Computation of the Lovász Theta Function for Circulant Graphs

Valentin E. Brimkov[*]   Bruno Codenotti[†]   Valentino Crespi[‡]
Reneta P. Barneva[§]   Mauro Leoncini[¶]

Abstract

The Lovász theta function $\theta(G)$ of a graph G has attracted a lot of attention for its connection with diverse issues, such as communicating without errors and computing large cliques in graphs. Indeed this function enjoys the remarkable property of being computable in polynomial time, despite being sandwitched between clique and chromatic number, two well known hard to compute quantities.

In this paper we deal with the computation of the Lovász function of certain circulant graphs, i.e., graphs whose adjacency matrix is circulant. Such graphs are important for both theoretical and practical reasons, and indeed arise in many different contests. The simplest circulant graph is the cycle; for the cycle, Lovász showed a simple formula expressing the value of the theta function. We consider the theta function of circulant graphs which can be viewed as the super-position of two cycles, i.e., circulant graphs of degree 4. We invertigate the possibility to take advantage of the specifis structure of the circulants in oreder to achieve higher efficiency. For a circulant graph $C_{n,j}$ on n vertices and with a chord length j, $2 \leq j \leq \lfloor n/2 \rfloor$, we propose an $O(j)$ time algorithm to compute $\theta(C_{n,j})$ if j is odd and an $O(n/j)$ time algorithm if j is even. This is a significant improvement over the best known algorithms for the theta function computation for general graphs which take $O(n^4)$ time. We also derive conditions under which $\theta(C_{n,j})$ can be computed in $O(1)$ time.

Keywords: *Lovász theta-function, Circulant graph, Linear programming, Time complexity of algorithm*

1   Introduction

Consider a graph G whose vertices represent letters from a given alphabet, and where adjacency indicates that two letters can be "confused". The zero-error *capacity* of G is the number $\Theta(G)$ of messages that can be communicated without any error. This notion was introduced in 1956 by Shannon [13], and has generated a lot of interest over the years. It was understood quite early that the exact determination of the Shannon capacity is a very difficult problem, even for small and simple graphs. In 1979

[*]Department of Mathematics and Computer Science, SUNY Fredonia, Fredonia NY, USA. E-mail: brimkov@cs.fredonia.edu.

[†]Istituto di Matematica Computazionale del CNR, Pisa, Italy. Currently with Department of Computer Science, University of Iowa, Iowa City, USA. E-mail: bcodenott@cs.uiowa.edu

[‡]Department of Computer Science, College of Engineering, Computer Science and Technology, California State University, Los Angeles, USA. E-mail: Valentino.Crespi@calstatela.edu

[§]Department of Mathematics and Computer Science, SUNY Fredonia, Fredonia NY, USA. E-mail: barneva@cs.fredonia.edu.

[¶]Università di Modena e Reggio Emilia, Italy. E-mail: leoncini@unimo.it.

Lovász [8] introduced a related function, to become soon thereafter known as Lovász theta function or Lovász number, with the aim of estimating the Shannon capacity.

The Lovász theta function (which we denote by $\theta(G)$, and we call theta function for short) is computable in polynomial time, although it is "sandwiched" between the clique number $\omega(G)$ and the chromatic number $k(G)$, whose computation is NP-hard. Because of this remarkable property and also of its relevance to communication issues, the Lovász number is widely studied (see the surveys by Knuth and Alizadeh [5, 1] and the bibliography therein).

Despite a lot of work in the field, very little is known about classes of graphs for whose theta function either a formula or a very efficient algorithm is available. A rare example of such a result is Lovász's formula $\theta(C_n) = \frac{n \cos \frac{\pi}{n}}{1 + \cos \frac{\pi}{n}}$ for n-cycles, with n odd [8]. Recently Brimkov et al. [3] obtained formulae for the more general cases of circulant graphs with chord length two and three.

We remark here that circulant graphs arise in a variety of counting problems and in other questions in combinatorics [12], as well as in telecommunication networks, VLSI design, and distributed computing [2, 6, 7, 9]. The relevance to distributed computing is due to the fact that circulant graphs are a natural extension of rings, and provide high connectivity and fault tolerance at low cost.

One may expect problems on circulant graphs to be significantly easier than in the general case. However, this is not always the case, in particular due to the fact that a circulant graph on n nodes contains an *arbitrary* graph on $\lfloor \sqrt{n} \rfloor$ nodes. So it becomes a challenge to find a way for taking advantage of some specific properties of the circulant graphs in order to achieve significant algorithmic benefits.

In this paper we use a geometric approach to devise a very efficient algorithm for computing the theta function of arbitrary circulant graphs of degree four. For a circulant graph $C_{n,j}$ with n vertices, degree four, and chord length $2 \leq j \leq n$, the algorithm performs $O(j)$ operations if j is odd, or $O(n/j)$ operations if j is even. This is a significant improvement over the best known algorithms for the computation of the theta function for general graphs, which take $O(n^4)$ time.

2 Preliminaries

2.1 Some graph-theoretical notions and facts

Let us recall some well-known definitions from graph theory. Given a graph $G(V, E)$, its *complement graph* is the graph $\bar{G}(V, \bar{E})$, where $\bar{E}$ is the complement of E to the set of edges of the complete graph on V. An *automorphism* of the graph G is a permutation p of its vertices such that two vertices $u, v \in V$ are adjacent iff $p(u)$ and $p(v)$ are adjacent. G is *vertex symmetric* if its automorphism group is vertex transitive, i.e., for given $u, v \in V$ there is an automorphism p such that $p(u) = v$.

A graph $G'(V', E')$ is an *induced subgraph* of $G(V, E)$, if E' contains all edges from E that join vertices from $V' \subseteq V$. G is called *perfect* if $\omega(G_A) = k(G_A), \forall A \subseteq V$, where G_A is the induced subgraph of G on the vertex set A.

An $n \times n$ matrix $A = (a_{i,j})_{i,j=0}^{n-1}$ is called *circulant* if its entries satisfy $a_{i,j} = a_{0,j-i}$, where the subscripts belong to the set $\{0, 1, \ldots, n-1\}$ and are calculated modulo n. In other words, any row of a circulant matrix can be obtained from the first one by a number of consecutive cyclic shifts, and thus the matrix is fully determined by its first row. A *circulant graph* is a graph with a circulant adjacency matrix. By $C_{n,j}$ we will denote a circulant graph of degree four, with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{(i, i+1 \bmod n), (i, i+j \bmod n), i = 0, 1, \ldots, n-1\}$, where $1 < j \leq \frac{n-1}{2}$ is the *chord length*.

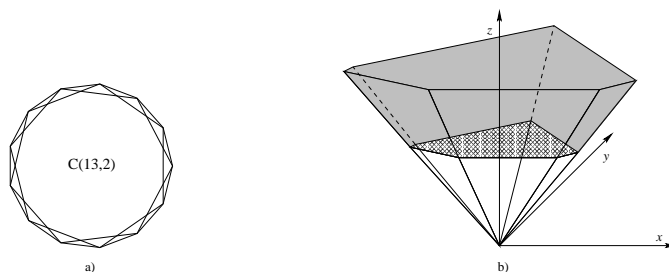Figure 1: a) The circulant graph $C_{13,2}$. b) The truncated polyhedral cone related to $C_{13,2}$, cut at $z = 2$.

See for illustration Fig. 1a presenting the circulant graph $C_{13,2}$.

Several equivalent definitions of the Lovász number are available [5]. We present here the one which requires only little technical machinery.

Definition 1 *Given a graph G, let $\mathbf{A}$ be the family of matrices A such that $a_{ij} = 0$ if v_i and v_j are adjacent in G. Let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ be the eigenvalues of A. Then $\theta(A) = \max_{A \in \mathbf{A}} \{1 - \frac{\lambda_1(A)}{\lambda_n(A)}\}$.*

For various results related to the theta function we refer to [5, 1]. In particular, the following propositions hold.

Proposition 1 (see [5]) *For every graph G with n vertices, $\theta(G) \cdot \theta(\bar{G}) \geq n$. If G is vertex symmetric, then $\theta(G) \cdot \theta(\bar{G}) = n$.*

2.2 LP formulation and related geometric constructions

Taking advantage of the particular properties of circulant matrices whose eigenvalues can be expressed in closed formulae and so generalizing the approach in [5], the validity of the following minmax formulation of the θ-function of circulant graphs of degree 4 can be easily derived.

Lemma 1 (see [3]) *Let $f_0(x,y) = n + 2x + 2y$ and, for some fixed value of j, $f_i(x,y) = 2x \cos \frac{2\pi i}{n} + 2y \cos \frac{2\pi ij}{n}$, $i = 1, 2, \ldots, n-1$. Then*

$$\theta(C_{n,j}) = \min_{x,y} \max_i \left\{ f_i(x,y), i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor \right\}. \tag{1}$$

This in turn is equivalent to the following Linear Programming problem, that we will refer to, from now thereon:

$$\theta(C_{n,j}) = \min\{z \ : \ f_i(x,y) - z \leq 0, \ i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor, z \geq 0\}. \tag{2}$$

We observe that the equalities $f_1(x,y) - z = 0, \ldots, f_{n-1}(x,y) - z = 0$ define planes through the origin. Having in mind the specific coefficients of these planes in the different ortants, as well as the relations between the coefficients of two consecutive planes, one can see that the set $\max_i \{f_1(x,y), \ldots, f_{n-1}(x,y)\}$ is a polyhedral surface, namely a polyhedral cone C with apex at the origin. The cone belongs to the positive halfspace $z \geq 0$ and the Oz axis is contained inside the cone. The faces of the cone are portions

of certain planes with equations $z = f_i(x, y)$, where i is in the range $1 < i \leq n - 1$. The rays of C are intersections of planes, obtained for no more than $n - 1$ pairs of indices i_1, i_2, where $1 \leq i_1, i_2 \leq n - 1$. The other intersections are not of interest, since they all fall "below" the conic surface $\max_i\{f_i\}$ and thus are not part of it.

We now consider the plane $f_0(x, y) = n + 2x + 2y$. Its intersection with the cone C produces a new polyhedral surface, that is a truncated cone. This is the upper part of the cone C, i.e., the one above the plane f_0 (see Fig. 1b).

Clearly, the intersection points of the plane f_0 with C are the possible candidates for solution of the problem. The theta function is the intersection point with minimal z.

Consider the intersection of C and f_0. This intersection is the boundary of some 2D convex polyhedron P (possibly unbounded). As mentioned above, the solution is at some of the vertices of this intersection. Let this be the point $A = (x_0, y_0, z_0)$ (and thus $\vartheta = z_0$). Let us now assume that we have intersected C by the plane $z = z_0$ (parallel to the xy-plane). The intersection is a (bounded) convex polygon Q_{z_0}. By construction, it follows that the polyhedron P and the polygon Q_{z_0} intersect at a single point, i.e., the point $A = (x_0, y_0, z_0)$. We will determine A using the sides of Q_{z_0}, rather than the sides of P. Since the coefficients of x and y of the plane $z = n + 2x + 2y$ are equal (indeed they are both equal to 2), then it is not difficult to see that A will be the vertex of Q_{z_0}, obtained as the intersection of the two sides of Q_{z_0} which "sandwich" the straight line in $z = z_0$ passing through A, and with a slope of 45 degrees. These lines have equations $2x \cos \alpha + 2y \cos(\alpha j) = z_0$ and $2x \cos \beta + 2y \cos(\beta j) = z_0$, where $\alpha = \frac{2\pi i_1}{n}$ and $\beta = \frac{2\pi i_2}{n}$, for some indices i_1 and i_2. Once i_1 and i_2 are known, z_0 can be computed by solving the linear system

$$\begin{cases} z &= 2x \cos \alpha + 2y \cos(2\alpha) \\ z &= 2x \cos \beta + 2y \cos(2\beta) \\ z &= n + 2x + 2y. \end{cases}$$

Note that one can use any horizontal intersection of the cone, since all such intersections are homothetic to each other.

Through a nontrivial analysis of the structure of the admissible region defined by the linear constraints, it was possible to obtain closed formulae for some special cases of circulant graphs of degree four [3]. For example, we report the least complex formula for the simplest case $j = 2$:

$$\theta(C_{n,2}) = n \left(1 - \frac{\frac{1}{2} - \cos(\frac{2\pi}{n} \lfloor \frac{n}{3} \rfloor) - \cos(\frac{2\pi}{n}(\lfloor \frac{n}{3} \rfloor + 1))}{(\cos \frac{2\pi}{n} \lfloor \frac{n}{3} \rfloor) - 1)(\cos(\frac{2\pi}{n}(\lfloor \frac{n}{3} \rfloor + 1)) - 1)} \right). \tag{3}$$

In this paper we measure the complexity of a computation by evaluating the number of operations from the set $\{+, -, *, /, \lfloor . \rfloor, \cos(\cdot)\}$ as a function of the number of constraints in the three-dimensional LP formulation 2. We notice that such kind of algebraic computation models have been traditionally used in algebraic complexity, computational geometry, and scientific computing. For related discussion and complexity issues see, e.g., [4, 10].

Relatively to the adopted model the above formula, whose length is constant with respect to the graph size n, provides implicitly a constant time algorithm for the computation of $\theta(C_{n,3})$. In the remainder of this paper we will describe our effort to determining explicit formulae and devising algorithms that perform either in constant time, with respect to the value of n, or in time proportional to j, which results in constant time for j fixed, or, at worst, in time proportional to n/j, which still results in constant time when $j = \Theta(n)$. The general idea is to significantly reduce the set of constraints in the

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LP problem 2 and then apply existing efficient algorithms for LP in 3D. This includes, for example, the well-known Megiddo's algorithm [11] that, in 3D, performs in time linear with respect to the number of constraints of the problem.

3   Computation of $\theta(G)$

Relying on formulation 2 we will focus on the geometric unintuitive regularities of the polygon defined by the lines l_k of equation

$$x \cos(\alpha_k) + y \cos(j\alpha_k) \ = \ 1, \tag{4}$$

with $\alpha_k = \frac{2\pi}{n}k$. Let $a(k) = 1/\cos(\alpha_k)$ and $b(k) = 1/\cos(2\pi kj/n)$ be their x and y coordinates (axes cuts) respectively. We will refer to angle α_k as to the angle of line l_k. We distinguish two cases: j even and j odd.

3.1 j odd

As a first result let us prove the following lemma that provides an immediate solution to the case n even and j odd:

Lemma 2 *Let $C(n; j_1, j_2, \ldots, j_k)$ be a circulant graph with n vertices and chord lengths: $2 < j_1 < j_2 < \ldots < j_k$. Assume that n is even and all the chord lengths j_i are odd. Then $C(n; j_1, \ldots, j_k)$ is perfect and $\theta(C(n; j_1, \ldots, j_k)) = n/2$.*

Proof Since every circulant graph is vertex symmetric, by Proposition 1 we have

$$\theta(C(n; j_1, \ldots, j_k)) \cdot \theta(\bar{C}(n; j_1, \ldots, j_k)) = n.$$

Thus it is enough to show that $\theta(\bar{C}(n; j_1, \ldots, j_k)) = 2$. Bearing in mind the inequality $\omega(G) \leq \theta(\bar{G}) \leq \chi(G)$ which applies to any graph G, we obtain that it suffices to show that $\omega(C(n; j_1, \ldots, j_k)) = \chi(C(n; j_1, \ldots, j_k)) = 2$. In fact, the clique number is 2 since for $j \geq 3$ the minimal cycle in $C(n; j_1, \ldots, j_k)$ has length at least 4 (which bound is reached for $j = 3$). Moreover, it is also not hard to see that if n is even and the j_i's are odd, then the vertices of $C(n; j_1, \ldots, j_k)$ can be alternatively colored only with two colors. This completes the proof. $\qquad\square$

From now on we will focus on circulant graphs of degree 4.

Theorem 1 *Let n, j be two odd numbers with $j \leq \frac{n-1}{2}$. Then $\theta(C_{n,j})$ can be computed by solving a 3D LP problem having $O(j)$ constraints.*

Proof Here we prove that we can identify in constant time a set of at most $\lfloor \frac{j}{2} \rfloor + 1$ lines that define the polygon Q_{z_0}.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{j+1}\}$ be a set of adjacent intervals covering $[0, \pi]$ defined as

$$S_1 = [\pi - \frac{\pi}{2j}, \pi], S_{j+1} = [0, \frac{\pi}{2j}], \text{ and } S_{k+1} = [\pi - (2k+1)\frac{\pi}{2j}, \pi - (2k-1)\frac{\pi}{2j}],$$

for $k = 1, 2, \ldots, j - 1$. So, S_1, S_{j+1} are intervals of width $\frac{\pi}{2j}$, whereas $S_2, S_3, \ldots, S_j$ are intervals of width $\frac{\pi}{j}$ (see Fig. 2a). A quick analysis of the function $\cos(j\alpha)$ reveals that:
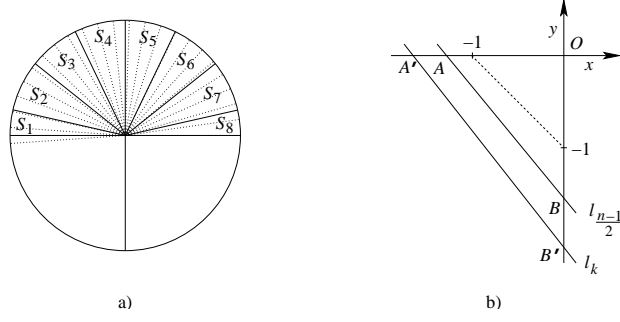
Figure 2: a) Pictorial description of the S-intervals for $j = 7$. b) Position of line $l_{(n-1)/2}$ with respect to any l_k with $\alpha_k \in S_1$.

a) it is periodic of period $2\pi/j$;

b) it nullifies on $S_k \cap S_{k+1}$, for $k = 1, 2, \ldots, j$, and

c) it is negative on the odd numbered intervals attaining -1 on their middle points.

Consider line $l_{\lfloor \frac{n}{2} \rfloor}$ in interval S_1. This verifies

$$
\begin{cases}
a(\lfloor \frac{n}{2} \rfloor) &=& \frac{1}{\cos(\pi - \frac{\pi}{n})} &=& \max\{a(i) \mid a(i) < 0 \wedge i \geq 0\} < -1 \\[2ex]
b(\lfloor \frac{n}{2} \rfloor) &=& \frac{1}{\cos(j(\pi - \frac{\pi}{n}))} &=& \frac{1}{\cos(\pi - j\frac{\pi}{n})}.
\end{cases}
$$

This line defines a face of Q_{z_0} because it is the one that intersects the Ox axis in the closest point to $(-1, 0)$ (see Fig.2b). Furthermore, its inclination w.r.t. the Oy axis is less than 45 degrees and all other lines in S_1 have lower x-cut and y-cut therefore falling outside Q_{z_0}.

Consider the even numbered intervals. Lines whose angle τ falls in these intervals all have positive y coordinate because $\cos(j\tau) > 0$. Their x-coordinate can be either positive, and in that case they would not even cross the third quadrant, or negative, in which case it must be

$$
\frac{1}{\cos(\tau)} < \frac{1}{\cos(\alpha_{(n-1)/2})} = a(\lfloor \frac{n}{2} \rfloor).
$$

So, we can conclude that those lines cannot affect the solution.

Consider the odd numbered intervals S_{2k-1}, for $k = 1, 2, \ldots, (j+1)/2$ and let $\beta^{(k)} = \pi - (2k - 2)\pi/j$. Angle $\beta^{(1)}$ is π whereas, for $k > 1$, angles $\beta^{(k)}$ correspond to the centers of intervals S_{2k-1} and all verify $\cos(j\beta^{(k)}) = -1$. We can observe that the only lines l_i that intersect $l_{\lfloor \frac{n}{2} \rfloor}$ in the third quadrant are those for which

$$
b(\lfloor \frac{n}{2} \rfloor) < b(i) < -1. \tag{5}
$$

Now, focus for a moment on the function $f(x) = 1/\cos(jx)$. It is periodic and assumes the same values within the odd numbered intervals $S_{2k-1} = [\beta^{(k)} - \pi/2j, \beta^{(k)} + \pi/2j]$. Furthermore, it is increasing over

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$[\beta^{(k)} - \pi/2j, \beta^{(k)}]$, decreasing over $[\beta^{(k)}, \beta^{(k)} + \pi/2j]$ and verifies:

$$f(\beta^{(k)}) = -1, \qquad \lim_{x \to (\beta^{(k)} - \pi/2j)^+} f(x) = \lim_{x \to (\beta^{(k)} + \pi/2j)^-} f(x) = -\infty.$$

Observe that $b(i) = f(2i\pi/n)$, i.e., condition 5 can be rephrased as

$$f\left(\pi - \frac{\pi}{n}\right) < f\left(\frac{2\pi}{n} \cdot i\right) < -1 .$$

Since the behavior of $f(x)$ on the interval $[\pi - \pi/2j, \pi + \pi/2j]$ is the same as for all the other odd numbered intervals $[\beta^{(k)} - \pi/2j, \beta^{(k)} + \pi/2j]$, the condition

$$f\left(\pi - \frac{\pi}{n}\right) < f(x) < -1$$

will be verified only for

$$\mid x - \beta^{(k)} \mid < \frac{\pi}{n}, \tag{6}$$

where $\{\beta^{(k)} \mid k = 1, 2, \ldots, (j+1)/2\}$ is the set of the solutions to equation $f(x) = 1/\cos(jx) = -1$ on $[0, \pi]$.

Given that the angle does not vary with continuity, but assumes only a discrete set of values $\alpha_i = 2\pi i/n$, for $0 \le i \le (n-1)/2$, we can see that, if for some u, α_u satisfies condition 6, then $\alpha_{u+1} = \alpha_u + 2\pi/n$ cannot.

Thus we can deduce that for each odd numbered interval S_{2k-1} there can be at most one line verifying condition 5 and since we have $\lfloor \frac{j}{2} \rfloor$ odd numbered intervals to consider, there will be at most as many lines to select. $\square$

3.2 j even

The case for j even is a little more delicate. Consider once more the family of intervals $\mathcal{S} = \{S_1, S_2, \ldots, S_{j+1}\}$. Now the $j/2$ even numbered ones, S_{2k}, for $k = 1, 2, \ldots, j/2$, are those in which $\cos(j\alpha)$ is negative. Let $\beta^{(k)}$ denote the angle corresponding to the center of S_{2k}. Thus $\cos(j\beta^{(k)}) = -1$ for all k. First, notice that each interval contains no more than $\lceil \frac{n}{j} \rceil$ lines.

Let us focus on $S_1 \cup S_2$ and on the following sequence of lines $l_{i_1}, l_{i_1+1}, \ldots, l_s$, where l_{i_1} is the line whose angle, α_{i_1} is the closest to the center of S_2, and l_s is the line whose angle is the largest within S_1, i.e., $s = \lfloor n/2 \rfloor$. It is not hard to see that those lines define a set C_1 of segments that, together with the x and y negative axes, bind a convex polygon Q.

We can apply the same idea to the other even numbered intervals S_{2k}, $k = 2, 3, \ldots, j/2$, and study the sequence of lines $l_{i_k}, l_{i_k+1}, \ldots, l_{s_k}$, where l_{i_k} is the line whose angle is the closest to the center of S_{2k}, whereas l_{s_k} is the line whose angle is the largest in S_{2k-1}. Now, as in the case for j odd, it turns out that for all k only l_{i_k} might intersect Q. Furthermore, this would occur only when the angle of l_{i_k}, α_{i_k}, satisfies

$$\mid \alpha_{i_k} - \beta^{(k)} \mid < \mid \alpha_{i_1} - \beta^{(1)} \mid \le \pi/n . \tag{7}$$

As a consequence, the search for the solution can be restricted to the vertices of the polygon formed by the two axes, the lines in $S_1 \cup S_2$ plus, possibly, the lines whose angles verify property 7. Thus the total number of lines to be considered is $O(n/j)$.
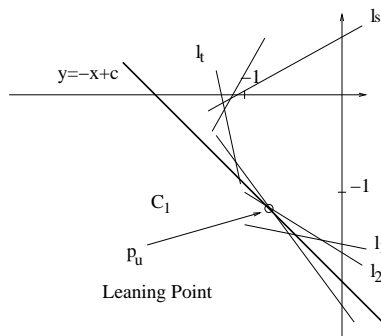
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Figure 3: Chain Property. Note that in this case also the Leaning Property holds.

3.2.1 Explicit formulae

The case for j even is very intriguing and deserves more attention on our side. In particular, we are now going to explore conditions for which the problem can be actually solved in constant time through the evaluation of explicit short formulae.

The approach is to understand when and how to generalize the case $j = 2$, that was fully investigated in [3].

To this purpose, let us introduce the following definitions that will be be instrumental to our case analysis.

Definition 2 (Chain Property) *Let $u < v$ be two positive integers. We say that the sequence of lines $\{l_i\}, i = u, u + 1, \ldots, v$, possesses the Chain Property if the intersection points p_i between consecutive lines l_i and l_{i+1} are all vertices of the convex polygon Q defined by $x, y \leq 0$, and $\forall i \in \{u, u+1, \ldots, v\}$, $x \cdot \cos \alpha_i + y \cdot \cos j\alpha_i \leq 1$.*

Definition 3 (Leaning Property) *Let $u < v$ be two positive integers. We say that the sequence $\{l_i\}, i = u, u + 1, \ldots, v$, possesses the Leaning Property if it possesses the Chain Property and, in addition, there exists an index k, $u \leq k \leq v$, such that line l_k forms, with the Oy axis, an angle larger than 45° and l_{k+1} forms, with the Oy axis, an angle smaller than 45°.*

The sense of the Chain Property is that all the intersection points p_i lie on a convex curve while the Leaning Property implies, in addition, that Q *leans* on a line of equation $y = -x + c$ for a proper $c < 0$ (see Fig. 3). As we will see, this makes point p_u a candidate solution for the problem in Q_{z_0}. Observe that the Leaning Property holds in the case $j = 2$ and that allowed us to establish the closed formula (3) for $\theta(C_{n,2})$ [3].

The following lemma establishes a condition for the Leaning Property to hold:

Lemma 3 *Let j be an even number and let $n > 2(1 + j)j$. Consider the sequence of lines $\{l_i\}$, for $\lceil \frac{n(j-1)}{2j} \rceil \leq i \leq \frac{n-1}{2}$. If it possesses the Chain Property then it possesses also the Leaning Property.*

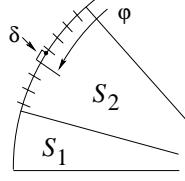**Proof** We prove that, if $n > 2(1 + j)j$, two conditions hold:

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Figure 4: Smallest angle greater than the center of S_2 (black dot).

1. Line $l_{\lceil \frac{n(j-1)}{2j} \rceil}$, that has the smallest angle greater than or equal to $\pi - \pi/j$, the center of S_2, makes with the Oy axis an angle $\gamma \geq 45$ degrees;

2. Line l_t whose angle is the largest in S_2 forms instead with the Oy axis an angle $\gamma_t \leq 45$ degrees.

The claim will then follow from the fact that the Chain Property implies that the angle formed by line l_i with Oy strictly decreases with the index i and so at some point it will necessarily cross the border of 45 degrees.

1. Let $A = (1/a, 0)$ and $B = (0, 1/b)$ be the intersection points of $l_{\lceil \frac{n(j-1)}{2j} \rceil}$ with the axes Ox and Oy, respectively. The angle in question will be larger than 45 degrees if $|a| < |b|$.

   Let $\phi = \alpha_{\lceil \frac{n(j-1)}{2j} \rceil} = \frac{2\pi}{n} \lceil \frac{n(j-1)}{2j} \rceil$ and $\delta = \phi - (\pi - \pi/j)$. We can see that $|a| = |\cos(\phi)|$ and $|b| = |\cos(j\delta)|$. Thus to prove the lemma, we need to show that $|\cos(\phi)| < |\cos(j\delta)|$ (see Fig. 4). Note that if $\delta = 0$ then the lemma is clearly true. Let $\delta \neq 0$. We can safely assume that $j > 2$, so that $\pi - (\pi - \pi/j) < \pi/2$. Then our problem reduces to solving the inequality $\cos(\pi - \phi) < \cos(j\delta)$ or equivalently:

   $$\cos\left(\pi - \frac{2\pi}{n}\left\lceil \frac{n(j-1)}{2j}\right\rceil\right) < \cos\left(j\left(\frac{2\pi}{n}\left\lceil \frac{n(j-1)}{2j}\right\rceil - (\pi - \pi/j)\right)\right).$$

   Since function cos is monotone decreasing in $[0, \pi/2]$ the above expression is verified for

   $$2\left\lceil \frac{n(j-1)}{2j}\right\rceil - n < n(j-1) - 2j\left\lceil \frac{n(j-1)}{2j}\right\rceil,$$

   and a fortiori for

   $$2\left(\frac{n(j-1)}{2j} + 1\right) - n < n(j-1) - 2j\left(\frac{n(j-1)}{2j} + 1\right)$$

   from which the claim follows.

2. By definition $t = \arg\max\{2\frac{\pi}{n}i \mid 2\frac{\pi}{n}i < \pi - \frac{\pi}{2j}\} = \lfloor \frac{n}{2} - \frac{n}{4j} \rfloor$. As before, let $A = (1/a, 0)$ and $B = (0, 1/b)$ be the intersection points of l_t with the axes Ox and Oy, respectively. The angle in question will be smaller than 45 degrees if $|a| > |b|$, i.e., if

   $$\left|\cos\left(\frac{2\pi}{n} \cdot \left\lfloor \frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right| > \left|\cos\left(j \cdot \frac{2\pi}{n} \cdot \left\lfloor \frac{n}{2} - \frac{n}{4j}\right\rfloor\right)\right|.$$

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But this can be easily verified by observing that, for $j > 2$, it is certainly true that

$$\left| \cos\left(\frac{2\pi}{n} \cdot \left\lfloor \frac{n}{2} - \frac{n}{4j} \right\rfloor \right) \right| > \cos \frac{\pi}{4} > \left| \cos\left(j \cdot \frac{2\pi}{n} \cdot \left\lfloor \frac{n}{2} - \frac{n}{4j} \right\rfloor \right) \right| .$$

(Intuitively, as n increases, one gets $\left| \cos\left(\frac{2\pi}{n} \cdot \left\lfloor \frac{n}{2} - \frac{n}{4j} \right\rfloor \right) \right| \approx |\cos(\pi - \frac{\pi}{2j})| = \cos \frac{\pi}{2j}$, and $\left| \cos\left(j \cdot \frac{2\pi}{n} \cdot \left\lfloor \frac{n}{2} - \frac{n}{4j} \right\rfloor \right) \right| \approx |\cos(j(\pi - \frac{\pi}{2j}))| = 0$.) $\qquad\qquad\square$

Interesting are the cases for which the Leaning Property holds and in addition the leaning vertex p_u belongs to Q_{z_0}. This means that p_u is never cut off by any other line not in $\{l_{i_1}, l_{i_1+1}, \ldots, l_s\}$ and that leaves it as the only solution to the problem.

Theorem 2 *Let n and j be integer numbers. Assume that*

a) *j is even and $n > 2(1 + j)j$;*

b) *Let L_b^a denote the line of equation $x \cos \frac{2\pi b}{a} + y \cos \frac{2\pi b j}{a} = 1$. Then for all $m \geq n$ the sequences of lines $\{L_i^m\}$, for $\lceil \frac{m(j-1)}{2j} \rceil \leq i \leq \frac{m-1}{2}$, all possess the Chain Property.*

Then $\theta(C_{n,j}) = z_0$, where (x_0, y_0, z_0) is the only solution to the following 3×3 linear system:

$$\begin{cases} n + 2x + 2y & = z, \\[2mm] 2x \cos(\frac{2\pi k}{n}) + 2y \cos(\frac{2\pi k j}{n}) & = z, \\[2mm] 2x \cos(\frac{2\pi(k+1)}{n}) + 2y \cos(\frac{2\pi(k+1)j}{n}) & = z, \end{cases}$$

for $k = \lfloor \frac{nj}{2(j+1)} \rfloor$.

Proof The hypothesis and Lemma 3 imply that the sequence of lines $\{L_i^n = l_i\}$, for $\lceil \frac{n(j-1)}{2j} \rceil \leq i \leq \frac{n-1}{2}$, possesses the Leaning Property. So to prove the claim we need to show that the leaning vertex p_u of Q is the intersection point $P(k)$ between lines l_k and l_{k+1} and it is not cut off by any other lines defining Q_{z_0}.

Let us start with the first claim. The Leaning Property reduces the problem to identifying the two angles across the zero of the trigonometric equation $\cos(j\alpha) - \cos(\alpha) = 0$. Basic trigonometric calculations show that the only solution to the given equation within the interval S_2 is given by

$$\alpha = \frac{\pi j}{j + 1}.$$

To see this, we first exploit the following classical identity

$$\cos(j\alpha) - \cos(\alpha) = -2 \sin \frac{\alpha(j + 1)}{2} \cdot \sin \frac{\alpha(j - 1)}{2} = 0 .$$

Then we solve for each term in the interval $[\pi - \pi/j, \pi - \pi/2j]$, obtaining the following candidates

$$\alpha_{1,s} = \frac{2\pi s}{j + 1} \quad \text{and} \quad \alpha_{2,h} = \frac{2\pi h}{j + 1},$$

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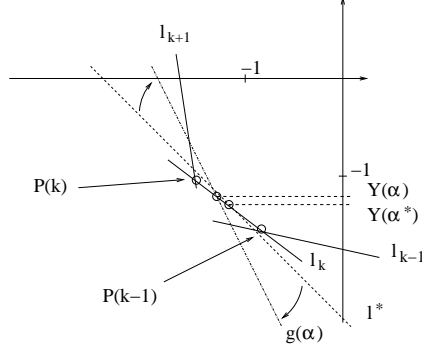
Figure 5: Motion of $g(\alpha)$: $Y(\alpha)$ increases with α.

for h, s nonnegative integers. Thus the first term nullifies for all integers s such that

$$\frac{j^2 - 1}{2j} < s < \frac{j^2 + \frac{j}{2} - \frac{1}{2}}{2j} \ .$$

We can see that there must exist only one solution given by

$$s = \left\lceil \frac{j^2 - 1}{2j} \right\rceil = \left\lfloor \frac{j^2 + \frac{j}{2} - \frac{1}{2}}{2j} \right\rfloor = \frac{j}{2} \ .$$

The second term, instead, nullifies for all integers h such that

$$\frac{(j-1)^2}{2j} < h < \frac{j^2 - \frac{3}{2}j + \frac{1}{2}}{2j} \ ,$$

and we can easily see that no such integer values can exist since, for $j \geq 4$,

$$\left\lceil \frac{(j-1)^2}{2j} \right\rceil = \left\lfloor \frac{(j-1)^2}{2j} + 1 \right\rfloor = \left\lfloor \frac{j^2 + 1}{2j} \right\rfloor = \frac{j}{2} > \left\lfloor \frac{j^2 - \frac{3}{2}j + \frac{1}{2}}{2j} \right\rfloor \ .$$

So the first of the two lines we are looking for will be found determining the largest integer k such that $\frac{2\pi k}{n} < \alpha$. This determines line l_k. The second one will be just the next one: l_{k+1}.

The second claim is a little more complex. First observe that the only lines that might interfere are those whose angles verify property 7. Now the worst possible case is represented by a line l' that cuts the Oy axis in the point $(0, -1)$ and the Ox axis in the point $(1/\cos(\beta^{(2)} + 2\pi/n), 0)$. That would have equation $x \cos(\beta^{(2)} + 2\pi/n) - y = 1$, where $\beta^{(2)} = \pi - 3\pi/j$, $\cos(j \cdot \beta^{(2)}) = -1$. This comes from the fact that the two values $1/\cos(\beta^{(2)} + 2\pi/n)$ and -1 are upper bounds to the x and y cuts of any possible "harmful" line, where, by harmful, we mean a line that verifies property 7.

Let $P(i) = (A_i, B_i)$ be the intersection point between line l_i and line l_{i+1}. Let l^* be the line of equation $x \cos \alpha^* + y \cos j\alpha^* = 1$, where α^* verifies $\cos \alpha^* = \cos j\alpha^*$. This line belongs to the family $y = -x + c$. Condition b) on the Chain Property implies that it must pass between points $P(k-1)$ and $P(k)$.
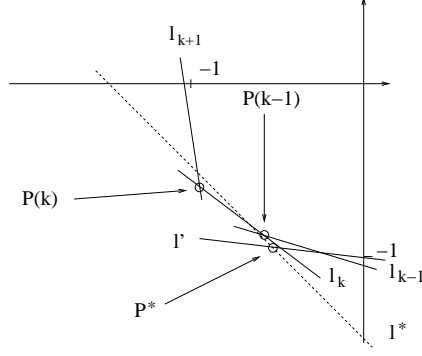
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Figure 6: $P(k)$ is not cut off by l'.

To see this, consider the family of lines

$$L = \{g(\alpha): \ x \cdot \cos \alpha + y \cdot \cos j\alpha = 1 \mid \alpha \in [\alpha_k, \alpha_{k+1}]\}.$$

Clearly, $L_i^n = l_i = g\left(\frac{2\pi i}{n}\right)$. Now, imagine line $g(\alpha)$ that moves with continuity from l_k to l_{k+1}, as α ranges from α_k to α_{k+1}, and focus on the ordinate $Y(\alpha)$ of the intersection point between l_k and $g(\alpha)$ (see Fig. 5). Since the Chain Property holds for all m, then $Y(\alpha)$ must increase monotonically from B_{k-1}^+ to B_k^-. Furthermore, $\alpha_k \leq \alpha^* \leq \alpha_{k+1}$ and so l^* must cut somewhere the segment joining $P(k-1)$ and $P(k)$.

Let $P^* = (A^*, B^*)$ be the intersection point between l' and l^*. We shall prove that $B^* < B_{k-1}$. Thus, since $B_{k-1} < B_k$, a fortiori, $P(k)$ cannot be compromised by line l' (see Fig. 6).

Solving the intersection problems for lines l', l^* and lines l_{k-1}, l_k, we obtain

$$
\begin{aligned}
B_{k-1} &= \frac{\cos \alpha_{k-1} - \cos \alpha_k}{\cos \alpha_{k-1} \cos j\alpha_k - \cos j\alpha_{k-1} \cos \alpha_k}, \\
B^* &= \frac{\cos \alpha^* - \cos(\pi - 3\pi/j + 2\pi/n)}{-\cos \alpha^* - \cos j\alpha^* \cos(\pi - 3\pi/j + 2\pi/n)} \ .
\end{aligned}
$$

Let us now compare the numerators. Clearly, it holds that

$$\cos \alpha_{k+1} < \cos \alpha^* < \cos \alpha_k < \cos \alpha_{k-1} < \cos(\pi - 3\pi/j + 2\pi/n) < 0,$$

$$|\cos \alpha_{k+1}| > |\cos \alpha^*| > |\cos \alpha_k| > |\cos \alpha_{k-1}| > |\cos(\pi - 3\pi/j + 2\pi/n)| > 0,$$

so it must be $\cos \alpha^* - \cos(\pi - 3\pi/j + 2\pi/n) < \cos \alpha_{k-1} - \cos \alpha_k$. Let us now compare the denominators. Clearly, it holds that

$$\cos j\alpha_{k-1} < \cos j\alpha_k < \cos j\alpha^* < \cos j\alpha_{k+1} < 0 \ ,$$

$$|\cos j\alpha_{k-1}| > |\cos j\alpha_k| > |\cos j\alpha^*| > |\cos j\alpha_{k+1}| > 0,$$

and so $\frac{\cos j\alpha_{k-1}}{\cos j\alpha^*} > 1$ and $\frac{\cos \alpha_{k-1}}{\cos \alpha^*} < 1$. We need to show that

$$-\cos \alpha^* - \cos j\alpha^* \cos(\pi - 3\pi/j + 2\pi/n) > \cos \alpha_{k-1} \cos j\alpha_k - \cos j\alpha_{k-1} \cos \alpha_k \ ,$$

or equivalently

$$-\cos\alpha^* + \cos j\alpha_{k-1}\cos\alpha_k > \cos\alpha_{k-1}\cos j\alpha_k + \cos j\alpha^*\cos(\pi - 3\pi/j + 2\pi/n) \ .$$

Dividing by $-\cos\alpha^* > 0$ and remembering that $\cos\alpha^* = \cos j\alpha^*$, the above inequality translates into

$$1 - \frac{\cos j\alpha_{k-1}}{\cos\alpha^*}\cos\alpha_k > \frac{\cos\alpha_{k-1}}{\cos\alpha^*}(-\cos j\alpha_k) - \cos(\pi - 3\pi/j + 2\pi/n) \ .$$

But this follows from the fact that

$$1 > \frac{\cos\alpha_{k-1}}{\cos\alpha^*}(-\cos j\alpha_k) > 0$$

and

$$\frac{\cos j\alpha_{k-1}}{\cos\alpha^*}(-\cos\alpha_k) > -\cos\alpha_k > -\cos(\pi - 3\pi/j + 2\pi/n) > 0 \ .$$

$\square$

It is now clear that the Chain Property is crucial for the determination of explicit formulae in the case j even. Surprisingly, we have observed, experimentally, that the Chain Property is always verified. Given the extent of our massive experimentation we are led to state the following conjecture.

Conjecture 1 *Let j be an even number and $n > 2(1+j)j$. Let L_b^a denote the line of equation $x\cos\frac{2\pi b}{a} + y\cos\frac{2\pi bj}{a} = 1$. Then for all $m \geq n$ the sequences of lines $\{L_i^m\}$, for $\left\lceil\frac{m(j-1)}{2j}\right\rceil \leq i \leq \frac{m-1}{2}$, all possess the Chain Property.*

If this conjecture were proven to be true, we would have determined closed formulae for $\theta(C_{n,j})$, for all j even and $n > 2(1+j)j$.

4 Concluding remarks

We have presented efficient ways to compute the theta function of circulant graphs of degree four. In particular, for j odd the problem could be reduced, in the worst case, to a 3-variable L.P. problem having at most $O(j)$ constraints, whereas for j even the bound on the number of significant constraints was shown to be $O(n/j)$. Consequently, an immediate application of Megiddo's algorithm [11] allows one to compute $\theta(C_{n,j})$ in constant time, when n and j are odd and j fixed or when j is even and $j = \Theta(n)$. It is indeed known that Megiddo's algorithm solves any L.P. problem in linear time with respect to the number of constraints, provided that the number of variables is fixed. Unfortunately, its complexity includes a factor of the order of $O(2^{s^2})$, where s is the number of variables. This makes it useless for the solution to the more general problem of computing the theta function of circulant graphs of higher degrees.

We believe that this paper is just a first step towards a better understanding of the theta function of structured graph. Further work includes the investigation of different techniques applicable to handle less sparse circulant graphs.

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