Electronic Colloquium on Computational Complexity, Report No. 34 (2004)

Network Coding: Does the Model Need Tuning?

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April 9, 2004

Abstract

We consider the *general network information flow problem*, which was introduced by Ahlswede et. al[1]. We show a *periodicity effect*: for every integer $m \geq 2$, there exists an instance of the network information flow problem that admits a solution if and only if the alphabet size is a perfect m^{th} power. Building on this result, we construct an instance with $O(m)$ messages and $O(m)$ nodes that admits a solution if and only if the alphabet size is an enormous $2^{\exp\left(\Omega\left(m^{1/3}\right)\right)}$. In other words, if we regard each message as a length-k bit string, then k must be *exponential* in the size of the network. For this same instance, we show that if edge capacities are slightly increased, then there is a solution with a modest alphabet size of $O(2^m)$. In light of these results, we suggest that a more appropriate model would assume that the network operates at slightly under capacity.

1 Introduction

Network information theory considers the information carrying capacity of a network. Formally, in the *network information flow problem*, introduced by Ahlswede at. al [1], a network is represented by a directed acyclic multigraph containing sources (each with a set of available messages), intermediate nodes, and sinks (each demanding a set of messages). Messages are single symbols from an alphabet Σ of size q. An edge with capacity c can transmit c symbols from the alphabet. Information can be duplicated and encoded at internal nodes. A solution is a set of encoding functions for internal nodes and a decoding function for each sink that collectively allow the sinks to receive all their requested messages.

To give perspective on this model, we consider two other classic problems which have been used to model communication networks. First, in the framework of multi-commodity flow, a message is a commodity that must be shipped from a sender to a receiver. Flow must be conserved at internal nodes, and two commodities can only share an edge if the sum of their flows is at most the capacity of the edge [3]. In contrast, in the information flow problem we view a message as information. Thus, flow conservation is no longer an issue; an internal node receiving one copy of a message can transmit exact copies of this message on two or more outgoing links. Similarly, messages can

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ISSN 1433-8092