynomial-time algorithms, Chvátal observed that Edmonds' matching theory led to polytopes of bounded rank. Quite remarkably, Chvátal was anticipating on the later development of computational complexity:

> *One may be tempted to believe that each class of zero-one linear programming problems having a bounded rank possesses a polynomial-time algorithm. [...]. In this context, it may be interesting to note that each class of integer linear programming problems with bounded rank admits a good characterization.*

Then he goes on to prove the last sentence. In current terminology, problems with good characterizations are those in NP $\cap$ co-NP, which means that the solutions can be characterized by an existential and a universal statement. Thus, according to Chvátal's result, 0/1 linear programming problems corresponding to NP-complete problems have unbounded rank, unless NP = co-NP. It is perhaps even more impressive that Chvátal proved, unconditionally, that polytopes arising from the search of large independent sets in graphs (an NP-complete problem) had unbounded rank.

The successful theory of Chvátal-Gomory cuts has been revisited a number of times. From the algorithmic perspective, they form the basis of popular algorithms such as Branch and Bound, or Gomory's Cutting Planes algorithm (see [Sch86]). From the combinatorial optimization perspective, they are a powerful tool for discovering the structure of integer hulls (see [Sch86] again, and [CCH89]). Cook, Coullard, and Turán [CCT87] introduced a new perspective. They saw Chvátal-Gomory cuts as a rule for inferring valid inequalities, and defined a proof system for proving the unsatisfiability of formulae in propositional logic. The resulting proof system is called cutting planes (CP). It is well-known that sets of Boolean clauses can be represented as systems of linear inequalities over the 0-1 cube. Thus, unsatisfiable sets of clauses translate into polytopes without integer points. It was proved that from such systems it is always possible to derive a false inequality, such as $1 \leq 0$, by repeated application of Chvátal-Gomory inferences and linear combinations of previously derived inequalities. Such a derivation is then viewed as a refutation of the original set of clauses. In fact, Cook, Coullard, and Turán proved that the *length* of such a refutation, measured by the number of inferences, is at most proportional to the length of the minimal resolution refutation, and that in some important cases, it might be exponentially shorter. Thus, the cutting planes system simulates resolution, and in fact, it is exponentially stronger.

According to Chvátal's results mentioned in the quotation above, the rank of the polytopes corresponding to unsatisfiable sets of clauses should, in general, be unbounded. This is because testing unsatisfiability is a co-NP-complete problem and we do not expect it to be in NP. Until recently, this measure of complexity of sets of clauses had not been studied. The purpose of the present paper is to put forward the rank measure for its study in propositional proof complexity. See [BOGH+03] for a recent independent work with interesting results on the same topic. The rank lower bounds that were known before [BOGH+03] and the present work, were for polytopes arising from combinatorial optimization problems having integer points [CCH89], or for polytopes that do not arise from an unsatisfiable set of clauses [ES99, GHP02]. The study of the rank measure suggests several new interesting questions. What is the relationship between the Chvátal rank of a set of clauses and the length of its minimal cutting planes refutation? Are there explicit sets of clauses requiring large Chvátal rank? Can the rank measure be used for finding cutting planes refutations?

The first observation is that the Chvátal rank of the polytope corresponding to a set of clauses with n variables is at most n. This follows easily from the known simulation of resolution by cutting planes. Thus, the Chvátal rank is a measure that ranges from 0 to n. Then we study the relationship between rank and length. Our first result is that the Chvátal rank is indeed somehow related to the length of cutting planes refutations. We show that if the polytope of an unsatisfiable set of clauses F has Chvátal rank d, then F has a cutting planes refutation of size $O(n^d)$. In fact, the resulting cutting planes refutation is tree-like, which means that every derived inequality is used at most once in the refutation. The proof of this result uses duality theory and follows an analogous argument by Chvátal, Cook, and Hartmann [CCH89] appropriately modified to work with polytopes with empty integer hull.

It follows from the above that sufficiently strong lower bounds on the lengths of cutting planes refutations implies lower bounds on the Chvátal rank. From the known exponential lower bounds for the lengths of refutations in cutting planes via monotone interpolation we obtain explicit sets of clauses with n variables requiring Chvátal rank $n^{\Omega(1)}$. The interesting question at this point is whether the reverse direction holds. Do Chvátal rank lower bounds imply length lower bounds for cutting planes refutations? For the sake of argument, suppose for the moment that cutting planes refutations could be *balanced*, that is, suppose that a cutting planes refutation of length m could be transformed to a cutting planes refutation of height $O(\log m)$. We note that this sort of structural result holds for proof systems such as Frege, or even bounded-depth Frege [Kra94]. Obviously, the height of a refutation is a bound on the Chvátal rank since it bounds the number of rounds of applications of the Chvátal-Gomory cut. This would imply that if the set of clauses requires Chvátal rank $\Omega(d)$, then it requires length $2^{\Omega(d)}$ to refute in cutting planes. Unfortunately, this approach is condemned to fail in its full generality. We show an example of class of formulas requiring Chvátal rank $n^{\Omega(1)}$ and having cutting planes proofs of length $n^{O(1)}$. Thus, we answer the question above negatively: rank lower bounds do not imply, in general, length lower bounds. Let us note that a result of this type has also been obtained in [BOGH+03] independently using different techniques. An immediate corollary is that cutting planes proofs cannot be balanced. What remains as an interesting open problem is whether rank lower bounds imply length lower bounds for tree-like cutting planes refutations.

Then we turn our attention to proving lower bounds for the Chvátal rank directly without resorting to the relationship with refutation-length. We consider the usual encoding of the Pigeonhole Principle, and prove that its Chvátal rank is $\Omega(\log n)$. In order to prove this result we develop a general technique for deriving rank lower bounds directly from the syntactical expression of the inequalities. We also observe that the Pigeonhole polytope has cutting planes refutations of height

$O(\log n)$. Thus, the Chv´atal rank of the Pigeonhole polytope is $\Theta(\log n)$. The other rank lower bounds that are known for sets of clauses are for random 3-CNF and Tseitin formulas [BOGH+03].

2 Preliminary Definitions

We first give the definition of the Cutting Planes proof system.

Definition 1 *The* Cutting Planes proof system, *CP from now on, is a refutational proof system defined as follows. The allowed formulas are linear inequalities with integer coefficients. There are three rules of inference,*

Addition*:*

$$\frac{a_1x_1 + \cdots + a_nx_n \le a_0 \qquad b_1x_1 + \cdots + b_nx_n \le b_0}{(a_1+b_1)x_1 + \cdots + (a_n+b_n)x_n \le a_0 + b_0}$$

Scalar Multiplication*: For a positive integer b,*

$$\frac{a_1x_1 + \cdots + a_nx_n \le a_0}{(b \cdot a_1)x_1 + \cdots + (b \cdot a_n)x_n \le b \cdot a_0}$$

Integer Division*: For a positive integer b,*

$$\frac{(b \cdot a_1)x_1 + \cdots + (b \cdot a_n)x_n \le a_0}{a_1x_1 + \cdots + a_nx_n \le \lfloor a_0/b \rfloor}$$

and the following axioms $-x_i \le 0$ and $x_i \le 1$, for any variable x_i. The goal of the system is to refute a given set of linear inequalities by deriving $1 \le 0$.

Next we give a method to translate clauses into inequalities. A clause

$$x_{j_1} \vee \cdots \vee x_{j_k} \vee \neg x_{l_1} \vee \cdots \vee \neg x_{l_m}$$

is translated into the inequality

$$x_{j_1} + \cdots + x_{j_k} + (1 - x_{l_1}) + \cdots + (1 - x_{l_m}) \ge 1.$$

It was proved by Cook, Coullard, and Tur´an [CCT87] that CP polynomially simulates Resolution when clauses are presented as linear inequalities according to the previous translation. Therefore, the system is complete.

Next we define a few complexity measures of the proof system.

Definition 2 *The* length *of a CP proof is the number of inequalities or formulas in it. The* size *of a CP proof is the number of binary symbols needed to represent the proof. The* height *of a CP proof is the maximum length of a path from the root to a leaf, representing the proof as a directed acyclic graph.*

As usual, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, and $\mathbb{Z}^n$ denote the set of vectors in $\mathbb{R}^n$ whose components are all integer. Let P be a polyhedron in the space $\mathbb{R}^n$. Chvátal defined P as follows:

$$P' = \{\mathbf{x}\in P \ : \ \forall \mathbf{a}\in\mathbb{Z}^n \, \forall b\in\mathbb{Z} \, ((\forall \mathbf{y}\in P \ \ \mathbf{a}^T\mathbf{y} < b+1) \Rightarrow \mathbf{a}^T\mathbf{x} \leq b)\}.$$

We let P_I be the convex hull of $P \cap \mathbb{Z}^n$. If we define $P^{(0)} = P$, then we can recursively define $P^{(j+1)} = (P^{(j)})'$.

We will deal with *rational polyhedra*, that is, sets of the form $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, where $\mathbf{b} \in \mathbb{Z}^m$, and $\mathbf{A} \in \mathbb{Z}^{m\times n}$. We say that the set of inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ *defines* the polyhedron P. Sometimes we identify P with the set of equations that define it. Chvátal [Chv73] proved that for bounded polyhedra P there exists a d such that $P^{(d)} = P_I$. Schrijver [Sch80] extended this to unbounded rational polyhedra, and showed that if P is a rational polyhedron, then every $P^{(j)}$ is also a rational polyhedron.

Theorem 3 (Theorem 1 of [Sch80]) *If P is a rational polyhedron, then $P^{(j)}$ is also a rational polyhedron for every integer j. Moreover, $P^{(j)}$ is defined by a finite subset of inequalities of the form $\mathbf{a}^T\mathbf{x} \leq b$, with integer coefficients $\mathbf{a} \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, that satisfy $\mathbf{a}^T\mathbf{y} < b+1$, for any $\mathbf{y} \in P^{(j-1)}$.*

We are now ready to define an important complexity measure that we study in this paper.

Definition 4 *For a rational polyhedron P, its* Chvátal rank *is the minimum j such that $P^{(j)} = P_I$.*

3 Relationship Between Rank and Size

The Cutting Planes proof system can be reduced to a single inference rule:

Generalized Step:

$$\frac{\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \leq b_n \end{array}}{\left(\sum_{i=1}^n \lambda_i a_{i1}\right) x_1 + \cdots + \left(\sum_{i=1}^n \lambda_i a_{in}\right) x_n \leq c}$$

where c is an integer such that $\lfloor \sum_{i=1}^n \lambda_i b_i \rfloor \leq c$, and λ_i's are non-negative rational coefficients satisfying $\sum_{i=1}^n \lambda_i a_{ij} \in \mathbb{Z}$, for any $1 \leq j \leq n$.

Notice that this rule has the same number of premises as variables. Every generalized step can be simulated by, at most, $n+1$ scalar multiplications, n additions, and one integer division.

The idea behind this rule is the following: If P is a rational polytope, then there exist a definition of P' as $P' = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, such that every inequality of $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ can be derived, from inequalities defining P, using just one generalized step. In what follows we will give technical justification of this fact.

The first Lemma of this Section states and proves a strong form of Farkas' Lemma. In the course of its proof we will use the following form of the Duality Theorem of Linear Programming:

$$\max\{\mathbf{a}^T\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\} = \min\{\mathbf{b}^T\mathbf{y} : \mathbf{A}^T\mathbf{y} = \mathbf{a}, \ \mathbf{y} \geq 0\}.$$

Its proof can be found in any standard textbook that treats linear programming (see [Sch86], for example).

Lemma 5 *Let* $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \le \mathbf{b}\}$ *be a non-empty rational polytope, let* $\mathbf{a} \in \mathbb{Z}^n$*, and let* $c \in \mathbb{Z}$*. If every* $\mathbf{x} \in P$ *satisfies* $\mathbf{a}^T\mathbf{x} < c$*, then there exists* $\mathbf{y} \ge 0$ *such that* $\mathbf{y}^T\mathbf{A} = \mathbf{a}$ *and* $\mathbf{y}^T\mathbf{b} < c$*. Moreover,* $\mathbf{y}$ *can be chosen to have rational components of which at most* n *are non-zero.*

Proof: Consider the following linear program

$$\max\{\mathbf{a}^T\mathbf{x} : \mathbf{Ax} \le \mathbf{b}\}.$$

Since P is bounded and non-empty, the linear program has an optimum $\mathbf{x}^*$. By assumption, $\mathbf{a}^T\mathbf{x}^* < c$. The dual linear program is

$$\min\{\mathbf{by} : \mathbf{A}^T\mathbf{y} = \mathbf{a},\ \mathbf{y} \ge 0\}.$$

By the Duality Theorem, the optimum $\mathbf{y}^*$ of the dual exists and has the same objective value than the primal. Thus, $\mathbf{b}^T\mathbf{y}^* = \mathbf{a}^T\mathbf{x}^* < c$ and $\mathbf{y}^*$ is feasible dual. Thus, $\mathbf{A}^T\mathbf{y}^* = \mathbf{a}$ and $\mathbf{y}^* \ge 0$. For the second part of the lemma we use the fact that the optima of linear programs in standard form are achieved at the basic feasible solutions. The optimum of the dual program, which is in standard form, is achieved at a basic feasible solution that has the form

$$\mathbf{y}^* = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{a} \\ 0 \end{bmatrix},$$

where $\mathbf{B}$ is a matrix with n columns of $\mathbf{A}^T$ that are linearly independent. Since the entries of $\mathbf{A}$ are integers, the entries of $\mathbf{B}^{-1}$ are rationals. Thus, $\mathbf{y}^*$ can be chosen to have rational components of which at most n are non-zero. ■

The next result relates rank with length. In particular it shows that constant rank implies polynomial tree-like cutting plane length. We prove this relationship for polyhedra without integer points. This differs from Theorem 6.1 in [CCH89] in that they only consider polyhedra with integer points. Our proof requires an additional argument.

Theorem 6 *Let* $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \le \mathbf{b}\}$ *be a rational polytope without integer points and Chvátal rank* d*. Then, there exists a tree-like generalized steps refutation of* P *of height* $d+2$ *and length* $n^{O(d)}$*.*

Proof: The result about height implies the bound on the length by resolving a simple recursion, because each generalized step uses n premises. We first prove the following statement by induction on d:

Claim 7 *Let* $\mathbf{a} \in \mathbb{Z}^n$ *and* $c \in \mathbb{Z}$*. If* $P^{(d)}$ *is non-empty and* $\mathbf{a}^T\mathbf{x} < c+1$ *for every* $\mathbf{x} \in P^{(d)}$*, then* $\mathbf{a}^T\mathbf{x} \le c$ *has a tree-like generalized steps proof from* P *of height* $d+1$*.*

Proof: The case $d = 0$ is seen as follows. Assume $P^{(0)} = P \neq \emptyset$ and $\mathbf{a}^T\mathbf{x} < c+1$ for every $\mathbf{x} \in P$. By Lemma 5, there exists $\mathbf{y} \geq 0$, with at most n non-zero rational components, such that $\mathbf{A}^T\mathbf{y} = \mathbf{a}$ and $\mathbf{b}^T\mathbf{y} < c+1$. It follows that one can derive $\mathbf{a}^T\mathbf{x} \leq c$ in one generalized step, because $\lfloor \mathbf{b}^T\mathbf{y} \rfloor \leq c$.

For the inductive case $d > 0$, reason as follows. By Theorem 3, the set $P^{(d)}$ is a rational polyhedron, defined by $P^{(d)} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}'x \leq \mathbf{b}'\}$, for some integer matrix $\mathbf{A}'$ with rows $\mathbf{a}'_1, \ldots, \mathbf{a}'_m$ and some column vector $\mathbf{b}'$ with integer components $b'_1, \ldots, b'_m$. Moreover, by the same theorem, the inequality $\mathbf{a}'_i\mathbf{x} < b'_i + 1$ holds for every $\mathbf{x} \in P^{(d-1)}$. We also have $P^{(d-1)} \supseteq P^{(d)} \neq \emptyset$. Then, by induction hypothesis, each $\mathbf{a}'_i\mathbf{x} \leq b'_i$ has a tree-like generalized steps proof from P of height d. Now, by Lemma 5, there exists $\mathbf{y} \geq 0$, such that $\mathbf{A}'^T\mathbf{y} = \mathbf{a}$ and $\mathbf{b}'^T\mathbf{y} < c+1$. Moreover, $\mathbf{y}$ has at most n non-zero rational components. Hence, we only need n of the inequalities $\mathbf{a}'_i\mathbf{x} \leq b'_i$. Now, $\mathbf{a}^T\mathbf{x} \leq \lfloor \mathbf{b}'^T\mathbf{y} \rfloor \leq c$ can be deduced from these inequalities in one generalized step. This gives a tree-like generalized steps proof of height $d+1$. ■

To complete the proof of the Theorem, we need to see what happens when $P^{(d)}$ is empty. We use the same notation as in the proof of Claim 7. For every $i \in \{0, \ldots, m\}$, let $P_i^{(d)}$ be the set of $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\begin{bmatrix} \mathbf{a}'_1 \\ \vdots \\ \mathbf{a}'_i \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} b'_1 \\ \vdots \\ b'_i \end{bmatrix}.$$

In the following, let $\mathbf{A}'_i$ be the matrix with rows $\mathbf{a}'_1, \ldots, \mathbf{a}'_i$, and let $\mathbf{b}'_i$ be the column vector with components $b'_1, \ldots, b'_i$. Since $P^{(d)} = P_m^{(d)}$ is empty and $P_0^{(d)} = \mathbb{R}^n$, there exists a maximal $i \in \{0, \ldots, m\}$ such that $P_i^{(d)}$ is not empty. Observe that for that maximal i, the inequality $-\mathbf{a}'_{i+1}\mathbf{x} < -b'_{i+1}$ holds for every $\mathbf{x} \in P_i^{(d)}$. By Claim 7, there exists a proof of $-\mathbf{a}'_{i+1}\mathbf{x} \leq -b'_{i+1} - 1$ of height $d+1$. Combined with $\mathbf{a}'_{i+1}\mathbf{x} \leq b'_{i+1}$, this gives a proof of $0 \leq -1$ from P. The height of this proof is $d+2$. ■

An interesting question is whether the converse to Theorem 6 is true, i.e. whether high rank implies high length. Actually this is false. To be able to argue this, we need to present a class of formulas that has high rank but short cutting planes refutations. We will take our example from [BEGJ00]. We give a short presentation below.

Let n and d be natural numbers. Let us first define the pyramid of depth d,

$$Pyr_d = \{(i,j) : 1 \leq j \leq i \leq d\}.$$

Our set of clauses will have three types of variables p, q, and r. The variable $q_{i,j,a}$ is intended to mean that the element $a \in \{1, \ldots, n\}$ is in position (i,j) of the pyramid. The variable $p_{a,b,c}$ for $a, b, c \in \{1, \ldots, n\}$ means that a and b generate c in a pyramidal fashion. If $a = b$ we say that a generates c. Finally, the variable r_a for $a \in \{1, \ldots, n\}$ indicates that a gets colored 1.

The contradictory set of clauses $Gen(p,q) \cup Col(p,r)$ say: (1) Every position of the pyramid has an element from $\{1, \ldots, n\}$. (2) The elements in positions (d,j) for $1 \leq j \leq d$ are generated by 1. (3) The element in position $(1,1)$ generates n. (4) If a, b and c are in positions $(i+1,j)$, $(i+1,j+1)$, and (i,j) respectively, then a and b generate c. (5) If a and b are colored 0 and a and b generate c, then c is colored 0. (6) 1 gets colored 0 and n gets colored 1.

Theorem 8 *([BEGJ00], theorems 4.1 and 4.2) Let d and n be natural numbers. Then for some $\epsilon > 0$, every tree-like cutting planes refutation of the clauses $Gen(p,q) \cup Col(p,r)$ has to be of length $2^{\Omega(n^\epsilon)}$. On the other hand, there are (dag-like) resolution refutations of length $n^{O(1)}$ of the clauses $Gen(p,q) \cup Col(p,r)$.*

Directly from Theorems 8 and 6 we obtain the following:

Corollary 9 *There is a class of formulas with Chvátal rank $n^{\Omega(1)}$ and having CP refutations of length $n^{O(1)}$.*

In general all size lower bounds (tree-like or dag-like) in cutting planes give rank lower bounds.

Definition 10 *A refutation of length m can be* balanced *if there exists another refutation of the same set of clauses of height $O(\log m)$.*

The following lemma relates the rank with the height.

Lemma 11 *Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ be a rational polytope without integer points. Then, the rank of P is smaller or equal than the height of any refutation using generalized steps.*

Proof: It suffices to prove that for any application of the generalized step, if all the premises are satisfied by every point of $P^{(i)}$, then the conclusion is satisfied by every point of $P^{(i+1)}$. This is obvious since any scalar multiplication of the premises, and addition of them will be satisfied by $P^{(i)}$, and by definition of P', if the premise of an integer division is satisfied by $P^{(i)}$, then the conclusion is satisfied by $P^{(i+1)}$. Then, we can prove that $1 \leq 0$ is satisfied by every point of $P^{(h)}$, being h the height of the refutation, thus $P^{(h)}$ is empty, and the rank of P is not greater than h. ∎

We note that the converse inequality was proved in Theorem 6. Also as an immediate consequence of Theorems 8 and 6 and Lemma 11 we obtain:

Corollary 12 *Refutations in the cutting planes proof system cannot be balanced.*

Note that weaker systems like Resolution cannot be balanced either. Stronger systems like Frege can, and even Bounded Depth Frege which is incomparable with Cutting Planes.

4 The Rank of the Pigeonhole Principle

In this Section we will present matching upper and lower bounds for the rank of the Pigeonhole Principle. This shows that in this case the polynomial-size proof in cutting planes can be balanced with a modest increase in size. We start with a Lemma.

Lemma 13 *There is a tree-like cutting planes proof of $x_0 + \cdots + x_{n-1} \leq 1$ from the set of hypothesis $\{x_i + x_j \leq 1 : 0 \leq i, j \leq n-1,\ i \neq j\}$ in height $O(\log(n))$ and size $n^{O(\log(n))}$.*

Proof: Let k be such that $2k-1 \leq n$. We obtain a proof of $x_0+\cdots+x_{2k-2} \leq 1$ from the following set of inequalities:

$$\left\{ \sum_{i=0}^{k-1} x_{f(j+i)} \leq 1 : j = 0, \ldots, 2k-2 \right\},$$

where $f(z) = z \mod (2k-1)$. The proof is simple: add all inequalities in the set to get $\sum_{i=0}^{2k-2} kx_i \leq 2k-1$, and divide by k to obtain $\sum_{i=0}^{2k-2} x_i \leq \lfloor (2k-1)/k \rfloor = 1$. Also, we may obtain a proof of $\sum_{i \in I} x_i \leq 1$ for every set I of cardinality $2k-1$ from $2k-1$ inequalities of the form $\sum_{i \in J} x_i \leq 1$ with J of cardinality k. Each of these proofs requires a unique division by k.

Now we form a tree rooted by $x_0 + \cdots + x_{n-1} \leq 1$. The immediate successors are the $\lceil n/2 \rceil$ inequalities of the form $\sum_{i \in I} x_i \leq 1$ with $|I| = \lceil n/2 \rceil$ that are needed to prove $x_0+\cdots+x_{n-1} \leq 1$ in the argument above (we may assume that n is odd). The immediate successors of each of these are the $\lceil \lceil n/2 \rceil /2 \rceil$ inequalities of the from $\sum_{i \in J} x_i \leq 1$ with $|J| = \lceil \lceil n/2 \rceil /2 \rceil$ that are needed to prove them (again, we may assume that $\lceil n/2 \rceil$ is odd). Repeating this until we arrive at inequalities of the form $x_i + x_j \leq 1$, we obtain a tree of height $O(\log(n))$ and size $n^{O(\log(n))}$. This tree can be used as the skeleton of a tree-like cutting planes proof of height $O(\log(n))$. ■

Theorem 14 *There is a tree-like cutting-planes proof of* PHP_n^{n+1} *of height* $O(\log(n))$ *and length* $n^{O(\log(n))}$*. Therefore, the* PHP_n^{n+1} *polytope has rank* $O(\log(n))$*.*

Proof: From the clauses $\sum_{j=1}^{n} p_{i,j} \geq 1$ for $i \in \{1, \ldots, n+1\}$, we deduce that $\sum_{i=1}^{n+1} \sum_{j=1}^{n} p_{i,j} \geq n+1$ in one (generalized) step. From the clauses $p_{i,j} + p_{k,j} \leq 1$ for a fixed $j \in \{1, \ldots, n\}$, we deduce $\sum_{i=1}^{n+1} p_{i,j} \leq 1$ by Lemma 13 in height $O(\log(n))$ and $n^{O(\log(n))}$ steps. Adding up these for all $j \in \{1, \ldots, n\}$ and commutativity gives $\sum_{i=1}^{n+1} \sum_{j=1}^{n} p_{i,j} \leq n$. Finally, we obtain $0 \geq 1$ easily from this and $\sum_{i=1}^{n+1} \sum_{j=1}^{n} p_{i,j} \geq n+1$ in one generalized step. The whole proof has height $O(\log(n))$ and size $n^{O(\log(n))}$. ■

An independent proof of the upper bound in Theorem 14 can be found in [BOGH$^+$03]. Now we turn to proving a matching lower bound. For that, we need the following Lemma.

Lemma 15 *Let* $s \leq k \leq 2n$*. Let* P *be a polytope defined by the set of linear inequalities* $\{a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i : i = 1, \ldots, m\}$ *with integer coefficients, and let* $a_1x_1 + \ldots + a_nx_n \leq b$ *be a linear inequality with integer coefficients obtained from* P *by a generalized step. If the following conditions are satisfied*

- *(i) if* $b_i \geq 1$ *then* $\left(\sum_{j=1}^{n} a_{ij}\right)/b_i \leq s \leq k$
- *(ii) if* $b_i = 0$ *then* $\sum_{j=1}^{n} a_{ij} \leq 0$
- *(iii) if* $b_i \leq -1$ *then* $\left(\sum_{j=1}^{n} a_{ij}\right)/b_i \geq k \geq s$
- *(iv)* $\max\{a_{i1}, \ldots, a_{in}\} \leq b_i$,

then

- *(i) if* $b \geq 1$ *then* $\left(\sum_{j=1}^{n} a_j\right)/b \leq 2s$

(ii) *if* $b = 0$ *then* $\sum_{j=1}^n a_j \leq 0$

(iii) *if* $b \leq -1$ *then* $\left(\sum_{j=1}^n a_j\right)/b \geq k/2$

(iv) $\max\{a_1, \dots, a_n\} \leq b$.

Proof: By hypothesis, there exist rational coefficients $\lambda_1, \dots, \lambda_m \geq 0$ such that $a_j = \sum_{i=1}^m \lambda_i a_{ij}$ and $b \geq \lfloor \sum_{i=1}^m \lambda_i b_i \rfloor$. Let $b' = \sum_{i=1}^m \lambda_i b_i$. Since $a_{ij} \leq b_i$ and $\lambda_i \geq 0$, we have that $a_j = \sum_{i=1}^m \lambda_i a_{ij} \leq \sum_{i=1}^m \lambda_i b_i = b'$. Moreover, since each a_j is an integer, we have $a_j \leq \lfloor b' \rfloor \leq b$. This proves conditions (iv) and (ii) of the conclusion of the lemma. Suppose next that $b \neq 0$. Our first goal is to show that $\sum_{j=1}^n a_j \leq sb'$ and $\sum_{j=1}^n a_j \leq kb'$. Let $A = \{i : b_i \geq 1\}$, $B = \{i : b_i = 0\}$ and $C = \{i : b_i \leq -1\}$. Then,

$$\begin{aligned}
\sum_{j=1}^n a_j &= \sum_{j=1}^n \left(\sum_{i\in A} \lambda_i a_{ij} + \sum_{i\in B} \lambda_i a_{ij} + \sum_{i\in C} \lambda_i a_{ij} \right) = \\
&= \sum_{i\in A} \lambda_i \sum_{j=1}^n a_{ij} + \sum_{i\in B} \lambda_i \sum_{j=1}^n a_{ij} + \sum_{i\in C} \lambda_i \sum_{j=1}^n a_{ij} \leq \\
&\leq \sum_{i\in A} \lambda_i s b_i + \sum_{i\in C} \lambda_i k b_i.
\end{aligned}$$

For the inequality we use the fact that $\sum_{j=1}^n a_{ij} \leq b_i s$ for $i \in A$, $\sum_{j=1}^n a_{ij} \leq 0$ for $i \in B$, and $\sum_{j=1}^n a_{ij} \leq b_i k$ for $i \in C$. Now, observe that

$$\sum_{i\in A} \lambda_i s b_i + \sum_{i\in C} \lambda_i k b_i \leq \sum_{i\in A} \lambda_i s b_i + \sum_{i\in C} \lambda_i s b_i = s \left(\sum_{i\in A} \lambda_i b_i + \sum_{i\in C} \lambda_i b_i \right) = sb',$$

$$\sum_{i\in A} \lambda_i s b_i + \sum_{i\in C} \lambda_i k b_i \leq \sum_{i\in A} \lambda_i k b_i + \sum_{i\in C} \lambda_i k b_i = k \left(\sum_{i\in A} \lambda_i b_i + \sum_{i\in C} \lambda_i b_i \right) = kb'.$$

The first inequality follows from the fact that $s \leq k$, $\lambda_i \geq 0$ and $b_i < 0$ for every $i \in C$, while the second inequality follows from the fact that $s \leq k$, $\lambda_i \geq 0$ and $b_i > 0$ for every $i \in A$.

We are ready to prove the lemma. If $b \geq 1$, since $b \geq \lfloor b' \rfloor$, then $b' \leq 2b$ and so $\sum_{j=1}^n a_j \leq b's \leq 2bs$. Hence, $\left(\sum_{j=1}^n a_j\right)/b \leq 2s$ as required. If $b \leq -1$, there are two cases: $b' < -1$ and $-1 \leq b' < 0$. If $b' < -1$, then $b' \leq b/2$ since $b \geq \lfloor b' \rfloor$, and so $\sum_{j=1}^n a_j \leq b'k \leq bk/2$. Hence, $\left(\sum_{j=1}^n a_j\right)/b \geq k/2$. If $-1 \leq b' < 0$, then $b \geq \lfloor b' \rfloor = -1 \geq b$, so $b = -1$. Therefore, $\sum_{j=1}^n a_j \leq -n$ because $a_j \leq b$ as proved before. It follows again that $\left(\sum_{j=1}^n a_j\right)/b \geq n \geq k/2$. This completes the proof of the lemma. ■

Theorem 16 *The Pigeonhole Principle polytope has rank* $\Theta(\log(n))$.

Proof: The upper bound has been proved in Theorem 14. For the lower bound, we use Lemma 15. The inequalities of the Pigeonhole Principle polytope are the following

$$\sum_{j=1}^n -p_{ij} \leq -1 \qquad i \in \{1, \dots, n+1\}$$

$$p_{ik} + p_{jk} \leq 1 \qquad i, j \in \{1, \ldots, n+1\}, i \neq j, k \in \{1, \ldots, n\}$$
$$p_{ij} \leq 1$$
$$-p_{ij} \leq 0.$$

Observe that each of these equations satisfies the conditions of Lemma 15 with $s = 2$ and $k = n$. Consider the inequality $\sum_{i,j} p_{ij} \leq (n+1)n/8$ (notice that simply adding the necessary $p_{ik} + p_{jk} \leq 1$ equations we get $\sum_{i,j} p_{ij} \leq (n+1)n/2$). By Lemma 15, the inequality is not obtained by one generalized step. On the other hand, the inequalities we can obtain with one generalized step satisfy the conditions of Lemma 15 with $s = 4$ and $k = n/2$. Hence, we can iterate this reasoning. In general, for any $d < (\log_2(n))/2$, the inequality $\sum_{i,j} p_{ij} \leq (n+1)n/2^{d+2}$ can not be obtained by a proof of height d generalized steps. Therefore, $1 \leq 0$ can not be obtained by a proof of height $d-2$ generalized steps. (Notice that from $1 \leq 0$, and $-x_i \leq 0$, $x_i \leq 1$, we can obtain any inequality with integer coefficients using two generalized steps). Thus, the rank is at least $\lfloor (\log_2(n))/2 \rfloor - 3$. ■

5 Open Problems

Let us give now a list of questions related to this paper that remain open.

- Does Theorem 6 give an automatization algorithm for cutting planes in time $O(n^d)$ for polytopes of rank d? If so, PHP would be an example were this algorithm performs better than any algorithm based on resolution (or even on bounded-depth Frege). In general we consider very important the problem of studying whether cutting planes can have a good proof search algorithm, and we think that if so, a possible algorithm would be based on getting equations of decreasing rank.

- Notice that lemma 15 gives a $O(\log(\log n))$ rank lower bound for the Ramsey principle. It would be interesting to have a better lower bound, and a matching upper bound. It is also open whether the Ramsey principle has polynomial size cutting planes proofs.

- We know from our results that rank lower bounds do not give size lower bounds. But the following is still open: Do rank lower bounds give tree-like size lower bounds? Probably not either.

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