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Linear Upper Bounds for Random Walk on Small Density Random 3-CNFs

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Abstract

We analyze the efficiency of the random walk algorithm on random 3-CNF instances, and prove *linear* upper bounds on the running time of this algorithm for small clause density, less than 1.63. Our upper bound matches the observed running time to within a multiplicative factor. This is the first sub-exponential upper bound on the running time of a *local improvement* algorithm on random instances.

Our proof introduces a simple, yet powerful tool for analyzing such algorithms, which may be of further use. This object, called a *terminator*, is a weighted satisfying assignment. We show that any CNF having a good (small weight) terminator, is assured to be solved quickly by the random walk algorithm. This raises the natural question of the *terminator threshold* which is the maximal clause density for which such assignments exist (with high probability).

We use the analysis of the pure literal heuristic presented by Broder, Frieze and Upfal [8, 27] and show that for small clause densities good terminators exist. Thus we show that the Pure Literal threshold (≈ 1.63) is a lower bound on the terminator threshold. (We conjecture the terminator threshold to be in fact higher).

One nice property of terminators is that they can be found efficiently, via linear programming. This makes tractable the future investigation of the terminator threshold, and also provides an efficiently computable certificate for short running time of the simple random-walk heuristic

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1 Introduction

The phenomena we explain in this paper is best described by the following figure 1.

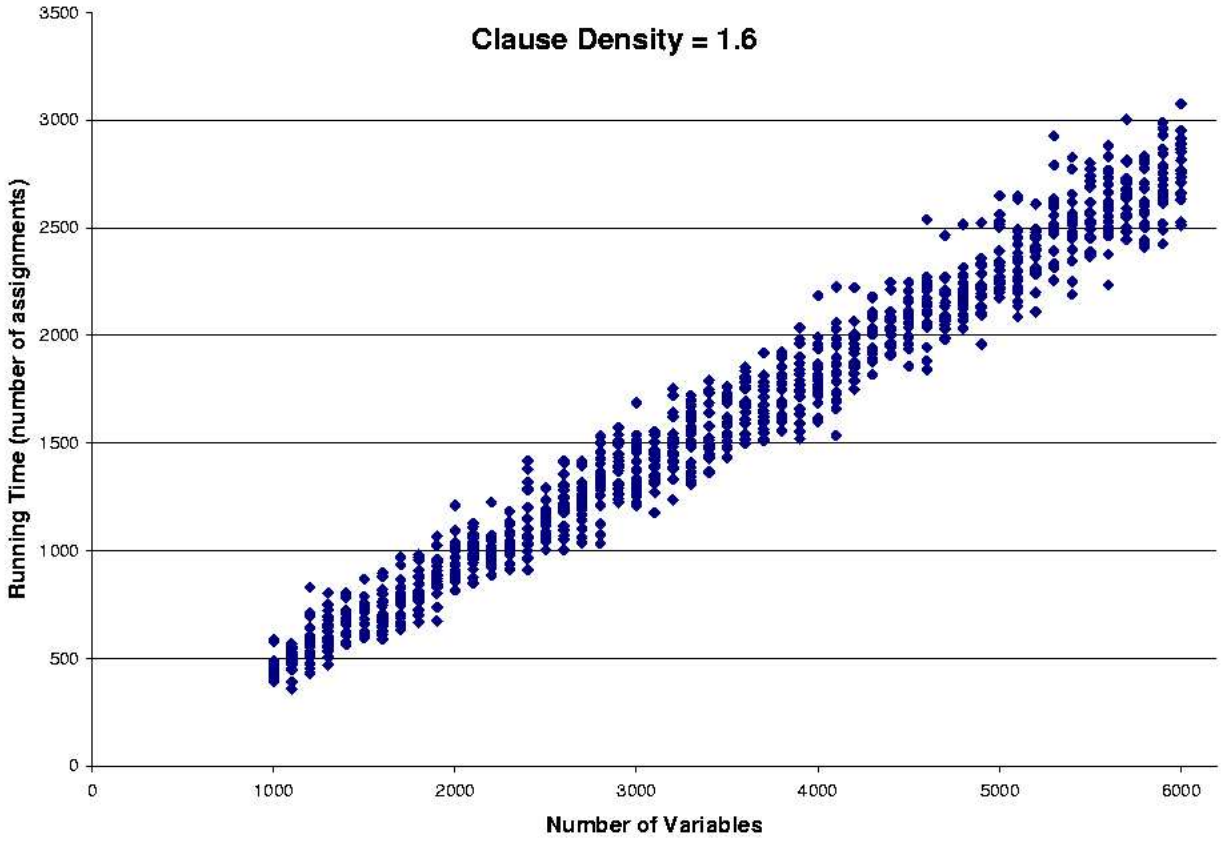

Figure 1: Running time of `RWalkSAT` on random 3-CNF formulas with clause density 1.6

`RWalkSAT` , originally introduced by Papadimitriou [29], tries to find a satisfying assignment for a CNF $\mathcal{C}$ by the following method. We start with a random assignment and as long as the assignment at hand does not satisfy the CNF, an unsatisfied clause $C \in \mathcal{C}$ is picked and the assignment to a random literal in this clause is flipped. The new assignment satisfies C, but may "ruin" the satisfiability of other clauses. We repeat this process (of flipping a bit in the current assignment according to some unsatisfied clause) until either a satisfying assignment is found (success), or we get tired and give up (failure).

In figure 1 we selected 1,020 random 3-CNF formulas and gave them as inputs to `RWalkSAT` . All instances have clause density 1.6 (i.e. the clause/variable ratio is 1.6). The number of variables n ranges between $1,000$ and $6,000$, with twenty instances for each value of n, and jumps of 100 between successive values of n. For each formula, we display the number of assignments checked until a satisfying assignment was found (the algorithm succeeded on all inputs).

Looking at this figure raises the conjecture that the running time is linear. Actually, it is even less than the number of variables (and clauses) and seems to have a slope of $\approx 1/2$. In this paper we offer an explanation for the seemingly linear running time of Figure 1. We prove that random 3-CNFs with clause density less than 1.63 require (with high probability) a *linear* number

of RWalkSAT steps. We leave the determination of the exact constant as well as explaining the seemingly linear running time at higher clause densities as intriguing open problems.

1.1 Techniques - Terminators

Our technique can be viewed as a generalization of the analysis of RWalkSAT on satisfiable 2-CNF formulas [29], so we briefly review this result. Papadimitriou showed that the Hamming distance of the assignment at time t from some fixed satisfying assignment α, is a random variable that decreases at each step with probability at least 1/2. Thus, in at most $O(n^2)$ steps this random variable will reach 0, implying we have found α. (The algorithm may succeed even earlier, by finding some other satisfying assignment).

We look at *weighted satisfying assignments*, i.e. we give non-negative weights to the bits of α. Instead of Hamming distance, we measure the *weighted distance* between α and the current assignment α^t. We show that in some cases, one can find a satisfying assignment α and a set of weights w such that for *any* unsatisfied clause at time t, the expectation of the weighted distance (between α and α^t) decreases by at least 1. Moreover, the maximal weight given to any variable is constant. In this case the running time of RWalkSAT will be linear with high probability (even better than the quadratic upper bound of [29] for 2-CNFs). We call such weighted assignments *terminators*, as their existence assures us that RWalkSAT will terminate successfully in linear time.

Two parameters of a terminator bound the running time of RWalkSAT. The *total weight* (sum of weights of all variables) bounds the distance needed to be traversed by the random walk, because the weighted distance of α^0 from α can be as large as this sum. The second parameter is the maximal weight of a variable, that bounds the variance of our random walk. Thus we define the termination weight of $\mathcal{C}$ (denoted Term($\mathcal{C}$)) to be the minimal product of these two parameters, taken over all terminators for $\mathcal{C}$. As stated above, the running time of RWalkSAT is linear (at most) in the termination weight of $\mathcal{C}$. Not all satisfiable CNFs have these magical terminators, and if $\mathcal{C}$ has no terminator we define its termination weight to be ∞.

1.2 Results

With the terminator concept in hand, we examine the running time of RWalkSAT on random 3-CNF formulas. If $\mathcal{C}$ is a random 3-CNF then Term($\mathcal{C}$) is a random variable. Understanding this variable and bounding it from above bounds the running time of RWalkSAT . Our main result (Theorem 4.1) is that for clause density ≤ 1.63, a random 3-CNF has *linear* termination weight (hence RWalkSAT succeeds in linear time). This matches the behavior depicted in Figure 1 up to a multiplicative constant. We also present a deterministic version of RWalkSAT and show it finds a satisfying assignment in linear time for the same clause density (Section 3.1).

Our result relies on previous analysis done for bounding a different SAT heuristic, called the *pure literal heuristic* [8] (see also [27] for a different and shorter analysis). This heuristic is known to succeed up to a clause density threshold of 1.63 and fail above this density. We believe terminators should exist even beyond the pure literal threshold, as the experimental data seems to indicate (see Appendix B). However, at clause density ≥ 2 only a negligible fraction of random CNFs have terminators (see Section 5), meaning we need to develop new techniques for explaining the observed linear running time at (say) density 2.5 depicted in Figure 2.

One nice property of terminators is that they can be found efficiently. A terminator is a solution to a linear system of inequalities, and thus linear programming can be used to find it. Thus, the behavior of the random variable Term($\mathcal{C}$) for random $\mathcal{C}$ can be estimated and analyzed efficiently

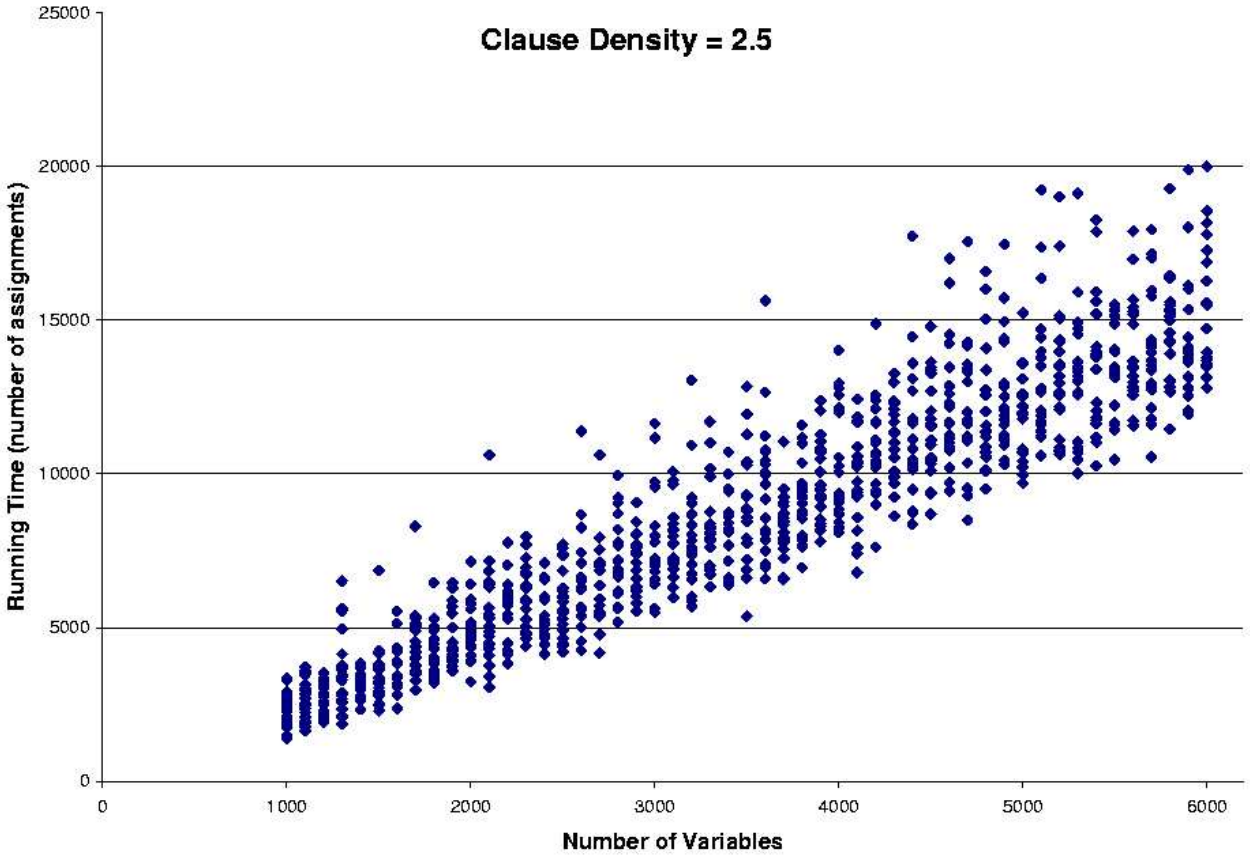

Figure 2: Running time of `RWalkSAT` on random 3-CNF formulas with clause density 2.5. Based on twenty random examples for values of $n = 1000, 1100, \ldots, 6000$.

(see Appendix B). These tests will allow us to better understand the behavior of `RWalkSAT` and its connection to the termination weight parameter.

The success of the pure literal heuristic does not necessarily imply polynomial running time for `RWalkSAT` . Indeed, in section 6 we provide a counter example that requires exponential time from `RWalkSAT` , although a solution can be found using the pure literal heuristic in linear time. Furthermore, for a random planted SAT instance with large enough clause density, `RWalkSAT` takes exponential time (Section 7). This is in contrast to the efficient performance of spectral algorithms for planted SAT presented by Flaxman [14].

1.3 History and Related Results

Local Improvement Algorithms: `RWalkSAT` was introduced by Papadimitriou, who showed it has quadratic running time on satisfiable 2-CNFs [29]. An elegant upper bound was given by Schöning [32], who showed that when restarted every $3n$ (unsuccessful) steps at a new random assignment, the expected number of restarts of `RWalkSAT` on a satisfiable k-CNF with n variables is at most $(2 \cdot \frac{k-1}{k})^n$ (compared with the exhaustive search upper bound of 2^n). The (worst case) upper bound of [32] was improved in a sequence of results, [12, 17, 6, 18, 31], and the best upper bound for 3-SAT is $(1.324)^n$ given by the recent paper [18].

`RWalkSAT` is one of a broad family of *local improvement* algorithms, (re)introduced in the 1990's with the work of [35]. These algorithms (the most famous of which is WalkSAT) are close relatives of the Simulated Annealing method, and were found to compete with DLL-type algorithms (also known as Davis-Putnam algorithms). Empirical results on random CNFs with up to $100,000$ variables seem to indicate that `RWalkSAT` requires linear time up to clause density ≤ 2.6 [30, 34]. More advanced algorithms such as WalkSAT (a Metropolis algorithm that is related to `RWalkSAT`), appear empirically to solve random instances with clause density ≤ 4, in quadratic time, and there is data indicating polynomial running time up to density ≤ 4.2 (the empirical SAT threshold is ≈ 4.26) [33].

Random 3-CNFs: Random CNFs have received much interest in recent years, being a natural distribution on NP-complete instances that seems (empirically as well as theoretically) computationally hard for a wide range of parameters. This distribution is investigated in such diverse fields as physics [26], combinatorics [22], proof complexity [11], algorithm analysis [3] and hardness of approximation [20], to mention just a few. One of the basic properties of random CNFs, is that for small density ($\Delta < 3.52...$ see [16, 25]) almost all formulas are satisfiable, whereas for large density ($\Delta > 4.506...$ see [13]) they are almost all unsatisfiable. Another interesting property is that the threshold between satisfiability and unsatisfiability is sharp [22]. It is conjectured that a *threshold constant* exists, and empirical experiments estimate it to be ≈ 4.26 [19]. The analysis of SAT solving algorithms on random CNFs has been extensively researched empirically, and random CNFs are commonly used as test cases for analysis and comparison of SAT solvers. From a theoretical point of view, several upper bounds were given on the running time of DPLL type algorithms of increasing sophistication [1, 2, 3, 8, 27, 9, 10, 15, 16, 25]. The best upper bound for random 3-CNF is given by the recent [16, 25]. An exponential lower bound on a wide class of DPLL algorithms for density ≈ 3.8 and above, was given by [3].

Upper bounds for algorithms imply lower bounds on the satisfiability threshold, and in fact, all lower bounds on the threshold (for $k = 3$) so far, have come from analyzing specific SAT solving algorithms. Most of the algorithms for which average case analysis has been applied so far are DPLL algorithms (and typically, with the exception of the recent papers [16, 25], when proving upper bounds on these algorithms, myopic[1] versions are considered). Much less is known about non-DPLL algorithms, and local improvement ones in particular. Our result is (to the best of our knowledge) the first rigorous theoretical analysis of a non-DPLL algorithm on random CNFs.

Paper Outline: After giving the necessary formal definition in Section 2, we discuss terminators in Section 3. Using terminators we prove our upper bound in Section 4. In Section 5 we give some empirical and theoretical upper bounds on the terminator threshold. We then discuss the tightness of the terminator method (Section 6). We conclude with exponential lower bounds on the running time of `RWalkSAT` on random CNFs from the "planted-SAT" distribution (Section 7).

2 Preliminaries

Random 3-CNFs For x_i a Boolean variable, a *literal* ℓ_i over x_i is either x_i or $\bar{x_i}$ (the negation of x_i), where x_i is called a *positive literal* and $\bar{x_i}$ is a *negative* one. A *clause* is a disjunction of literals, and a CNF formula is a set of clauses. Throughout this paper we reserve calligraphic notation for CNF formulas. For $\mathcal{C}$ a CNF, let $Vars(\mathcal{C})$ denote the set of variable appearing in $\mathcal{C}$ (we will

[1] See [3] for definition and tightest analysis of myopic algorithms.

always assume $Vars(\mathcal{C}) = \{x_1, \ldots, x_n\}$ for some n). An *assignment* to $\mathcal{C}$ is some Boolean vector $\alpha \in \{0,1\}^n$. A literal ℓ_i is satisfied by α iff $\ell_i(\alpha_i) = 1$. We study the following distribution.

Definition 2.1 *Let $\mathbb{F}^n_\Delta$ be the probability distribution obtained by selecting Δn clauses uniformly at random from the set of all $8 \cdot \binom{n}{3}$ clauses of size 3 over n variables. $\mathcal{C} \sim \mathbb{F}^n_\Delta$ means that $\mathcal{C}$ is selected at random from this distribution. We call such a $\mathcal{C}$ a* random 3-CNF

The Algorithm `RWalkSAT` is described by the following pseudo-code. $\mathcal{C}$ is the input CNF and T is the time bound, i.e. if no satisfying assignment is found in T steps we give up. We use the notation $UNSAT(\mathcal{C}, \alpha)$ for the set of clauses of $\mathcal{C}$ that are unsatisfied by α.

```
RWalkSAT(C, T)
  Select α ∈ {0,1}^n (uniformly) at random
  Initialize t = 0
  While t < T  {
    If C(α) = 1 Return ("INPUT SATISFIED BY " α)
    Else {
      Select C ∈ UNSAT(C, α) at random
      Select literal ℓ ∈ C at random
      Flip assignment of α at ℓ
      t++ }
    }
  Return "FAILED TO FIND SATISFYING ASSIGNMENT IN T STEPS"
```

3 Terminators

In this section we develop the tools needed to bound the running time of `RWalkSAT` on various interesting instances.

Intuition Suppose a k-CNF $\mathcal{C}$ over n variables has a satisfying assignment α such that each clause of $\mathcal{C}$ is satisfied by at least $k/2$ literals under α. In this case `RWalkSAT` will terminate in quadratic time (with high probability). The reason is that if a clause C is unsatisfied at time t by α^t, then α^t must disagree with α on at least half of the literals in C. So with probability $\geq 1/2$ we decrease the Hamming distance between our current assignment and α. If we let sim^t be the similarity of α^t and α, i.e. the number of bits that are identical in both assignments (notice $0 \leq \mathrm{sim}^t \leq n$), then sim^t is a sub-martingale, i.e. $E(\mathrm{sim}^t | \mathrm{sim}^1, \ldots, \mathrm{sim}^{t-1}) \geq \mathrm{sim}^{t-1}$. (See e.g. [28] for more information on martingales). Standard techniques from the theory of martingales show that sim reaches n (so α^t reaches α) within $O(n^2)$ steps. One elegant example of this situation is when $\mathcal{C}$ is a satisfiable 2-CNF. Papadimitriou [29] proved quadratic upper bounds on the running time of `RWalkSAT` in this case, using the proof method outlined above.

For a general 3-CNF we don't expect a satisfying assignment to have two satisfying literals per clause. Yet all we need in order to prove good running time, is to set up a measure of similarity between α^t and some fixed satisfying assignment α such that *(i)* if sim^t reaches its maximal possible value, then $\alpha^t = \alpha$; and *(ii)* the random variables $\mathrm{sim}^1, \mathrm{sim}^2, \ldots$ are a sub-martingale. We achieve both these properties by giving *non-negative weights* $w_1, \ldots, w_n$ to the variables $x_1, \ldots, x_n$. Instead of similarity, we measure the *weighted similarity* between α and α^t, defined by $\mathrm{sim}_w(\alpha, \alpha^t) \stackrel{\mathrm{def}}{=} \sum_{\alpha^t_i = \alpha_i} w_i$. Now, suppose there exists a satisfying assignment α such that for any clause C, the

expected change in sim_w, conditional on C being unsatisfied, is non-negative. Suppose further more that all w_i are bounded by a constant. Then we may conclude as above that α^t will reach its maximal value $W = \sum_i w_i$ in time $O(W^2)$.

In some cases we can do even better. We set up a system of weights such that (for any clause C) the expected change in sim_w (conditional on C being unsatisfied) is *strictly positive*. In this case the running time is *linear* in $W = \sum w_i$ (instead of quadratic). As we shall later see, such a setting of weights is possible (with high probability) for random 3-CNFs. But first we formalize our intuition.

Notation In what follows Boolean variables range over $\{-1, 1\}$. A CNF $\mathcal{C}$ with n variables and m clauses is represented by an $m \times n$ matrix $A^{\mathcal{C}}$ with $\{-1, 0, 1\}$-entries. The ith clause is represented by $A_i^{\mathcal{C}}$ (the ith row of $A^{\mathcal{C}}$) and has a -1-entry in the jth position if $\bar{x_j}$ is a literal of the ith clause of $\mathcal{C}$, a 1-entry if x_j is a literal of C_i, and is zero otherwise. Thus, if $\mathcal{C}$ is a k-CNF then the support size of each row $A_i^{\mathcal{C}}$ is at most k. A Boolean assignment is $\alpha \in \{-1, 1\}^n$ and we say α satisfies $\mathcal{C}$ iff for all $i \in [m]$

$$\langle A_i^{\mathcal{C}}, \alpha \rangle > -\left|\left|A_i^{\mathcal{C}}\right|\right|_1 \tag{1}$$

where $\langle \alpha, \beta \rangle$ is the standard inner-product over $\mathbb{R}^n$ (defined by $\sum_{i=1}^n \alpha_i \cdot \beta_i$) and $||\cdot||_1$ is the ℓ_1 norm (defined by $||\beta||_1 = \sum_{i=1}^n |\beta_i|$). It is easy to see that this definition of satisfiability coincides with the standard one.

Terminator - definition A terminator is a generalization of a satisfying assignment. On the one hand, we allow α to be any vector in $\mathbb{R}^n$, but we require a stronger satisfying condition than (1).

Definition 3.1 (Terminators) *Let $\mathcal{C}$ be a k-CNF with n variables and m clauses represented by the matrix $A^{\mathcal{C}}$. $\alpha \in \mathbb{R}^n$ is a* terminating *satisfying assignment (or* terminator*) if for all $i \in [m]$:*

$$\langle A_i^{\mathcal{C}}, \alpha \rangle \geq 1 \tag{2}$$

The termination weight of $\mathcal{C}$ is

$$\text{Term}(\mathcal{C}) \stackrel{\text{def}}{=} \min\{||\alpha||_1 \cdot ||\alpha||_\infty : \alpha \text{ terminator for } \mathcal{C}\}$$

In case $\mathcal{C}$ has no terminator we define $\text{Term}(\mathcal{C})$ *to be ∞.*

One may think of $\text{sign}(\alpha_i)$ as the Boolean assignment to variable x_i (where $\text{sign}(\alpha_i)$ is 1 if $\alpha_i \geq 0$ and is -1 otherwise) and $|\alpha_i|$ as the weight given to x_i. Notice that if α is a terminator then the $\{-1, 1\}$-vector $\text{sign}(\alpha)$ satisfies $\mathcal{C}$. This is because by property (2) in each clause there is at least one literal that agrees in sign with α.

The decisive name given in the previous definition is justified by the following claim, which is the main theorem of this section.

Theorem 3.2 (Terminator Theorem) *If a k-CNF $\mathcal{C}$ has a terminator α, then* `RWalkSAT` *succeeds on $\mathcal{C}$ in time $O(||\alpha||_1 \cdot ||\alpha||_\infty)$ with probability $\geq 1 - \exp\left(-\Omega\left(||\alpha||_1 / ||\alpha||_\infty\right)\right)$.*

Notice that we do not claim that when `RWalkSAT` terminates, it finds the assignment $\text{sign}(\alpha)$, but rather the existence of any terminator of small weight implies short running time. We can say that `RWalkSAT` is "drawn to" α, but only when using the weighted distance measure given by $|\alpha|$. If

$|\alpha_i| = 1$, this means `RWalkSAT` indeed approaches α (as is the case when each clause is satisfied by two literals). But in general, being "close" according to the weighted measure $|\alpha|$ does not imply small Hamming distance.

Proof (of Theorem 3.2): Let $\mathcal{C}$ be a k-CNF and α be a terminator of minimal weight for $\mathcal{C}$, i.e. $\text{Term}(\mathcal{C}) = ||\alpha||_1 \cdot ||\alpha||_\infty < \infty$. Let $\beta^t \in \{-1,1\}^n$ be the sequence of assignments traversed by $\texttt{RWalkSAT}(\mathcal{C})$ starting from the random assignment β^0, where $t \leq T = c \cdot k \cdot ||\alpha||_1 \cdot ||\alpha||_\infty$ (c will be fixed later). Let Y^t be the random variable $\langle \beta^t, \alpha \rangle$. If `RWalkSAT` fails to find a satisfying assignment in T steps, then

$$Y^t < ||\alpha||_1 \quad \text{for all } t < T \tag{3}$$

This is because $\langle \beta^t, \alpha \rangle = ||\alpha||_1$ implies $\beta^t = \text{sign}(\alpha)$ and $\text{sign}(\alpha)$ satisfies $\mathcal{C}$. Thus we only need to bound the probability of (3). Suppose clause C_i is picked at time t (i.e. C_i is unsatisfied by β^{t-1}). We claim the expected change in Y^t (with respect to Y^{t-1}) is precisely

$$\frac{1}{k} \cdot \langle A_i^{\mathcal{C}}, \alpha \rangle \tag{4}$$

With probability $1/k$ we flip the assignment to each literal x_j of C_i, which amounts to multiplying β_j^{t-1} by -1. Thus the expected change in Y^t is $\frac{-1}{k} \cdot \langle \beta^{t-1}|_i, \alpha \rangle$, where $\beta^{t-1}|_i$ is the restriction of β^{t-1} to support of $A_i^{\mathcal{C}}$. But C_i being unsatisfied by β^{t-1} implies $\beta^{t-1}|_i = -A_i^{\mathcal{C}}$, so equation (4) is proved. Thus by property (2) in Definition 3.1

$$E[Y^t | Y^1, \ldots, Y^{t-1}] = Y^{t-1} + \frac{1}{k} \langle A_i^{\mathcal{C}}, \alpha \rangle \geq Y^{t-1} + 1/k$$

Let $\mu^t = E[Y^t]$, and notice that by linearity of expectation, $\mu^t \geq t/k - ||\alpha||_1$ (because $\mu^0 \geq -||\alpha||_1$). We conclude that the sequence of random variables $\{X_t \stackrel{\text{def}}{=} Y^t - \mu^t : t = 1, 2, \ldots\}$ is a *martingale* and by assumption $|X_t - X_{t-1}| \leq ||\alpha||_\infty$. In order to bound (3), it suffices to bound the probability of the event "$X_T < 2||\alpha||_1 - T/k$" (if this event does not occur then $Y^T \geq -||\alpha||_1 + X_T \geq ||\alpha||_1$). Recalling $T = c \cdot k \cdot ||\alpha||_1 \cdot ||\alpha||_\infty$ we will pick $c > \frac{4}{k}$ so that

$$2||\alpha||_1 - T/k = 2||\alpha||_1 - c \cdot ||\alpha||_1 \cdot ||\alpha||_\infty < -\frac{ck}{2} ||\alpha||_1 \cdot ||\alpha||_\infty$$

We now apply Azuma's inequality and get

$$\begin{aligned}
(3) &\leq \Pr\left[X_T < -\frac{ck}{2} ||\alpha||_1 ||\alpha||_\infty\right] \\
&\leq \exp\left(-\frac{(\frac{ck}{2} ||\alpha||_1 ||\alpha||_\infty)^2}{2T(||\alpha||_\infty)^2}\right) \\
&\leq \exp\left(-\frac{c^2k^2(||\alpha||_1)^2(||\alpha||_\infty)^2}{8ck \cdot ||\alpha||_1 (||\alpha||_\infty)^3}\right) \\
&\leq \exp\left(-\frac{ck||\alpha||_1}{8||\alpha||_\infty}\right) = \exp\left(-\Omega\left(\frac{||\alpha||_1}{||\alpha||_\infty}\right)\right)
\end{aligned}$$

The theorem is proved. ∎

3.1 A Deterministic Variant of RWalkSAT

Consider the following deterministic variant of RWalkSAT, which we will call DWalkSAT. Fix an ordering on clauses in $\mathcal{C}$. Initialize α_0 to be (say) the all zeros assignment. At each step t, select the smallest clause unsatisfied by α_t and flip the assignment to *all* literals in it. Repeat this process until all clauses are satisfied. Naturally, one can introduce a time bound T and declare failure if a satisfying assignment is not found within T steps. We immediately get the following result.

Theorem 3.3 *If a CNF $\mathcal{C}$ has a terminator α, then* DWalkSAT *succeeds on $\mathcal{C}$ within $2 \cdot ||\alpha||_1$ steps.*

Proof: We closely follow the proof of the Terminator Theorem 3.2. Let $\beta^0, \beta^1, \ldots$ be the (deterministic) sequence of assignments traversed by the algorithm. Let $Y^t = \langle \beta^t, \alpha \rangle$ (noticing Y^t is not random anymore). Clearly, $Y^0 \geq -||\alpha||_1$ and if $Y^t = ||\alpha||_1$ then β^t (equals sign(α), hence) satisfies $\mathcal{C}$. So we only have to show for all t

$$Y^t \geq Y^{t-1} + 1 \tag{5}$$

This follows from the fact that the clause $\mathcal{C}_i$ flipped at time t was unsatisfied at time $t-1$, hence $\beta^{t-1}|_i = -\alpha|_i$. Flipping all variables in $\mathcal{C}_i$ amounts to adding to Y^{t-1} the amount $\langle A_i^{\mathcal{C}}, \alpha \rangle$ and this, by definition of terminator, is at least one. We have proved Equation 5 and with it the theorem. □

4 Linear Upper Bounds on Random CNFs

In this section we show that for clause densities for which the pure literal heuristic succeeds, there exist linear weight terminators. Our current analysis uses insights into the structure of such pure CNFs, but we see no reason to believe that the terminator threshold is linked to the pure literal threshold. The main theorem of this section is the following.

Theorem 4.1 *For any $\Delta < 1.63$, there exists a constant c such that with high probability $\mathcal{C} \sim \mathbb{F}_\Delta^n$ has a terminator $\alpha \in \mathbb{R}^n$ with $||\alpha||_\infty \leq c$, hence $||\alpha||_1 \leq c \cdot n$.*

Corollary 4.2 *For any $\Delta < 1.63, \epsilon > 0$, there exists a constant c such that with high probability for $\mathcal{C} \sim \mathbb{F}_\Delta^n$,* RWalkSAT *succeeds on $\mathcal{C}$ in time $c \cdot n$ with probability $\geq 1 - \epsilon$.*

Corollary 4.3 *For any $\Delta < 1.63, \epsilon > 0$, there exists a constant c such that with high probability for $\mathcal{C} \sim \mathbb{F}_\Delta^n$,* DWalkSAT *succeeds on $\mathcal{C}$ in time $c \cdot n$.*

To prove our main theorem, we construct small weight terminators for pure and expanding CNFs, and then merge the two into one small weight terminator.

4.1 Terminators for Pure CNFs

A literal ℓ in $\mathcal{C}$ is called *pure* if it appears only as a positive literal, or only as a negative literal, in $\mathcal{C}$. A clause in $\mathcal{C}$ is said to be *pure* if it contains a pure literal. When seeking a satisfying assignment, a natural strategy is to start by assigning all pure literals their satisfying assignment, and thus remove all pure clauses. The removal of pure clauses may result in the emergence on new pure literals in the restricted CNF, and the process may be repeated. The *pure literal heuristic* is the heuristic that applies this removal process until no pure clauses remain. If the remaining CNF is empty, the pure literal heuristic has found a satisfying assignment, and otherwise it failed.

Let us introduce some notation. For $\mathcal{C}$ a CNF, define $\mathcal{C}_0 = \mathcal{C}$, L_0 to be the set of pure literals in $\mathcal{C}$, and P_0 to be the set of pure clauses in $\mathcal{C}$. Recursively define $\mathcal{C}_{i+1}$ to be $\mathcal{C}_i \setminus P_i$, and let L_{i+1}, P_{i+1} be respectively the set of pure literals, and pure clauses, in $\mathcal{C}_{i+1}$. Finally, let r be the minimal i such that $L_i = \emptyset$. Notice that the pure literal succeeds on $\mathcal{C}$ iff $\mathcal{C}_r = \emptyset$. If $\mathcal{C}_r = \emptyset$ we say $\mathcal{C}$ is *r-pure*.

Theorem 4.4 *Every r-pure k-CNF over n variables has a terminator $\alpha \in \mathbb{R}^n$ with $||\alpha||_\infty \leq k^r$ and $||\alpha||_1 \leq n \cdot (k)^r$, so* $\mathrm{Term}(\mathcal{C}) \leq n \cdot k^{2r}$. *Moreover α is supported only on $\cup_{i=0}^{r-1} L_i$.*

Notice that invoking Theorem 3.2 we bound the running time of `RWalkSAT` on an r-pure k-CNF by $n \cdot k^{2r}$ (with high probability).

Proof: Let $L_0, \ldots, L_{r-1}$ be the pure literals in $\mathcal{C}_0, \ldots, \mathcal{C}_{r-1}$. Notice that $\bigcup_{j=0}^{r-1} L_j$ does not necessarily cover all variables in $\mathcal{C}$, but assigning each pure literal to 1 (i.e. if ℓ_i is pure, then set $\mathrm{sign}(\alpha_i) = \mathrm{sign}(\ell_i)$) and assigning the other variables arbitrarily gives a satisfying assignment α. We now deal with the weights (absolute values) of α. Fix the weight of each variable in L_j to k^{r-j}. For any variable $x_i \notin \bigcup_{j=0}^{r-1} L_j$ fix its weight to 0.

To see α is a terminator (of weight nk^r), consider any clause $C_i \in P_j$. By definition of P_j there are no literals from $L_0, \ldots, L_{j-1}$ appearing in C. Thus all literals appearing in C have weight $\leq (k)^{r-j}$. There is at least one literal $\ell_s \in C$ that has weight k^{r-j} and agrees with α_s in sign, and any literal disagreeing with α must have weight $\leq (k)^{r-j-1}$. Hence

$$\langle A_i^{\mathcal{C}}, \alpha \rangle \geq k^{r-j} - (k-1) \cdot k^{r-j-1} \geq 1$$

□

Broder, Frieze and Upfal showed that with high probability the pure literal heuristic finds a satisfying assignment for a random 3-CNF with clause density < 1.63 [8] (for a simpler analysis of the same heuristic see [27]). In particular, the following theorem follows from the work of [8]. A proof of this theorem can be found in appendix A.

Theorem 4.5 *[8] For every $\Delta < 1.63$ there exists a constant c such that with high probability $\mathcal{C} \sim \mathbb{F}_\Delta^n$ is $c \log n$-pure.*

By applying Theorems 3.2,4.4 to Theorem 4.5 we conclude that the running time of `RWalkSAT` on a random instance (with small enough clause density) is at most polynomial.

4.2 Terminators for Expanding CNFs

Our next step in proving Theorem 4.1 starts with the following Theorem, which is a combination of a result of Broder, Frieze and Upfal [8] and (the now) standard analysis of random CNFs, originating in the work of Chvátal and Szemerédi [11]. Being standard and somewhat technical, we defer its proof to appendix A.

Definition 4.6 *For $\mathcal{C}$ a CNF, we say $\mathcal{C}$ is an* (r, c)-expander *if for all $\mathcal{C}' \subseteq \mathcal{C}$ $|\mathcal{C}'| \leq r$, $|Vars(\mathcal{C}')| \geq c \cdot |\mathcal{C}'|$.*

Theorem 4.7 *For every $\Delta < 1.63$ there exists an integer d such that for $\mathcal{C} \sim \mathbb{F}_\Delta^n$, with high probability $\mathcal{C}_d$ is a $(|\mathcal{C}_d|, 7/4)$-expander, where $\mathcal{C}_d$ is the CNF remaining of $\mathcal{C}$ after removing the d outermost pure layers.*

This Theorem assures us that after removing a *constant* number of the layers from a random $\mathcal{C}$ (with small clause density), we have in hand a residual CNF $\mathcal{C}_d$, such that *any* subset of it, including all of $\mathcal{C}_d$, has a very large set of neighbors. This in turn implies the existence of small weight terminators for $\mathcal{C}_d$.

Theorem 4.8 *If $\mathcal{C}$ is an $(|\mathcal{C}|, 7/4)$-expanding 3-CNF over n variables, then $\mathcal{C}$ has a terminator $\alpha \in \mathbb{R}^n$ with $||\alpha||_\infty \leq 4$ (hence $||\alpha||_1 \leq 4n$).*

Proof: Form the following bipartite graph G. On the left hand, put one vertex for each clause in $\mathcal{C}$. On the right hand side, put 4 distinct vertices for each variable appearing in $\mathcal{C}$. Connect the vertex labeled by the clause C to all 12 vertices labeled by variables appearing in C. We do not care if the appearance is as positive or negative literals.

Since $\mathcal{C}$ is an $(|\mathcal{C}|, \frac{7}{4})$-expander, G has expansion factor 7, i.e. for all subsets S on the left hand side, $|N(S)| \geq 7 \cdot |S|$, where $N(S)$ is the set of neighbors of S. By Hall's Matching Theorem we conclude there is an 7-matching from the left hand side to the right, i.e. each node C on the left hand can be associated with a set of 7 of its neighbors on the right hand side (denoted $N'(C)$), such that for all clauses $C \neq D$, $N'(C) \cap N'(D) = \emptyset$. We now use N' to define our terminator α. For any variable x, if there exists a clause C such that $N'(C)$ has at least 3 members labeled by x, then we say x is *associated* with C, and the weight of x is $|N'(C)|$ (notice this weight is either 3 or 4). For any variable x_i associated with a clause C, set $\text{sign}(\alpha_i)$ to the value that satisfies C and set $|\alpha_i|$ to the weight of x_i. Set all other variables to zero. α is well defined because a variable can be associated with at most one clause. We are left with verifying that it is a terminator. This follows by a case analysis, using the fact that each clause has a dozen neighbors, and seven of them are in $N'(C_i)$. There are three cases to consider.

C_i **has at least two associated variables:** In this case, $\text{sign}(\alpha)$ agrees with C on at least two variables, and each variable has weight at least 3. The remaining variable has weight at most 4, so $\langle A_i^{\mathcal{C}}, \alpha \rangle \geq 6 - 4 \geq 2$.

C_i **has one associated variable of weight three:** The remaining four neighbors of $N'(C_i)$ must be evenly split between the two remaining variables of C (otherwise C_i would have two associated variables). So the remaining pair of variables of C_i have weight zero. Hence $\langle A_i^{\mathcal{C}}, \alpha \rangle = 3$.

C_i **has one associated variable of weight four:** The remaining three neighbors of $N'(C_i)$ are split between the remaining two variables. One variable has two such neighbors (and hence zero weight) and the other has one neighbor, so the weight of this literal is at most 3. Thus, $\langle A_i^{\mathcal{C}}, \alpha \rangle \geq 4 - 3 = 1$.

Theorem 4.8 follows. □

4.3 Small Weight Terminators for Random CNFs

Proof (of Theorem 4.1): By Theorem 4.7, (with high probability) $\mathcal{C}$ can be partitioned into the d outermost pure layers $\mathcal{C}' \stackrel{\text{def}}{=} \cup_{i=0}^{d-1} P_i$ and the remaining residual inner core $\mathcal{C}'' = \mathcal{C}_d$. This inner core is a $(|\mathcal{C}''|, 7/4)$-expander. We know (by Theorems 4.4, 4.8 respectively) how to construct terminators for each of these formulas, so all we need to do is merge them into a single terminator for $\mathcal{C}$.

Let α', α'' be the respective terminators of $\mathcal{C}', \mathcal{C}''$. By Theorem 4.4 α' has all its support on pure literals, which do not appear in $\mathcal{C}''$. Thus the supports of α' and α'' are disjoint. We merge the two

assignments by defining α as the assignment that agrees with $9 \cdot \alpha'$ on the support of α', and agrees with α'' otherwise (the reason for multiplying α' by the scalar 9 will soon become clear) . By our previous remark (that α' and α'' have disjoint supports) α is well defined and we now prove it is a terminator.

Consider a clause $C_i \in \mathcal{C}$. If $C_i \in \mathcal{C}''$ then $\langle A_i^{\mathcal{C}}, \alpha \rangle = \langle A_i^{\mathcal{C}}, \alpha'' \rangle \geq 1$, because all literals appearing in $\mathcal{C}''$ are given zero weight by α'. Otherwise, $C_i \in \mathcal{C}'$ might have some of its (non-pure) literals in $Vars(\mathcal{C}'')$, but recall that the maximal weight of α'' is 4, so in the worst case C_i has two literals with weight 4 coming from α''. Thus $\langle A_i^{\mathcal{C}}, \alpha \rangle \geq 9 - 2 \cdot 4 = 1$. We have shown the existence of a terminator of linear total weight, and the proof of Theorem 4.1 is complete. ∎

5 Investigating the Terminator Threshold

Acknowledgements The contents of this section are due to Jeong Han Kim [Personal Communication] and we are grateful to him for allowing us to include them in this paper.

When $\mathcal{C}$ is a random CNF, Term($\mathcal{C}$) is a random variable. Since Term($\mathcal{C}$) bounds the running time of `RWalkSAT` , investigating this random variable is an interesting question. The property of having a terminator α with $||\alpha||_\infty \leq w$ is monotone with respect to addition of new clauses. Thus one can define the *terminator threshold* θ_n^w as the density for which a terminator α, $||\alpha||_\infty \leq w$ exists with probability 1/2.

Claim 5.1 *A CNF $\mathcal{C}$ with m clauses and n variables has some terminator iff $0 \notin$ convex-hull($\{A_i^{\mathcal{C}} : i = 1, \ldots, m\}$).*

Proof: Think of a terminator α as the normal of a hyper-plane in $\mathbb{R}^n$ passing through zero. This hyperplane partitions $\mathbb{R}^n$ into two parts. $\langle A_i^{\mathcal{C}}, \alpha \rangle > 0$ iff the point $A_i^{\mathcal{C}}$ lies in the positive half of $\mathbb{R}^n$. Thus $\langle A_i^{\mathcal{C}}, \alpha \rangle > 0, i = 1, \ldots, m$ iff zero is not in the convex hull of the points. □

Füredi proved the following general theorem (he gave a tighter bound than presented here, but the form we quote is sufficient for our purposes). A set of points $P \subset \mathbb{R}^n$ is *symmetric* if $p \in P \Rightarrow (-p) \in P$.

Theorem 5.2 *[23] Let $\{P_n \subset \mathbb{R}^n\}_{n \in \mathbb{N}}$ be an infinite family of finite symmetric sets of points. Suppose $(2 + \epsilon)n$ points are selected uniformly at random from P_n. Then*

$$\lim_{n \to \infty} \Pr[0 \notin \textit{ convex-hull of points}] = 0$$

In our case P_n is the symmetric set of $\{-1, 0, 1\}$-valued points with support size three. Thus, by Füredi's theorem when the clause density is greater than 2, whp there is no terminator. Notice this upper-bound on the terminator threshold holds for any k-CNF, even for non-constant k (e.g. $k = n$). Combining Theorem 4.1 with Füredi's Theorem gives for $k = 3$ the following bounds:

$$1.63 \leq \theta_n^\infty \leq 2$$

We leave the resolution of the terminator threshold for $k = 3$ as an interesting open problem.

For the case of 2-CNFs we can bound the terminator threshold from above by 1, because this is the satisfiability threshold for random 2-CNFs (and a terminator implies satisfiability). It seems reasonable to conjecture that for $k = 2$ the satisfiability and terminator threshold coincide. This could be used to prove that for random 2-CNFs below the satisfiability threshold, `RWalkSAT` terminates in linear time (as opposed to the quadratic upper bound guaranteed for any satisfiable 2-CNF by [29]).

6 Tightness of Terminator Based Bounds

In this section we show that the upper bound derived by the terminator method is tight, even for pure CNFs. We present pure CNFs such that the running time of `RWalkSAT` on them is exponential in the number of variables, and also lower bounded by the terminator weight.

Theorem 6.1 *For arbitrarily large n, there exist pure 3-CNFs over n variables, with total terminator weight $\geq 2^{n/2}$, and the running time of* `RWalkSAT` *on them is $2^{\epsilon n}$ for some $\epsilon > 0$.*

Proof: Use the following formula, which is a slight variation of the X-DAG contradiction used in [7].

Definition 6.2 *Let $\mathcal{G}_n$ be the following CNF over variables $x_1, \ldots, x_n, y_1, \ldots, y_n, z$:*

$$\{\bar{x}_1\} \wedge \{\bar{y}_1\} \wedge \bigwedge_{i=1}^{n-1} \{x_i \vee y_i \vee \bar{x}_{i+1}\}$$
$$\wedge \bigwedge_{i=1}^{n-1} \{x_i \vee y_i \vee \bar{y}_{i+1}\} \wedge \{x_n \vee y_n \vee \bar{z}\}$$

$\mathcal{G}_n$ has a unique satisfying assignment, $\vec{0}$. Moreover, $\mathcal{G}_n$ is n-pure, because $\bar{z}$ appears only in one clause, and once z is satisfied and removed, $\bar{y}_n, \bar{x}_n$ each appear in one clause in the remaining formula. Thus, one can repeatedly remove x_{i-1}, y_{i-1} until all the formula is satisfied. This implies the existence of a terminator of weight 3^n, and it is not hard to see that any terminator must have weight 2^n at least. We claim that `RWalkSAT` requires exponential time to succeed on $\mathcal{G}_n$.

Let X_t be the number of ones assigned by α_t to the variables $x_2, \ldots x_n, y_2, \ldots y_n$. Whp $X_0 > (1-\epsilon)n$, and if `RWalkSAT`$(\mathcal{G}_n, T)$ succeeds, we know $X_T = 0$. But for every step t, the probability of X_t decreasing is at most $1/3$. The theorem follows. □

7 Lower Bounds for Large Density Planted SAT

In this section, we show that `RWalkSAT` is not a good algorithm for random CNFs with large clause density. By definition, `RWalkSAT` gives the correct answer on any unsatisfiable formula. For large enough clause density ($\Delta > 4.6$), almost all formulas in $\mathbb{F}_\Delta^n$ are unsatisfiable [13]. Thus, one may argue that `RWalkSAT` operates very well for these densities. On second thought, on this distribution, even the constant time algorithm that fails on every input, without reading it, operates well. Thus, it makes sense to discuss the performance of `RWalkSAT` only on the uniform distribution over *satisfiable* formulas with Δn clauses (denoted $\mathbf{SAT}_n^\Delta$). The problem is that for small densities, $\mathbf{SAT}_n^\Delta$ is not well characterized, we don't know how to analyze it. Thus, we propose to look at the following pair of *planted SAT* distributions over satisfiable 3-CNFs.

Definition 7.1 (Planted SAT) *Let $\mathbb{S}_\Delta^n$ be the distribution obtained by selecting at random $\beta \in \{0,1\}^n$, and selecting at random Δn clauses out of all clauses of size 3 that are satisfied by β. Denote a random formula from this distribution by $\mathcal{C} \sim \mathbb{S}_\Delta^n$.*

Let $\mathbb{P}_\Delta^n$ be the distribution obtained by selecting at random $\beta \in \{0,1\}^n$, and for each clause C satisfied by β, select C to be in $\mathcal{C}$ with independent probability $p_n^\Delta = \frac{6\Delta}{7(n-1)(n-2)}$. Denote a random formula from this distribution by $\mathcal{C} \sim \mathbb{P}_\Delta^n$.

This distribution is highly interesting in its own right. It is the analog of the *planted clique* and *planted bisection* distributions, studied e.g. in [5, 21, 24]. There are efficient spectral algorithms for finding the satisfying assignment for the planted SAT distribution [14] and in this section we argue that `RWalkSAT` performs poorly (takes exponential running time) on this distribution.

Theorem 7.2 (Main Lower Bound) *There exists a constant $\Delta_0 > 0$, such that for all $\Delta \geq \Delta_0$, (Δ may be a function of n), whp for $\mathcal{C} \sim \mathbb{P}^n_\Delta$, or $\mathcal{C} \sim \mathbb{S}^n_\Delta$*

$$\mathbf{P}[\texttt{RWalkSAT}(\mathcal{C}, 2^{\epsilon n}) \text{ succeeds}] \leq 2^{-\epsilon n}$$

where $\epsilon > 0$ is some a constant, depending on Δ.

For simplicity, our proof will be only for the distribution $\mathbb{P}^n_\Delta$, and we point out that the same analysis can be carried over to the distribution $\mathbb{S}^n_\Delta$.

The rest of this section is devoted to the proof of theorem 7.2. We warm up by discussing the case of $\mathcal{C}$ being the maximal size CNF satisfying β, and then apply our insights to the case of a random CNF. For the rest of this section we assume without loss of generality that β, the random planted assignment, is the all zero vector, denoted $\vec{0}$.

The full CNF of size n has all clauses of size 3 that are satisfied by $\vec{0}$. In the next subsection we analyze the behavior of `RWalkSAT` on this CNF, and show its inefficiency. We then generalize our analysis to $\mathcal{C} \sim \mathbb{P}^n_\Delta$, claiming that $\mathcal{C}$ is a "random piece" of the full CNF, and hence the behavior of `RWalkSAT` on $\mathcal{C}$ will be close to its behavior on this simple formula.

7.1 Analysis of `RWalkSAT` on the Full CNF

Definition 7.3 (Full CNF) *$\mathcal{F}_n$ is the 3-CNF containing all clauses of size exactly 3 (without repetition of literals) satisfying $\vec{0}$.*

Theorem 7.4 $\mathbf{P}[\texttt{RWalkSAT}(\mathcal{F}_n, 2^{n/100}) \text{ succeeds}] \leq 2^{-n/100}$

For the proof of theorem, we need the following lemma. The *line* of length n is the graph $G =< V, E >$ where $V = \{0, 1, \ldots, n\}$ and $E = \{(i, i+1) : 0 \leq i < n\}$.

Lemma 7.5 *Let R be a random walk on the line of length n, starting at a position $b' \geq b$. Assume there is an interval $0 \leq a < b$ such that for all vertices $i \in \{a, a+1, \ldots, b\}$ the probability of making a move in the direction of 0 when standing on vertex i, is at most $1/2 - \gamma$, for some constant $\gamma > 0$. Then*

$$\mathbf{P}[R \text{ reaches } 0 \text{ in less than } e^{\gamma(b-a)} \text{ steps }] \leq e^{-\gamma(b-a)}$$

Proof: Let $T = 2^{\epsilon n}$. for $t \leq T$, let r_t be the position of R at time t, and let X_t be the random variable that is 1 if R does a good move (towards 0) at time t, and -1 otherwise, Formally, $X_t = r_t - r_{t+1}$. Assume R reaches 0 at time t_2. Since $r_1 \geq b$, there must be a time interval $[t_1, t_2]$ in which $0 \leq r_t \leq b \;\; t \in [t_1, t_2]$. This means $\sum_{t=t_1}^{t_2} X_t = c$, where $c = b - a$. Let $t_2 - t_1 = 2s + c$. Recalling that for all $t \in [t_1, t_2]$, $\mathbf{P}[X_t = -1] \leq 1/2 - \gamma$, we get

$$\mathbf{P}[\sum_{t=t_1}^{t_2} X_t = c] \quad \leq \quad \binom{2s+c}{s} \cdot (1/2 - e)^{s+c} \cdot (1/2 + \gamma)^s \tag{6}$$

$$\leq \binom{2s+c}{s+c/2} \cdot (1/4-\gamma^2)^s \cdot (1/2-\gamma)^c \tag{7}$$

$$\leq \frac{2^{2s+c}}{\sqrt{2s+c}} \cdot (1/4-\gamma^2)^s \cdot (1/2-\gamma)^c \tag{8}$$

$$\leq \frac{1}{\sqrt{2s+c}} \cdot (1-4\gamma^2)^s \cdot (1-2\gamma)^c \tag{9}$$

$$\leq \frac{1}{\sqrt{2s+c}} \cdot \exp(-(4\gamma^2 s + 2\gamma c)) \tag{10}$$

$$\leq \frac{\exp(-(2\gamma c))}{\sqrt{c}} \tag{11}$$

Since there are at most T^2 possible selections of t_1, t_2, setting $T = e^{\gamma c/2}$, and taking a union bound, completes the proof. □

Proof [Theorem 7.4]: For $i = 1 \ldots 3$, let $\mathcal{F}_n^i$ be the set of clauses having i literals satisfied under the unique satisfying assignment $\vec{0}$. For $\alpha \in \{0,1\}^n$, $|\alpha| = \delta n$, let $m_{\delta,i}$ be the number of clauses in $\mathcal{F}_n^i$ that are not satisfied by α. $m_{\delta,i}$ is well defined, because it depends only on δ, and not on the actual assignment α. $m_{\delta,1}$ is the set of all clauses having exactly one negative literal, such that this literal is assigned 1 by α, and the remaining positive literals are assigned 0 by α. Since $|\alpha| = \delta n$, there are $\delta n \cdot \binom{(1-\delta)n}{2}$ such clauses. analogous calculations performed for $m_{\delta,2}, m_{\delta,3}$ yield:

$$m_{\delta,1} = \delta n \cdot \binom{(1-\delta)n}{2} \tag{12}$$

$$m_{\delta,2} = \binom{\delta n}{2} \cdot (1-\delta)n \tag{13}$$

$$m_{\delta,3} = \binom{\delta n}{3} \tag{14}$$

Since $\mathcal{F}_n$ has only one satisfying assignment, one can characterize the moves of `RWalkSAT`$(\mathcal{F}_n)$ as either *good*, or *bad*. A *good move* is one that reduces the weight of α_t, bringing us closer to the satisfying assignment, whereas a *bad move* increases this weight. We now calculate the probability of making a good move on $\mathcal{F}_n$, assuming the current assignment has weight δn. Let us denote this probability by P_δ, and let $m_\delta = m_{\delta,1} + m_{\delta,2} + m_{\delta,3}$.

$$P_\delta = \frac{m_{\delta,3}}{m_\delta} + \frac{m_{\delta,2}}{m_\delta} \cdot \frac{2}{3} + \frac{m_{\delta,1}}{m_\delta} \cdot \frac{1}{3} \approx \frac{3\frac{\delta^3}{6} + 2\frac{\delta^2}{2}(1-\delta) + \delta\frac{(1-\delta)^2}{2}}{3\left(\frac{\delta^3}{6} + \frac{\delta^2(1-\delta)}{2} + \frac{\delta(1-\delta)^2}{2}\right)} \tag{15}$$

$$= \frac{\delta^2 + 2\delta(1-\delta) + (1-\delta)^2}{\delta^2 + 3\delta(1-\delta) + 3(1-\delta)^2} = \frac{1}{\delta^2 - 3\delta + 3} \tag{16}$$

Elementary analysis shows that P_δ is an increasing function of δ in the range $(0,1)$, with $\lim_{\delta\to 0} P_\delta = 1/3$, $\lim_{\delta\to 1} P_\delta = 1$, and $P_\delta = 1/2$ when $(\delta-1)^2 = \delta$, which happens for $\delta^\star = \frac{3-\sqrt{5}}{2} \approx 0.382$.

From here we are almost done. Whp $|\alpha_0| \geq (1/2-\epsilon)n$, and for any assignment α_t, the probability of a good move depends only on $|\alpha_t|$. Thus the probability `RWalkSAT` succeeds in T steps is at most the probability of a random walk on the line, starting at distance $> (1/2-\epsilon)n$, reaches 0 in T steps, where the probability of moving in the direction of 0 when standing at position i is $P_{i/n}$. Using (16), for $i \leq n/10$ we get $P_{i/n} < 1/2 - 1/10$. Lemma 7.5 completes our proof. □

7.2 Generalization to $\mathbb{P}^n_\Delta$

In this section we complete the proof of theorem 7.2. As previously mentioned, the lower bound follows from the observation that $\mathcal{C}$ is obtained by taking a "small piece" of $\mathcal{F}_n$. Thus, we expect the behavior of `RWalkSAT` on $\mathcal{C}$ to be similar to its behavior on $\mathcal{F}_n$. There are two points we need to be careful about. The first is that $\mathcal{C}$ may have other satisfying assignments. We argue in lemma 7.7 that although this is true, for large Δ these assignments are all found within small Hamming distance of $\vec{0}$. The second problem is that there might be small weight assignments for which the probability of making a good step is large. We show in lemma 7.8 that for large Δ, whp this does not happen. Thus `RWalkSAT` operates on $\mathbb{P}^n_\Delta$ almost as poorly as it operates on $\mathcal{F}_n$. The proofs of lemmas 7.7,7.8 follow the proof of the theorem. In our proofs we use the following Chernoff bounds [28, Theorems 4.1,4.2].

Theorem 7.6 *For $X \sim B[n,p]$, with $0 < p < 1$, $\mu = E[X] = np$, and for all $c > 0$:*

$$\mathbf{P}[X > (1+c)\mu] < \left(\frac{e^c}{(1+c)^{1+c}}\right)^\mu \tag{17}$$

For $0 < \epsilon \leq 1$:

$$\mathbf{P}[X < (1-\epsilon)\mu] < e^{-\epsilon^2\mu/2} \tag{18}$$

Additionally, we make often use of the standard bound

$$\binom{n}{\lambda n} \leq e^{H_e(\lambda)\cdot n} \tag{19}$$

where $H_e(q) = q\ln\frac{1}{q} + (1-q)\ln\frac{1}{1-q}$ is the entropy measure.

Proof [Theorem 7.2]: Let A_δ be the event that $\mathcal{C}$ has a satisfying assignment of Hamming distance greater than δn from β (where β is the planted assignment).

Lemma 7.7 *For any $\delta > 0$ there exists a constant Δ', such that for all $\Delta \geq \Delta'$, the probability that $\mathcal{C} \sim \mathbb{P}^n_\Delta$ has property A_δ is exponentially small.*

Define a *good move* of `RWalkSAT` as one that decreases the weight of the current assignment. Call a clause *good* iff it has at least two negative literals. A good clause, if selected, will make a good move with probability at least 2/3, whereas a *bad clause*, having only one negative literal, will make a good move only with probability 1/3. We claim that the numbers of bad and good clauses are tightly concentrated around their expected values, and thus the probability of a good move in $\mathcal{C}$ is roughly that of a good move on $\mathcal{F}_n$.

Fix $\alpha \in \{0,1\}^n$, $|\alpha| = \delta n$, and recall the definition of $m_{\delta,i}$ from equations (12). For $i = 1\ldots 3$ let $X_{\delta,i}$ be the number of clauses in $\mathcal{C}$ that have i negative literals, and are not satisfied by α. Since α is fixed, whereas $\mathcal{C}$ is random, $X_{\delta,i}$ is a random variable depending only on δ and independent of the particular assignment α. $X_{\delta,i}$ is a sum of $m_{\delta,i}$ independent coin tosses, each with probability p^Δ_n. Moreover, all $X_{\delta,i}$'s are completely independent of each other, because the $\mathcal{F}^i_n$'s are disjoint. Let $B[n,p]$ denote the binomial distribution over n coin tosses, each with success probability p. According to our definitions, $X_{\delta,i} \sim B[m_{\delta,i}, p^\Delta_n]$, and $\mu_{\delta,i} \stackrel{\text{def}}{=} E[X_{\delta,i}] = p^\Delta_n \cdot m_{\delta,i}$. Plugging in the values of $m_{\delta,i}$ we get

$$\mu_{\delta,1} = \frac{3\delta(1-\delta)^2\Delta n}{7} \tag{20}$$

$$\mu_{\delta,2} = \frac{3\delta^2(1-\delta)\Delta n}{7} \tag{21}$$

$$\mu_{\delta,3} = \frac{\delta^3 \Delta n}{7} \tag{22}$$

Let $B^1_\delta(\epsilon)$ be the event that the number of bad clauses not satisfied by α, $X_{\delta,1}$, is smaller than $(1-\epsilon)\mu_{\delta,1}$. Similarly, let $B^2_\delta(c)$ be the probability that the number of good clauses, $X_{\delta,2} + X_{\delta,3}$ is greater than $(1+c)\cdot(\mu_{\delta,2}+\mu_{\delta,3})$. Notice that by definition, $\epsilon \leq 1$, whereas c can be much larger. Let $Bad(\delta, c, \epsilon)$ be the event that there exists an assignment α, $|\alpha| = \delta' n$, $\delta/2 \leq \delta' \leq \delta$, such that $B^1_{\delta'}(\epsilon)$ or $B^2_{\delta'}(c)$ occurs, and hence the numbers of good or bad clauses are far from the expected. The next lemma shows that for large enough Δ, this is not likely to happen.

Lemma 7.8 *For any $\epsilon > 0, c > 2e^2$ and $0 < \delta \leq 1/2$ there exists a constant $\Delta'' > 0$ such that for all $\Delta \geq \Delta''$, the probability that $\mathcal{C} \sim \mathbb{P}^n_\Delta$ has property $Bad(\delta, c, \epsilon)$ is exponentially small.*

We do not attempt to optimize constants. Set $\epsilon = 1/2, c = 20$, and $\delta = c^{-3}$. By lemmas 7.7 and 7.8, there exists some Δ for which both $\mathbf{P}[A_{\delta/2}]$ and $\mathbf{P}[Bad(\delta, c, \epsilon)]$ are exponentially small. Fix this Δ. Whp for $\mathcal{C} \sim \mathbb{P}^n_\Delta$, all satisfying assignments have weight $< \delta n/2$. Notice that for $\delta < 1/2$, $\mu_{\delta,i}$ increases with δ. For any α with weight $\delta n/2 \leq |\alpha| \leq \delta n$, by lemma 7.8 the number of bad clauses unsatisfied by α is at least

$$1/2 \cdot \mu_{\delta/2,1} > \frac{3\delta\Delta n}{28} > \frac{c^{-3}\Delta n}{10}$$

Similarly, assuming $\mu_{\delta,2} > \mu_{\delta,3}$, which is true for $\delta < 1/2$, the number of good clauses unsatisfied by α is at most

$$21(\mu_{\delta,2} + \mu_{\delta,3}) \leq 42 \cdot \frac{3}{7}\delta^2(1-\delta)\Delta n < 20\delta^2\Delta n = c^{-5}\Delta n$$

Thus,

$$\mathbf{P}[\text{Good move}] \leq \frac{\left(\frac{1}{3}\cdot\frac{c^{-3}}{10} + c^{-5}\right)\Delta n}{\frac{c^{-3}}{10}\Delta n} = \frac{1}{3} + \frac{1}{40} << 1/2$$

Whp the initial assignment of `RWalkSAT` will have weight greater than δn. Any satisfying assignment must have weight less than $\delta n/2$. For any assignment α having weight $\delta n/2 \leq |\alpha| \leq \delta n$, the probability of `RWalkSAT` making a good move is very close to $1/3$. Lemma 7.5 completes the proof of the theorem. □

Proof [Lemma 7.7]: Assuming wlog $\beta = \vec{0}$, let us bound the probability of the event A_δ. For each fixed $\alpha \in \{0,1\}^n$ having weight δn, there are $\binom{n}{3} - \binom{(1-\delta)n}{3} \approx (1-(1-\delta)^3)n^3/6$ clauses that are satisfied by $\vec{0}$ and not by α.

$$\begin{aligned} \mathbf{P}[\mathcal{C}(\alpha) = 1] &\approx (1 - p_n^\Delta)^{(1-(1-\delta)^3)n^3/6} && (23)\\ &\leq \exp(-(1-(1-\delta)^3)\cdot\Delta/7) && (24)\end{aligned}$$

Using a union bound and the standard inequality (19), we get

$$\begin{aligned} \mathbf{P}[A_\delta] &\leq n \cdot \binom{n}{\delta n} \cdot \exp(-(1-(1-\delta)^3)\cdot\Delta/7) && (25)\\ &\leq \exp((H_e(\delta) - \frac{\Delta(1-(1-\delta)^3)}{7}) \cdot n + \ln n) && (26)\end{aligned}$$

Thus, when

$$\Delta > \frac{7H_e(\delta)}{1-(1-\delta)^3} \tag{27}$$

we get that $\mathbf{P}[A_\delta] \leq \exp(-\epsilon n)$, for some constant $\epsilon > 0$. □

Proof [Lemma 7.8]: We bound the probabilities of $B^1_{\delta'}(\epsilon)$ and $B^2_{\delta'}(c)$ individually, for any $\delta/2 \leq \delta' \leq \delta$. We start with $B^1_{\delta'}(\epsilon)$. Plugging the value of $\mu_{\delta,1}$, given in (20), into the Chernoff bound (18) we get

$$\mathbf{P}[B^1_{\delta'}(\epsilon)] < \exp(-\epsilon^2 \mu_{\delta',1}/2) < \exp(\frac{-3\epsilon^2\delta'(1-\delta')^2\Delta n}{14}) \tag{28}$$

Now we deal with $B^2_{\delta'}(c)$. First notice that for any $\delta < 1/2$, $m_{\delta,3} < m_{\delta,2}$, so

$$\mathbf{P}[B^2_{\delta'}(c)] = \mathbf{P}[X_{\delta',2} + X_{\delta',3} > (1+c)\cdot(\mu_{\delta',2}+\mu_{\delta',3})] \leq 2\mathbf{P}[X_{\delta',2} > (1+c)\mu_{\delta',2}]$$

Assuming $c > 2e^2$, and plugging the value of $\mu_{\delta,2}$ given in (21) into the Chernoff bound (17) we get

$$\begin{aligned}
\mathbf{P}[B^2_{\delta'}(c)] &\leq 2\mathbf{P}[X_{\delta',2} > (1+c)\mu_{\delta',2}] &(29)\\
&< 2(\frac{e^c}{(1+c)^{1+c}})^{\mu_{\delta',2}} < (\frac{e}{c})^{c\mu_{\delta',2}} < \exp(-c\mu_{\delta',2}) &(30)\\
&= \exp(\frac{-3c\delta'^2(1-\delta')\Delta n}{7}) &(31)
\end{aligned}$$

Notice that for $\delta < 1/2$, the value of (28) and (31) is maximal when δ' is minimal (i.e $\delta' = \delta/2$). Using a union bound over all assignments of weight between $\delta n/2$ and δn, we get

$$\begin{aligned}
\mathbf{P}[Bad(\delta,c,\epsilon)] &\leq n\cdot\binom{n}{\delta n}\cdot\left(\mathbf{P}[B^1_{\delta/2}(\epsilon)] + \mathbf{P}[B^2_{\delta/2}(c)]\right)\\
&\leq 2n\cdot\exp\left(n\cdot\left(H_e(\delta) - \Delta\cdot\min\{\frac{-3\epsilon^2(\delta/2)(1-\delta/2)^2}{14}, \frac{-3c(\delta/2)^2(1-\delta/2)}{7}\}\right)\right)
\end{aligned}$$

Notice that once δ, ϵ, c are fixed, we can select Δ such that

$$H_e(\delta) - \Delta\cdot\min\{\frac{-3\epsilon^2(\delta/2)(1-\delta/2)^2}{14}, \frac{-3c(\delta/2)^2(1-\delta/2)}{7}\} < 0$$

Lemma 7.8 follows. □

8 Open Problems

1. What is the largest Δ for which one can prove `RWalkSAT` to have polynomial running time on $\mathcal{C} \sim \mathbb{F}^n_\Delta$? As a first step, can one go beyond the pure literal threshold (1.63)? See Figure 2 for empirical evidence that the running time of `RWalkSAT` is linear even for higher clause densities than 1.63.

2. What are the statistics of the random variable Term($\mathcal{C}$) as a function of the clause density? Does Term($\mathcal{C}$) $< \infty$ have a sharp threshold? Is there a terminator threshold independent of n? How does Term($\mathcal{C}$) correspond to n (number of variables) above density 1.63 (below 1.63 it is linear).

3. Can one use "non-rigorous" methods coming from theoretical physics (e.g. replica symmetry analysis) to better understand the random variable Term($\mathcal{C}$) on random CNFs?

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A Proofs

In this section we prove theorems 4.5 and 4.7. Our starting point is the following theorem and lemma proved implicitly in [8]. The lemma is a slight generalization of lemma 4.4 in [8], so we provide its proof. (The original lemma 4.4 of [8] only needed expansion factor of 3/2, whereas we need a constant fraction more than 3/2. The proof is essentially the same).

Theorem A.1 *[8] For every $\Delta < 1.63$ there exists an integer d such that with high probability for $\mathcal{C} \sim \mathbb{F}_\Delta^n$, $|\mathcal{C}_d| \leq \frac{n}{600\Delta^2}$.*

Lemma A.2 *[8] Let $\Delta_0 = 1.63$. For any constant $\Delta \leq \Delta_0$ whp $\mathcal{C} \sim \mathbb{F}_\Delta^n$ is a $(\frac{n}{600\Delta_0^2}, 3/2 + 10^{-3})$-expander.*

Proof (of Lemma A.2): Set $\epsilon = 10^{-3}$. Let A_k be the event that there exists a set of k clauses having less than $3/2 + \epsilon$ variables. Let us bound the probability of these bad events, using a union bound. Let $r = \frac{n}{600\Delta_0^2}$, and $c = 3/2 + \epsilon$. We make use of the following well-known inequality $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$.

$$\begin{aligned}
\mathbf{P}[Bad] &\leq \sum_{k=1}^{r} \mathbf{P}[A_k] \leq \sum_{k=1}^{r} \binom{\Delta n}{k} \cdot \binom{n}{ck} \cdot \left(\frac{ck}{n}\right)^{3k} \\
&\leq \sum_{k=1}^{r} \left(\frac{e\Delta_0 n}{k}\right)^k \cdot \left(\frac{en}{ck}\right)^{ck} \cdot \left(\frac{ck}{n}\right)^{3k} \\
&\leq \sum_{k=1}^{r} \left[(e^{1+c} c^{3-c} \Delta_0) \cdot \left(\frac{k}{n}\right)^{2-c} \right]^k \\
&\leq \sum_{k=1}^{r} \left[37 \cdot \left(\frac{k}{n}\right)^{\frac{1}{2}-\epsilon} \right]^k = o(1)
\end{aligned}$$

Where the last inequality holds for $r \leq \frac{n}{600\Delta_0^2}$. ∎

Notice that if $\mathcal{C}_d$ is a $(|\mathcal{C}_d|, \frac{3}{2} + \epsilon)$-expander then every subset of $\mathcal{C}_d$ (including $\mathcal{C}_d$ itself) has at least ϵ *unique neighbors* (i.e. literals appearing in exactly one clause), and these unique neighbors are pure. Thus, $\mathcal{C}_d$ is $O(\log n)$-pure. Hence $\mathcal{C}$ is $O(\log n)$-pure (remember d is a constant) and this proves theorem 4.5. In order to prove theorem 4.7 we need the following lemma from [11].

Lemma A.3 *[11] For all constants $\Delta > 0, c < 2$ there exists some constant $\delta > 0$ such that whp $\mathcal{C} \sim \mathbb{F}_\Delta^n$ is a $(\delta\Delta n, c)$-expander.*

Let δ be the constant promised by lemma A.3, for $\Delta = 1.63$ and $c = 7/4$. By theorem 4.5, $|\mathcal{C}_d| \leq n/(600\Delta_0^2)$, for some constant d. By lemma A.2, $\mathcal{C}_d$ is a $(|\mathcal{C}_d|, \epsilon)$-boundary expander, for some $\epsilon > 0$. Remove an additional d' layers from $\mathcal{C}$ (each containing at least an $\epsilon/3$ fraction of the remaining clauses) so that $|\mathcal{C}_{d+d'}| \leq \delta n$, and by lemma A.3 this remaining CNF is (with high probability) a $(|\mathcal{C}_{d+d'}, 7/4)$-expander. This proves theorem 4.7.

B Terminator Threshold - Empirical Results via Linear Programming

Acknowledgements The code used in this section was written by Jon Feldman. We thank him for assisting us with the empirical tests and allowing us to include these results in our paper.

A nice property of a terminator is that if it exists, it can be found by linear programming in polynomial time, and if it doesn't exist, we can efficiently find a proof of this fact. Since by definition a terminator contains a satisfying assignment, this provides us with an efficient approach for SAT-solving using linear programming. As far as we know, this is the first use of linear programming in this context.

Empirical Threshold Lower Bound As discussed earlier (Section 5), we know that the terminator threshold must lie between 1.63 and 2 (for 3-CNFs). Empirically it seems the terminator threshold is at least 1.8. This is based on the fact that only 5% of the inputs we tested failed to have a terminator, out of a sample of 120 random CNFs with 4,000-4,100 variables each. On instances with a smaller number of variables (450-1,000) the failure rate is significantly higher (around 25%). We believe this is an effect of small instance size and asymptotically at density 1.8 a terminator exists.

Terminator weight vs. Running Time There seems to be a large gap between the empirical terminator weight (that bounds from above the running time of `RWalkSAT`) and the actual running time of `RWalkSAT`. Whereas the existence of a terminator is decidable in polynomial time, finding the *minimal weight* terminator seems to be a much harder problem. Thus, it is not clear if the empirical gap displayed below (between terminator weight and `RWalkSAT` running time) really exists, or is merely an artifact of our limited computational power.

Recall the data of figure 2 showing an empirical running time of $\leq n$ on instances with n variables (and clause density ≤ 2.5). The following figure 3 shows an empirical upper bound on the average termination weight for instances with 450-1,000 variables, at clause densities $1.65, 1.7$ and 1.75. For each instance size between 450 and $1,000$ (jumps of 50 variables), we tested thirty random instances and their average termination weight is plotted in figure 3.

Experimental Methods For each instance $\mathcal{C}$ we searched for a small weight terminator as follows. First we searched for a solution for the linear program $A^{\mathcal{C}} \cdot \alpha > 0$ with a *random* objective function. Having found a terminator α (that is minimal with respect to the random objective function), we minimized its ℓ_1-norm by solving the linear program with the *new* objective function $\text{sign}(\alpha)$. For each instance $\mathcal{C}$ we repeated this two-phase process three times, each time starting with a fresh random objective function. Finally we took the minimum value (ℓ_1-norm) as an upper bound on the terminator weight for $\mathcal{C}$.

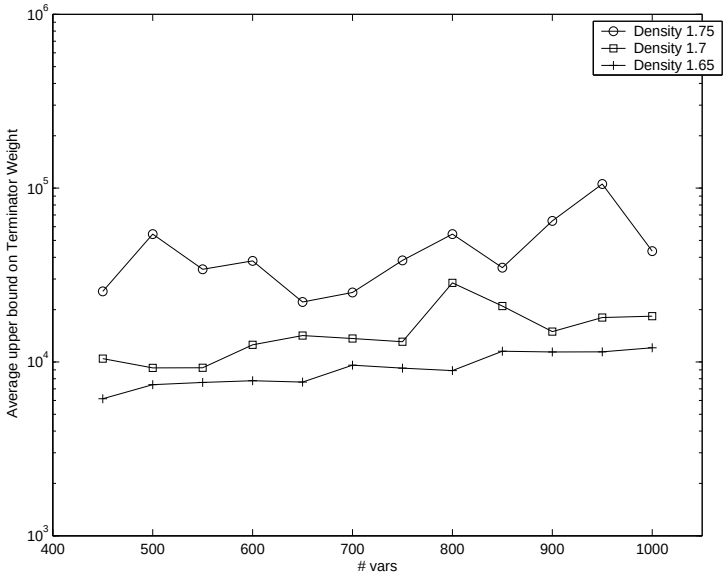

Figure 3: Empirical upper bounds on the average termination weight. Average taken over thirty random instance per variable size.

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