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# On Closure Properties of **GapL** *

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**Abstract**

We show necessary and sufficient conditions that certain algebraic functions like the rank or the inertia of an integer matrix can be computed in **GapL**.

## 1 Introduction

Valiant [Val79b, Val79a] introduced the counting class #**P** based on polynomial-time, nondeterministic Turing machines. It characterizes the computational complexity to compute the number of perfect matchings in a graph or the permanent of matrix over the natural numbers. Fenner, Fortnow, and Kurtz [FFK94] extended #**P** to the class **GapP** that can handle negative numbers. The permanent of an integer matrix is a complete problem for **GapP**. Closure properties of #**P** and **GapP** have been investigated in many papers, see for example [BRS91, FR96, HO93, OTTW96, TTW94].

By altering polynomial time to logspace computations, Allender and Ogihara [AO96] defined the counting class **GapL**. It characterizes the computational complexity to compute powers of an integer matrix or, most prominently, the determinant of an integer matrix [Dam91, Tod91b, Vin91, Val92].

The motivation for this paper comes from the fact that some related interesting problems with respect to integer matrices are *not known* to be computable in **GapL**. Examples are the rank or the signature of a matrix. These problems are just known to be computable with several queries to a **GapL**-oracle [ABO99, HT02b]. We investigate the question, whether these functions can be computed in **GapL**. Our main results are that these questions are equivalent to the collapse of certain complexity classes.

Complexity classes based on **GapL** are $\mathbf{C_{=}L}$ (*exact counting in logspace*) and **PL** (*probabilistic logspace*). Complete problems for these classes are to

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decide whether the determinant of an integer matrix is zero (singularity), or greater than zero, respectively. Another class is **SPL** which is based on 0-1-valued **GapL**-functions. Intuitively, this is a very small class. Nonetheless, Allender, Reinhardt, and Zhou [ARZ99] showed that the perfect matching problem is located in a nonuniform version of **SPL**.

We show in Section 3 that the rank of a matrix can be computed in **GapL** if and only if $\mathbf{C_=L} = \mathbf{SPL}$. In Section 4 we show that the signature of a matrix can be computed in **GapL** if and only if $\mathbf{PL} = \mathbf{SPL}$.

Note that $\mathbf{NL} \subseteq \mathbf{C_=L}$ and $\mathbf{SPL} \subseteq \oplus\mathbf{L}$. Hence, as a corollary of our results we get: if the rank or the signature of a matrix can be computed in **GapL** then $\mathbf{NL} \subseteq \oplus\mathbf{L}$ and $\mathbf{C_=L}$ is closed under complement. Both consequences are famous open problems right now.

We also consider a relaxed version of the above question. **GapL** is not known to be closed under division. Hence it is natural to ask whether we can write the rank or the signature of a matrix as a quotient of two **GapL**-functions. We show in Section 3 that this is true for the rank of a matrix if and only if $\mathbf{C_=L} = \mathbf{coC_=L}$. In Section 4 we show that this is true for the signature of a symmetric matrix if and only if $\mathbf{PL} = \mathbf{C_=L}$.

Finally, in Section 5 we characterize the case that the absolute value of any **GapL**-function can be computed in **GapL** too.

## 2 Preliminaries

For a nondeterministic Turing machine $M$ on input $x$, we denote the number of accepting and rejecting computation paths by $acc_M(x)$ and $rej_M(x)$, respectively. The difference of these two quantities is denoted by $gap_M(x)$. That is, $gap_M(x) = acc_M(x) - rej_M(x)$. In the polynomial time setting, complexity classes $\#\mathbf{P}$ and **GapP** are defined via these functions. We are interested in the logspace versions:

$$\begin{aligned} \#\mathbf{L} &= \{\, acc_M \mid M \text{ is a nondeterministic logspace Turing machine} \,\}, \\ \mathbf{GapL} &= \{\, gap_M \mid M \text{ is a nondeterministic logspace Turing machine} \,\}. \end{aligned}$$

In analogy to the polynomial time setting, we define the following counting complexity classes [AO96, ARZ99]:

$$\begin{aligned} \mathbf{C_=L} &= \{\, S \mid \exists f \in \mathbf{GapL},\ \forall x:\ x \in S \Longleftrightarrow f(x) = 0 \,\}, \\ \mathbf{PL} &= \{\, S \mid \exists f \in \mathbf{GapL},\ \forall x:\ x \in S \Longleftrightarrow f(x) > 0 \,\}, \\ \mathbf{SPL} &= \{\, S \mid \chi_S \in \mathbf{GapL} \,\}, \end{aligned}$$

where $\chi_S$ is the characteristic function of set $S$. It is known that

$$\mathbf{SPL} \subseteq \mathbf{C_=L} \subseteq \mathbf{PL} \subseteq \mathbf{TC}^1 \subseteq \mathbf{NC}^2.$$

Also we have $\mathbf{NL} \subseteq \mathbf{C_=L}$.

These classes are interesting because of the complete problems therein. We give some examples. When nothing else is said, by matrices we mean square integer matrices. We use $n$ as the order of the matrices.

Problems complete for **GapL** are to compute (one element of) the $m$-th power of a matrix and the determinant [Tod91a, Dam91, Vin91, Val92]. It follows that

$$\textsc{Singularity} = \{\, A \mid \det(A) = 0 \,\}$$

is complete for $\mathbf{C_=L}$. The set

$$\textsc{PosDeterminant} = \{\, A \mid \det(A) > 0 \,\}$$

is complete for **PL**. More general, the sets

$$\begin{aligned} \text{v-}\textsc{PowerElement} =& \{\, (A, a, m) \mid (A^m)_{1,n} = a \,\}, \\ \text{v-}\textsc{Determinant} =& \{\, (A, a) \mid \det(A) = a \,\}, \text{ and} \\ \textsc{Rank}_< =& \{\, (A, r \mid \operatorname{rank}(A) < r \,\} \end{aligned}$$

are complete for $\mathbf{C_=L}$. Consequently $\textsc{Rank}_\geq = \{\, (A, r \mid \operatorname{rank}(A) \geq r \,\}$ is complete for $\mathbf{coC_=L}$. The verification of the rank can be written as the intersection of a set in $\mathbf{C_=L}$ and in $\mathbf{coC_=L}$: v-$\textsc{Rank} = \{\, (A, r \mid \operatorname{rank}(A) = r \,\} = \textsc{Rank}_< \cap \textsc{Rank}_\geq$. This means that v-$\textsc{Rank} \in \mathbf{C_=L} \wedge \mathbf{coC_=L}$.

Allender, Beals, and Ogihara [ABO99] investigated the complexity of computing (one bit of) the rank. That is

$$\textsc{Rank} = \{\, (A, k, b) \mid \text{ the } k\text{-th bit of } \operatorname{rank}(A) \text{ is } b \,\}.$$

They showed that $\textsc{Rank}$ is a complete problem for $\mathbf{AC^0}(\mathbf{C_=L})$, the $\mathbf{AC^0}$-closure of $\mathbf{C_=L}$.

The *inertia* of a $n \times n$ matrix $A$ is defined to be the triple $i(A) = (i_+(A), i_-(A), i_0(A))$, where $i_+(A)$, $i_-(A)$, and $i_0(A)$ are the number of eigenvalues of $A$, counting multiplicities, which are positive, negative, and zero, respectively. Hoang and Thierauf [HT05] used the Routh-Hurwitz Theorem to show that the (bits of the) inertia of a matrix can be computed in **PL**.

The inertia is closely related to an equivalence relation on symmetric matrices: two symmetric matrices $A$ and $B$ are *congruent*, if there exists a nonsingular (real) matrix $S$ such that $A = SBS^T$. We denote the set of congruent symmetric matrices by $\textsc{Congruence}$. Sylvester's law of inertia says

$$A \text{ is congruent to } B \Longleftrightarrow i(A) = i(B).$$

$\textsc{Congruence}$ is contained in **PL** [HT02b] and hard for $\mathbf{AC^0}(\mathbf{C_=L})$ [HT00].

The *signature of* $A$ is defined as $\operatorname{sig}(A) = i_+(A) - i_-(A)$. The signature is closely related to the inertia. For symmetric matrices, the rank and the signature determine the inertia.

In analogy to the polynomial time setting, Allender, Reinhardt, and Zhou [ARZ99] defined the class **SPL**. They showed that the perfect matching problem is located in a nonuniform version of **SPL**.

**GapL** possesses some closure properties. In particular, it is closed under exponential summations and polynomial multiplications.

**Theorem 2.1** [AO96] *Let any $f \in$ **GapL** the following functions are in **GapL** too:*

1. $f(g(\cdot))$, *for any* $g \in \mathbf{FL}$,
2. $\sum_{i=0}^{2^{|x|^c}} f(x,i)$, *for any constant* $c$,
3. $\prod_{i=0}^{|x|^c} f(x,i)$, *for any constant* $c$,
4. $\binom{f(x)}{g(x)}$, *for any* $g \in \mathbf{FL}$ *such that* $g(x) = O(1)$.

Allender, Arvind, and Mahajan [AAM03] have shown a very useful closure property of **GapL**.

**Theorem 2.2** [AAM03] *The determinant of a matrix having* **GapL**-*computable elements can be computed in* **GapL**.

Because the determinant is complete for **GapL** one might be tempted to think that this theorem provides closure of **GapL** under composition. But there are some subtleties here! Closure under composition is still an open problem.

With respect to the decision problems we have that **PL**, $\mathbf{C_{=}L}$, and **SPL** are closed under union and intersection. Furthermore, **PL** and **SPL** are closed under complement. For $\mathbf{C_{=}L}$, closure under complement is an open problem. In addition, **PL** is closed under $\mathbf{AC}^0$- and $\mathbf{NC}^1$-reductions, i.e. $\mathbf{AC}^0(\mathbf{PL}) = \mathbf{NC}^1(\mathbf{PL}) = \mathbf{PL}$ [BF97].

# 3 Matrix Rank

Assume that the rank of a matrix could be computed in **GapL**. Then the verification of the rank, v-Rank, would be in $\mathbf{C_{=}L}$. On the other hand v-Rank is complete for $\mathbf{C_{=}L} \wedge \mathbf{coC_{=}L}$. Hence this would imply $\mathbf{C_{=}L} = \mathbf{coC_{=}L}$.

The following theorem strengthens this collapse considerably.

**Theorem 3.1** $\mathbf{C_{=}L} = \mathbf{SPL} \iff \mathrm{rank} \in \mathbf{GapL}$.

*Proof*. Assume that $\mathbf{C_=L} = \mathbf{SPL}$. Then v-RANK $\in \mathbf{SPL}$. Hence, there is a function $g \in \mathbf{GapL}$ such that for a given matrix $A$ of order $n$ and a number $r$ we have

$$\begin{aligned} \operatorname{rank}(A) = r &\implies g(A,r) = 1, \\ \operatorname{rank}(A) \neq r &\implies g(A,r) = 0. \end{aligned}$$

It follows that $\operatorname{rank}(A) = \sum_{r=1}^{n} r\, g(A,r)$, and therefore rank $\in \mathbf{GapL}$.

Conversely, suppose that rank $\in \mathbf{GapL}$. We consider SINGULARITY which is complete for $\mathbf{C_=L}$. Let $A$ be a matrix. Allender, Beals, and Ogihara [ABO99] showed how to construct a $N \times N$ matrix $B$ from $A$ in logspace with the following property:

$$\begin{aligned} \det(A) = 0 \quad &\implies \quad \operatorname{rank}(B) = N - 1, \text{ and} \\ \det(A) \neq 0 \quad &\implies \quad \operatorname{rank}(B) = N, \end{aligned}$$

Define function $g$ as $g(A) = N - \operatorname{rank}(B)$. Then $g \in \mathbf{GapL}$ and we have

$$g(A) = \begin{cases} 1, & \text{if } \det(A) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

This shows that SINGULARITY $\in \mathbf{SPL}$. $\square$

Next we weaken the assumption for the rank-function: instead of one **GapL**-function that computes the rank directly, suppose there are two **GapL**-functions $g$ and $h$ such that the rank can be written as the quotient of $g$ and $h$, i.e., $\operatorname{rank}(A) = g(A)/h(A)$. We show that this is a necessary and sufficient condition for $\mathbf{C_=L}$ being closed under complement.

**Theorem 3.2** $\mathbf{C_=L} = \mathbf{coC_=L} \iff \exists g, h \in \mathbf{GapL}$ $\;$ *rank* $= g/h$.

*Proof*. Assume that $\mathbf{C_=L} = \mathbf{coC_=L}$. Then the problem of verifying the rank of a matrix, v-RANK, is in $\mathbf{coC_=L}$. That is, there is a function $f \in \mathbf{GapL}$ such that for any symmetric matrix $A$ and any $r$,

$$\operatorname{rank}(A) = r \iff f(A,r) \neq 0.$$

Define functions $g(A) = \sum_{r=0}^{n} r f(A,r)$ and $h(A) = \sum_{r=0}^{n} f(A,r)$. Then we have $g, h \in \mathbf{GapL}$ and rank $= g/h$ as claimed.

Conversely, let $g, h \in \mathbf{GapL}$ such that rank $= g/h$. For a given symmetric matrix $A$ and an integer $k \geq 0$, define $f(A,k) = g(A) - k\, h(A)$. Then $f \in \mathbf{GapL}$ and we have

$$\operatorname{rank}(A) = r \iff f(A,r) = 0.$$

It follows that the rank of a matrix can be verified in $\mathbf{C_=L}$. Hence $\mathbf{C_=L} = \mathbf{coC_=L}$. $\square$

It was shown in [HT02a] that the degree of the minimal polynomial is computationally equivalent to matrix rank. Therefore, we can formulate the above theorems also in terms of the degree of the minimal polynomial.

There is an interesting alternative way of representing the rank of a matrix. Consider an $n \times n$ symmetric matrix $A$ and let its characteristic polynomial be

$$\chi_A(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

It is well known from linear algebra that $\text{rank}(A) = k \Longleftrightarrow c_{n-k} \neq 0$ and $c_{n-k-1} = c_{n-k-2} = \cdots = c_0 = 0$. Furthermore, all coefficients $c_i$ are computable in **GapL** [Ber84].

Define a vector $\boldsymbol{w} = (w_n, w_{n-1}, \cdots, w_1, w_0)^T$, where $w_j = \sum_{i=0}^{j} c_i^2$, for $j = 0, 1, \ldots, n$. Hence every element of $\boldsymbol{w}$ is computable in **GapL**. Furthermore we have $\text{rank}(A) = k$ if and only if

(i) $\boldsymbol{w}$ has precisely $k+1$ positive elements, $w_n, w_{n-1}, \ldots, w_{n-k}$, and

(ii) precisely $n-k$ zero elements, $w_{n-k-1} = w_{n-k-2} = \cdots = w_0 = 0$.

Conversely, for a given nonnegative **GapL**-vector $v$, the number of its positive elements is exactly the rank of the diagonal matrix whose diagonal is $v$.

In summary, the problem of determining the rank of a matrix is (logspace) equivalent to the problem of determining the number of consecutive zeros at the right end in a **GapL**-vector.

# 4 Matrix Inertia

Recall that the inertia of a matrix $A$ consists of the three values $i_+(A)$, $i_-(A)$, and $i_0(A)$, which are the numbers of eigenvalues of $A$, counting multiplicities, which are positive, negative, and zero, respectively. The (bits of the) inertia of a matrix can be computed in **PL** [HT05].

We summarize some properties of the inertia functions. Clearly, for a $n \times n$ matrix $A$ we have $i_+(A) + i_-(A) + i_0(A) = n$. Functions $i_+$ and $i_-$ are computationally equivalent, because $i_+(A) = i_-(-A)$. Therefore the signature $\text{sig}(A) = i_+(A) - i_-(A)$ can be reduced to $i_+$ (or $i_-$), $\text{sig}(A) = i_+(A) - i_+(-A)$.

The rank can be reduced to the signature and to $i_+$. If matrix $A$ is symmetric, then the rank of $A$ is the number of nonzero eigenvalues, $\text{rank}(A) = i_+(A) + i_-(A) = n - i_0(A)$. Hence the rank and the signature determine uniquely the inertia of a symmetric matrix, and conversely. For non-symmetric $A$ we consider the matrix $A^T A$, which is symmetric and has the same rank as $A$. Moreover, if $\lambda$ is an eigenvalue of $A$, then $|\lambda|^2$ is an eigenvalue of $A^T A$. Therefore $i_-(A^T A) = 0$ and we have

$$\text{rank}(A) = \text{rank}(A^T A) = \text{sig}(A^T A) = i_+(A^T A) = n - i_0(A^T A).$$

The following theorem characterizes the case that the upper bound for the inertia or the signature can be improved from **PL** to **GapL**.

**Theorem 4.1** *The following conditions are equivalent.*

*(i)* $\mathbf{PL} = \mathbf{SPL}$,

*(ii)* $i_+ \in \mathbf{GapL}$,

*(iii)* $\mathrm{sig} \in \mathbf{GapL}$.

*Proof*. To show that *(i)* implies *(ii)* assume that $\mathbf{PL} = \mathbf{SPL}$. It is known that the verification of $i_+$ is in **PL** [HT05], and hence in **SPL**. That is, there exists a function $g \in \mathbf{GapL}$ such that for a matrix $A$ and for all $0 \leq k \leq n$.

$$\begin{aligned} i_+(A) = k &\implies g(A,k) = 1, \\ i_+(A) \neq k &\implies g(A,k) = 0. \end{aligned}$$

It follows that $i_+(A) = \sum_{k=1}^{n} k\ g(A,k)$, and therefore $i_+ \in \mathbf{GapL}$.

Since the signature can be reduced to $i_+$, we have that *(ii)* implies *(iii)*. For the reverse direction, we express $i_+$ in terms of the signature and the rank:

$$2i_+(A) = \mathrm{rank}(A) + \mathrm{sig}(A).$$

From the outline above, $\mathrm{sig} \in \mathbf{GapL}$ implies $\mathrm{rank} \in \mathbf{GapL}$, and therefore $\mathbf{C_=L} = \mathbf{SPL}$ by Theorem 3.1. Define sets $S_k$ for $k = 0, 1, \ldots n$,

$$S_k = \{\, A \mid \mathrm{rank}(A) + \mathrm{sig}(A) = 2k \,\}.$$

Sets $S_k$ are in $\mathbf{C_=L}$, and therefore in **SPL**. Hence, for each $k$, there is a function $f_k \in \mathbf{GapL}$ such that $f_k(A) = 1$, if $A \in S_k$, and $f_k(A) = 0$, otherwise. Then we can write

$$i_+(A) = \sum_{k=1}^{n} k\ f_k(A).$$

We conclude that $i_+ \in \mathbf{GapL}$.

To show that *(ii)* implies *(i)*, assume that $i_+ \in \mathbf{GapL}$. Then the verification of $i_+$ is in $\mathbf{C_=L}$. Because $i_+$ is complete for **PL** [HT05] we have $\mathbf{PL} = \mathbf{C_=L}$. Because it also follows that the rank is in **GapL**, we have $\mathbf{C_=L} = \mathbf{SPL}$ by Theorem 3.1. Therefore $\mathbf{PL} = \mathbf{SPL}$. □

Like for the rank in Section 3, we consider the weaker condition that we can express $i_+$ or the signature as a quotient of two **GapL**-functions.

**Theorem 4.2** *The following conditions are equivalent.*

*(i)* $\mathbf{PL} = \mathbf{C_=L}$,

*(ii)* $\exists g, h \in \mathbf{GapL}\ \ i_+ = g/h$.

*(iii)* $\exists g, h \in \mathbf{GapL}\ \ \mathrm{sig} = g/h$,

*Proof.* Conditions *(ii)* and *(iii)* are equivalent, because we can write $\mathrm{sig}(A) = i_+(A) - i_+(-A)$ and $i_+(A) = (\mathrm{rank}(A) + \mathrm{sig}(A))/2 = (\mathrm{sig}(AA^T) + \mathrm{sig}(A))/2$.

To show that *(i)* implies *(ii)*, assume that $\mathbf{PL} = \mathbf{C_=L}$. Note that $\mathbf{PL}$ is closed under complement. It follows that we can verify $i_+$ in $\mathbf{coC_=L}$. That is, there is a function $f \in \mathbf{GapL}$ such that for any symmetric matrix $A$ and any $k$,

$$i_+(A) = k \iff f(A, k) \neq 0.$$

Define functions $g(A) = \sum_{k=0}^{n} k f(A, k)$ and $h(A) = \sum_{k=0}^{n} f(A, k)$. Then we have $g, h \in \mathbf{GapL}$ and $i_+ = g/h$ as claimed.

To show that *(ii)* implies *(i)*, assume that $i_+ = g/h$ for $g, h \in \mathbf{GapL}$. For any symmetric matrix $A$ and any $k$, define $f(A, k) = g(A) - k\ h(A)$. Then $f \in \mathbf{GapL}$ and we have

$$i_+(A) = k \iff f(A, k) = 0.$$

That is, we can verify $i_+$ in $\mathbf{C_=L}$. Therefore, $\mathbf{PL} = \mathbf{C_=L}$. □

Recall that two symmetric matrices are congruent iff they have the same inertia. Congruence is contained in $\mathbf{PL}$ [HT02b] and hard for $\mathbf{AC}^0(\mathbf{C_=L})$ [HT00]. The following theorem characterizes the case that Congruence is in $\mathbf{C_=L}$.

**Theorem 4.3** *The following conditions are equivalent, when all functions are restricted to symmetric matrices.*

*(i)* Congruence $\in \mathbf{C_=L}$,

*(ii)* $\exists g, h \in \mathbf{GapL}\ \ i_+ = g/h$,

*(iii)* $\exists g, h \in \mathbf{GapL}\ \ \mathrm{sig} = g/h$.

*Proof.* The equivalence of *(ii)* and *(iii)* follows from the proof of Theorem 4.2. We show that *(i)* implies *(ii)*. Assume that Congruence $\in \mathbf{C_=L}$. Because Congruence is hard for $\mathbf{AC}^0(\mathbf{C_=L})$ [HT00], this imply $\mathbf{C_=L} = \mathbf{coC_=L}$ and therefore Congruence $\in \mathbf{coC_=L}$. That is, there is a function $f \in \mathbf{GapL}$ such that for any two symmetric matrices $A$ and $B$ we have

$$(A, B) \notin \text{Congruence} \iff f(A, B) = 0.$$

Now let $A$ be a symmetric matrix of order $n$. Define diagonal matrices $B_{k,l}$ of order $n$ that have $k$ times 1 on the main diagonal, $l$ times $-1$, and the rest 0. The we have

$$\begin{aligned}
i_+(B_{k,l}) &= k, \\
i_-(B_{k,l}) &= l, \\
i_0(B_{k,l}) &= n - k - l.
\end{aligned}$$

Define functions $g(A) = \sum_{k,l=0}^{n} k f(A, B_{k,l})$ and $h(A) = \sum_{k,l=0}^{n} f(A, B_{k,l})$. Then we have $g, h \in \mathbf{GapL}$ and $i_+ = g/h$ as claimed.

That *(ii)* implies *(i)* follows from Sylvester's law of inertia and the closure of $\mathbf{C_=L}$ under intersection. □

## 5 Absolute Value

For any function $f$ mapping to integers, by $\mathrm{abs}(f)$ we denote the function of absolute values of $f$. That is

$$\mathrm{abs}(f)(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0, \\ -f(x), & \text{otherwise.} \end{cases}$$

**Theorem 5.1** $\mathbf{PL} = \mathbf{SPL} \iff \forall f \in \mathbf{GapL} \;\; \mathrm{abs}(f) \in \mathbf{GapL}$.

*Proof*. Suppose $\mathbf{PL} = \mathbf{SPL}$ and let $f \in \mathbf{GapL}$. Define the set $S = \{\, x \mid f(x) > 0 \,\}$. By definition $S \in \mathbf{PL}$ and therefore $S \in \mathbf{SPL}$, by assumption. That is, there is $g \in \mathbf{GapL}$ such that

$$g(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write $\mathrm{abs}(f) = (2g - 1)f$, and therefore $\mathrm{abs}(f) \in \mathbf{GapL}$.

Conversely, let $S \in \mathbf{PL}$. That is, for some function $f \in \mathbf{GapL}$, we can write $S = \{\, x \mid f(x) > 0 \,\}$. We define the following functions

$$\begin{aligned}
g =& \mathrm{abs}(f) - \mathrm{abs}(f - 1), \\
h =& \binom{g+1}{2}.
\end{aligned}$$

We have $g \in \mathbf{GapL}$, by assumption. It follows that $h \in \mathbf{GapL}$ by the closure properties of $\mathbf{GapL}$. Now observe that

$$h(x) = \begin{cases} 1, & \text{if } f(x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

This shows that $S \in \mathbf{SPL}$, and therefore $\mathbf{PL} = \mathbf{SPL}$. □

## Open Problems

In the polynomial time setting it is known that $\mathbf{PP} \subseteq \mathbf{SPP}^{\mathbf{C_=P}}$. The proof is quite easy:

> Let $A = \{ x \mid f(x) > 0 \} \in \mathbf{PP}$, for some $f \in \mathbf{GapP}$. A nondeterministic machine $M$ on input $x$ guesses $k > 0$ and asks its $\mathbf{C_=P}$-oracle whether $f(x) = k$. If the answer is "yes", then $M$ accepts. If the answer is "no", then $M$ branches ones and accepts on one branch and rejects on the other branch. This shows that $A \in \mathbf{SPP}^{\mathbf{C_=P}}$.

Note that this proof doesn't work in the logspace setting: in the Ruzzo-Simon-Tompa model of space-bounded oracle machines, the machine has to be deterministic while writing a query. Hence we ask

- Is $\mathbf{PL} \subseteq \mathbf{SPL}^{\mathbf{C_=L}}$?

Because $\mathbf{SPP}^{\mathbf{SPP}} = \mathbf{SPP}$, the above inclusion implies that $\mathbf{C_=P} = \mathbf{SPP} \implies \mathbf{PP} = \mathbf{SPP}$. In the logspace setting, we also have $\mathbf{SPL}^{\mathbf{SPL}} = \mathbf{SPL}$ [ARZ99], but the above conclusion is open.

- Does $\mathbf{C_=L} = \mathbf{SPL} \implies \mathbf{PL} = \mathbf{SPL}$?

If the answer is "yes", the conditions in Theorem 3.1 and 4.1 would all be equivalent. In particular, this question is equivalent to finding a reduction from the signature to the rank of a matrix, and the latter functions don't look very different (in complexity).

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