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The Computational Complexity of Evolutionarily Stable Strategies

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Abstract

Game theory has been used for a long time to study phenomena in evolutionary biology, beginning systematically with the seminal work of John Maynard Smith. A central concept in this connection has been the notion of an *evolutionarily stable strategy* (ESS) in a symmetric two-player strategic form game. A *regular* ESS is an important refinement of the ESS concept which has also been studied extensively and is further motivated by Harsanyi's result that "almost all" strategic form games contain only regular equilibria.

There is a substantial literature on computing evolutionarily stable strategies, yet despite these efforts the precise computational complexity of determining the existence of an ESS in a game has remained elusive, although it has been speculated by some that the problem is **NP**-hard.

In this paper we show that determining the existence of an ESS is both **NP**-hard and **coNP**-hard, and that it is contained in $\mathbf{\Sigma_2^p}$, the second level of the polynomial time hierarchy. On the other hand, we show that determining the existence of a regular ESS is indeed **NP**-complete. Our upper bounds also yield algorithms for computing a (regular) ESS, if one exists, with the same complexities.

Our upper bounds combine known criteria for the existence of an ESS based on quadratic forms, together with known results about the complexity of quadratic programming decision problems. Our lower bounds employ, among other things, a classic characterization of maximum clique size via quadratic programming.

1 Introduction

Game theoretic methods and tools have been applied for a long time to the study of phenomena in evolutionary biology, most systematically since the pioneering work of Maynard Smith in the 1970's and 80's ([SP73, Smi82]).

Since then "evolutionary game theory" has been used to explain and to understand a diverse range of sometimes counter-intuitive phenomena in biology. For a more recent overview of evolutionary game theory, and a sampling of its many applications in zoology and botany, see the survey by Hammerstein and Selten [HS94]. They mention, among others, the following applications: animal

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fighting, cooperation, and mating; conflict between the sexes, and offspring sex ratios; plant seed dispersal, root competition, nectar production, and flower size.

A central concept in evolutionary game theory has been the notion of an evolutionarily stable strategy (ESS) in a symmetric two-player strategic form game, introduced by Maynard Smith and Price ([SP73]). An ESS is a particular kind of mixed (randomized) strategy, where the probabilities in the mixed strategy are now viewed as denoting percentages in a population exhibiting different possible behaviors. To be an ESS, a mixed strategy s must first constitute a Nash equilibrium (s, s) when played against itself. This means that s is a "best response" to itself, i.e., that the expected payoff for a player who plays s against s is the maximum possible payoff of any strategy against s.

Secondly, to be an ESS, s must in a precise sense be "impervious to invasion" by other strategies. Specifically, it must be the case that if a different strategy t is also a best response to s, then the expected payoff of playing s against t must be strictly greater than the payoff of playing t against t.

It was shown already by Nash [Nas51] that every symmetric strategic form game contains a symmetric Nash equilibrium (s, s). However, not all symmetric 2-player games contain an ESS: rock-paper-scissors is a simple counter-example.

Thus, one may ask: what is the computational complexity of determining whether an ESS exists in a 2-player strategic game (with, say, rational payoffs)? And, if an ESS does exist, what is the complexity of actually computing one?

The computational complexity of computing an arbitrary Nash equilibrium for a 2-player strategic form game is a well-known open problem (see [Pap01]). It is neither known to be **NP**-hard, nor known to be computable in polynomial time. However, **NP**-hardness is known for computing Nash equilibria that satisfy any of several additional desirable conditions, such as equilibria that optimize "social welfare", and this is so even for symmetric games ([GZ89, CS03]). It has thus been speculated that finding an ESS may also be **NP**-hard, but to the best of our knowledge no proof was known.

There are at least two possible motivations for determining the precise computational complexity of finding an ESS. The most direct motivation is, of course, that when we model and analyze a biological system in an evolutionary game setting, we will want to know the most efficient algorithm for finding ESSs in our model.

A second, much more speculative motivation, is the following: let us consider an evolutionarily stable strategy as an equilibrium of a biological system viewed as a dynamical system (see, e.g., [HS88]). That is, the dynamical system goes though a sequence of simple changes over time (i.e., "evolves"), perhaps converging through evolution to an ESS state. Such a dynamic view is indeed part of the original motivation for the definition of ESSs (see, e.g., [HS94, vD91] for specific formulations of such dynamics. We assume here discrete rather than continuous dynamics, but the two are coarsely related.) Thus, one may ask: how many iterations of "simple evolutionary steps" involving small local changes are required by the system to converge to an ESS? In other words, can one perhaps learn something about "how long evolution must take" to reach equilibrium, by studying the complexity of ESSs? Since convergence may occur only in the limit, "(in)approximability" seems more appropriate here than exact computation, and this raises questions about the approximability of ESSs. We defer further discussion of these issues to the last section of the paper.

A *regular* ESS is a important refinement of the ESS concept. This is an ESS,

s, where the "support set" of s, i.e., the set of those pure strategies that are played with non-zero probability in s, already contains all pure strategies that are best responses to s. There are several other equivalent definitions of regular ESSs. Harsanyi [Har73b], introduced regular equilibria as a refinement of the Nash equilibrium concept, and he showed the important result that "almost all" strategic form games contain only regular equilibria, where "almost all" here means that the games with irregular equilibria constitute a set of measure zero in a suitably defined measure space on games. There are other, weaker refinements of Nash equilibria, such as "quasi-strict" equilibrium, also introduced by Harsanyi [Har73a]. For symmetric 2-player games, it turns out that the definition of a regular ESS coincides with that of an ESS that is a quasi-strict Nash equilibrium. Other equivalent formulations of regular ESSs make the notion rather robust (see, e.g., [vD91, Sel83, Bom86]). See van Damme's excellent book [vD91] for a comprehensive treatment of refinements of Nash equilibria, and their ramifications for evolutionarily stable strategies.

In this paper, we show that determining the existence of an ESS is both **NP**-hard and **coNP**-hard, and that it is contained in $\mathbf{\Sigma_2^P}$, the second level of the polynomial time hierarchy. We show, moreover, that determining the existence of a regular ESS is **NP**-complete. Our upper bounds also yield algorithms to compute a (regular) ESS, if one exists, with the same complexities. From our lower and upper bounds, it also follows easily that computing the number of (regular) ESSs is **#P**-hard (**#P**-complete).

There is a substantial literature on algorithms for computing evolutionarily stable strategies, and its connections to mathematical programming. (See, e.g., [Bom92, BP89, Bom02]. See also [MWC$^+$97] for a different computational perspective based on dynamics.) In particular Bomze ([Bom92]) developed criteria for ESSs, based on "copositivity" of a matrix over a cone, and uses these to provide an algorithm for enumerating all ESSs in a game. His criteria build on earlier criteria for ESSs developed by Haigh ([Hai75]) and Abakuks ([Aba80]). Bomze's enumeration algorithm uses a recursive elimination procedure that involves some complications including possible numerical issues. We were thus unable to deduce our $\mathbf{\Sigma_2^P}$ upper bounds for ESSs directly from Bomze's algorithms. We instead provide a self-contained development of the criteria we need, building directly on the Haigh-Abakuks criteria, and we then employ a result by Vavasis [Vav90] on the computational complexity of the quadratic programming decision problem to obtain our $\mathbf{\Sigma_2^P}$ upper bounds for ESSs. For regular ESSs, our **NP** upper bound follows from simple modifications of the Haigh-Abakuks criteria, together with basic facts from matrix theory.

Our **NP**-hardness result for ESSs provides a reduction from SAT that yields a 1-1 correspondence between satisfying assignments of a CNF boolean formula and the ESSs in the game to which it is reduced (this is reminiscent, but substantially different from, the reduction of [CS03] for Nash equilibria). Furthermore, these ESSs will all be regular, and therefore **NP**-hardness for regular ESSs also follows. For our **coNP**-hardness result for ESSs, we provide a reduction from coCLIQUE to the ESS problem. In doing so, we make use of a classic characterization of maximum clique size via quadratic programs, due to Motzkin and Straus [MS65].

An outline of the paper is as follows: In section 2 we provide necessary definitions and background. In section 3 we provide our hardness results for both ESS and regular ESS computation, and in section 4 we provide our upper bounds for both. We conclude with some discussion of open issues and future directions in section 5.

2 Definitions and Notation

For a $n \times n$-matrix A, and subsets $I, J \subseteq \{1, \ldots, n\}$, let $A_{I,J}$ denote the submatrix of A defined by deleting the rows with indexes not in I and deleting the columns with indexes not in J. Likewise, for (row) vector x, define $x_I := x_{I,\{1\}}$ ($x_{\{1\},I}$), viewing x as a $n \times 1$-matrix ($1 \times n$-matrix, respectively).

By A^T we denote the transpose of a matrix A. Likewise, x^T denotes the transpose of a vector x. Unless stated otherwise, we assume that all vectors are column vectors.

A real symmetric $n \times n$-matrix A is **positive definite** if $x^T A x > 0$ for all $x \in \mathbb{R}^n - \{\mathbf{0}\}$. Recall the determinant criterion for positive definiteness: a symmetric matrix A is positive definite if and only if $\det(A_{I,I}) > 0$ for all $I = \{1, \ldots, i\}$, $1 \leq i \leq n$, where det denotes the determinant of a square matrix (see, e.g., [LT85]). Thus, in particular, positive definiteness of a rational symmetric matrix can be detected in polynomial time. A real symmetric matrix A is called **negative definite** if $(-A)$ is positive definite. Note that for any matrix A and vector x, $x^T A x = x^T A' x$, where $A' := \frac{1}{2}(A + A^T)$ is a symmetric matrix. We thus say a general $n \times n$ matrix A is *positive (negative) definite* if A' is positive (negative) definite, and we can use the determinant criterion on A' to detect this.

We now recall some basic definitions of game theory (see, e.g., [OR94]). A **finite two-person strategic form game** $\Gamma = (S_1, S_2, u_1, u_2)$ is given by finite sets of strategies S_1 and S_2 and utility (or payoff) functions $u_1 : S_1 \times S_2 \mapsto \mathbb{R}$ and $u_2 : S_1 \times S_2 \mapsto \mathbb{R}$ for player one and two, respectively. Such a game is called **symmetric** if $S_1 = S_2 =: S$ and $u_1(i,j) = u_2(j,i)$ for all $i, j \in S$. We write (S, u_1) as shorthand for (S, S, u_1, u_2), with $u_2(j,i) = u_1(i,j)$ for $i, j \in S$. We assume for simplicity that $S = \{1, \ldots, n\}$. That is, the pure strategies in the game are identified with an initial segment of the positive integers.

In what follows we only consider finite symmetric two-person strategic form games. The **payoff matrix** $A_\Gamma = (a_{i,j})$ of $\Gamma = (S, u_1)$ is given by $a_{i,j} = u_1(i,j)$ for $i, j \in S$. (Note that A_Γ is not necessarily symmetric, even if Γ is a symmetric game.) A **mixed strategy** $s = (s(1), \ldots, s(n))^T$ for $\Gamma = (S, u_1)$ is a vector that defines a probability distribution on S. Thus, $s \in X$, where $X = \left\{s \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n s(i) = 1\right\}$ denotes the set of mixed strategies in Γ. s is called **pure** iff $s(i) = 1$ for some $i \in S$. In that case we identify s with i. For brevity, we generally use the word "strategy" to refer to a mixed strategy s, and indicate otherwise when the strategy is pure.

In our notation, we alternatively view a mixed strategy s as either a vector $(s_1, \ldots, s_n)^T$ of length n, or as a function $s : S \mapsto \mathbb{R}$, depending on which is more convenient in the context.

The **expected payoff** function, $U_i : X \times X \mapsto \mathbb{R}$ for player $i \in \{1, 2\}$ is given by $U_i(s,t) = \sum_{i,j \in S} s(i) t(i) u_i(i,j)$, for all $s, t \in X$. Note that $U_1(s,t) = s^T A_\Gamma t$ and $U_2(s,t) = s^T A_\Gamma^T t$. Let s be a strategy for $\Gamma = (S, u_1)$. A strategy $t \in X$ is a **best response** to s if $U_1(t,s) = \max_{t' \in X} U_1(t', s)$. The **support** $\text{supp}(s)$ of s is the set $\{i \in S : s(i) > 0\}$ of pure strategies which are played with non-zero probability. The **extended support** $\text{ext-supp}(s)$ of s is the set

$\{i \in S : U_1(i,s) = \max_{x' \in X} U_1(x,s)\}$ of all pure best responses to s.

A pair of strategies (s,t) is a **Nash equilibrium** for Γ if s is a best response to t and t is a best response to s. Note that (s,t) is a Nash equilibrium if and only if $\text{supp}(s) \subseteq \text{ext-supp}(t)$ and $\text{supp}(t) \subseteq \text{ext-supp}(s)$. A Nash equilibrium (s,t) is **symmetric** if $s = t$. It was shown already in [Nas51] that every symmetric game contains a symmetric Nash equilibrium.

Definition 1 *A mixed strategy $s \in X$ in a 2-player symmetric game Γ is an* ***evolutionarily stable strategy (ESS)*** *of Γ if:*

1. *(s,s) is a symmetric Nash equilibrium of Γ, and*
2. *if $t \in X$ is any best response to s and $t \neq s$, then $U_1(s,t) > U_1(t,t)$.*

An ESS s is ***regular*** *if $\text{supp}(s) = \text{ext-supp}(s)$.*

Definition 2 *Let* ESS *(*REG-ESS*) denote the decision problem of whether a game $\Gamma = (S, u_1)$ with rational payoff matrix A_Γ has at least one (regular) evolutionarily stable strategy. Let* #ESS *(*#REG-ESS*) denote the counting problem of how many (regular) ESSs a game $\Gamma = (S, u_1)$ with rational payoff matrix A_Γ has.*

As usual, an undirected graph $G = (V, E)$ has vertices V and a symmetric edge set $E \subset V \times V$ where $(i,j) \in E \Rightarrow (j,i) \in E$, and $(i,i) \notin E$, for all $i,j \in V$. Let A_G denote the (symmetric) adjacency matrix of undirected graph G. A **clique** $C \subseteq V$ of $G = (V,E)$ is a subset of V such that $(C \times C) - E = \{(i,i) \mid i \in C\}$. Let $\omega(G)$ denote the clique number of G, i.e., the maximum cardinality of a clique in G. Let $\mathsf{coCLIQUE} = \{(G,c) \mid c \in \mathbb{N} \text{ and } \omega(G) < c\}$. Thus coCLIQUE denotes the decision problem of, given an undirected graph G and $c \in \mathbb{N}$, determining whether G does not have a clique of size c.

We omit formal definitions of the standard computational complexity classes **NP**, **coNP**, $\mathbf{\Sigma_2^P}$, and **#P**. For a comprehensive introduction to computational complexity theory including these definitions see [Pap94].

3 Hardness results

We will show in this section that deciding whether a symmetric game Γ has an ESS is both **NP**-hard and **coNP**-hard, and that deciding whether there exists a regular ESS is **NP**-hard.

3.1 ESS is coNP-hard

We show that ESS is **coNP**-hard by reducing coCLIQUE to it. In doing so, we make essential use of the following classic result due to Motzkin and Straus [MS65].

Theorem 1 *([MS65]) Let $G = (V,E)$ be an undirected graph with maximum clique size c. Let $\Delta_1 = \left\{x \in \mathbb{R}_{\geq 0}^{|V|} : \sum_{i=1}^{|V|} x_i = 1\right\}$. Then $\max_{x \in \Delta_1} x^T A_G x = \frac{c-1}{c}$.*

Corollary 2 *Let $G = (V, E)$ be an undirected graph with maximum clique size c and let $l \in \mathbb{R}_{\geq 0}$. Let $\Delta_l = \left\{x \in \mathbb{R}_{\geq 0}^{|V|} : \sum_{i=1}^{|V|} x_i = l\right\}$. Then $x^T A_G x \leq \frac{c-1}{c} l^2$ for all $x \in \Delta_l$.*

Proof. For $l = 0$, $\Delta_0 = \{\mathbf{0}\}$ and thus $\mathbf{0}^T A_G \mathbf{0} = 0$. So suppose $l > 0$. Let $x \in \Delta_l$ and set $y = \frac{1}{l} x$. Then $y \geq 0$ and $\sum_{i=1}^{|V|} y_i = 1$, hence $y \in \Delta_1$. Therefore by Theorem 1 $x^T A_G x = (ly)^T A_G (ly) = l^2 y^T A_G y \leq l^2 \frac{c-1}{c}$. ■

Definition 3 *Let $G = (V, E)$ be an undirected graph. For $k \in \mathbb{N}$ define the game $\Gamma_k(G) = (S, u_1)$ where*

- *$S = V \cup \{a, b, c\}$ are the strategies for the players where $a, b, c \notin V$.*
- *The utilities are given by*
  - *$u_1(i, j) = 1$ for all $i, j \in V$ with $(i, j) \in E$.*
  - *$u_1(i, j) = 0$ for all $i, j \in V$ with $(i, j) \notin E$.*
  - *$u_1(z, a) = 1$ for all $z \in S - \{b, c\}$.*
  - *$u_1(a, i) = \frac{k-1}{k}$ for all $i \in V$.*
  - *$u_1(y, i) = 1$ for all $y \in \{b, c\}$ and $i \in V$.*
  - *$u_1(y, a) = 0$ for all $y \in \{b, c\}$.*
  - *$u_1(z, y) = 0$ for all $z \in S$ and $y \in \{b, c\}$.*

Theorem 3 *Let $G = (V, E)$ be an undirected graph. The game $\Gamma_k(G)$ has an ESS if and only if G has no clique of size k.*

Proof. Let $G = (V, E)$ be an undirected graph with maximum clique size c. We consider the game $\Gamma_k(G)$. Suppose s is an ESS of $\Gamma_k(G)$. Then $\text{supp}(s) \cap \{b, c\} = \emptyset$, because if not let $t \neq s$ be a strategy with $t(i) = s(i)$ for $i \in V$, $t(y) = s(b) + s(c)$ and $t(y') = 0$ where $y, y' \in \{b, c\}$ such that $y \neq y'$ and $s(y) = \min\{s(b), s(c)\}$. Since $u_1(b, z) = u_1(c, z)$ for all $z \in S$,

$$U_1(t, s) = \sum_{i \in V} \underbrace{t(i)}_{=s(i)} U_1(i, s) + \underbrace{(t(b) + t(c))}_{=s(b)+s(c)} U_1(b, s) = U_1(s, s)$$

and so t is a best response to s. An identical argument shows that also $U_1(s, t) = U_1(t, t)$, but this is a contradiction to s being an ESS.

Furthermore, $\text{supp}(s) \nsubseteq V$, because if not, by Theorem 1

$$U_1(s, s) = \sum_{i,j \in V} s(i)s(j)u_1(i, j) = x^T A_G x \leq \frac{c-1}{c} < 1 = U_1(b, s)$$

where $x = (s(v_1), \dots, s(v_{|V|}))^T \in \Delta_1$ and so (s, s) is not a Nash equilibrium.

Thus $s(a) > 0$. Suppose for contradiction $s(a) < 1$. Since (s, s) is a Nash equilibrium, a is a best response to s and $a \neq s$. Then $U_1(s, a) = \sum_{z \in \text{supp}(s)} s(z) u_1(s, a) = 1 = U_1(a, a)$ gives a contradiction. Therefore the only possible ESS of $\Gamma_k(G)$ is a. Note that (a, a) is a symmetric Nash equilibrium because $u_1(z, a) \leq 1 = u_1(a, a)$ for all $z \in S$.

Suppose $c < k$. Let $t \neq a$ be a best response to a. Then $\mathrm{supp}(t) \subseteq V \cup \{a\}$. Let $r = \sum_{i \in V} t(i)$. So $r > 0$ and $t(a) = 1 - r$. So using Corollary 2:

$$\begin{aligned} U_1(t,t) - U_1(a,t) = &\underbrace{\sum_{i,j \in V} t(i)t(j)u_1(i,j)}_{\leq \frac{c-1}{c}r^2} + r \cdot t(a)1 + t(a) \cdot r\frac{k-1}{k} + t(a)^2 1 \\ &- \left(r \cdot \frac{k-1}{k} + t(a) \cdot 1\right) \\ \leq &\frac{c-1}{c}r^2 - \frac{k-1}{k}r^2 + r(1-r) + (1-r)^2 - (1-r) \\ = &\left(\frac{c-1}{c} - \frac{k-1}{k}\right) r^2 = \frac{r^2}{ck}(c-k) < 0 \end{aligned}$$

So a is an ESS.

Now suppose $c \geq k$. Let $C \subseteq V$ be a clique of G of size k. Then t with $t(i) = \frac{1}{k}$ for $i \in C$ and $t(j) = 0$ for $j \in S - C$ is a best response to a and $t \neq a$, but $U_1(t,t) = \sum_{i,j \in C} t(i)t(j)u_1(i,j) = \frac{1}{k^2} \cdot (k-1)k \cdot 1 = \frac{k-1}{k} = U_1(a,t)$, so a is not an ESS. ∎

Corollary 4 ESS *is* **coNP**-*hard.*

Proof. Theorem 3 shows a reduction from coCLIQUE to ESS. The game $\Gamma_k(G)$ has an ESS if and only if G has no clique of size k. Clearly, $\Gamma_k(G)$ can be constructed from G in polynomial time. ∎

3.2 ESS and REG-ESS are both NP-hard

We now show that deciding whether a game with rational payoffs has an ESS is **NP**-hard by reducing SAT to ESS. We will moreover see that the same reduction shows that REG-ESS is **NP**-hard.

Lemma 5 *Let $n \in \mathbb{N}$ and $k \in \mathbb{R}_{\geq 0}$. Let A be the $n \times n$-matrix where all entries are 1 except diagonal entries which are all 0. Consider the mapping $f : \mathbb{R}^n \to \mathbb{R}, f(x) = x^T Ax$. Then, the only maximum of f subject to $\sum_{i=1}^n x_i = k$ is $x^* = \left(\frac{k}{n}, \frac{k}{n}, \ldots, \frac{k}{n}\right)^T$ with $f(\frac{k}{n}, \ldots, \frac{k}{n}) = \frac{n-1}{n}k^2$.*

Proof. Note $f(x) = \sum_{i=1}^n x_i \sum_{\substack{j=1 \\ j \neq i}}^n x_j$. Since $\sum_{j=1}^n x_j = k$, $f(x) = \sum_{i=1}^n x_i(k - x_i) = k\sum_{i=1}^n x_i - \sum_{i=1}^n x_i^2 = k^2 - \sum_{i=1}^n x_i^2$. Let $\langle x, y \rangle = \sum_i x_i y_i$ denote the standard inner product of vectors x and y. Let $\mathbf{1} = (1, \ldots, 1)^T$ denote the all 1 vector of length n. We thus want to minimize $\langle x, x \rangle = \sum_{i=1}^n x_i^2$, subject to $\langle x, \mathbf{1} \rangle = k$. It is well known that x^* is the unique such minimum. For completeness, we provide a proof. Suppose $\langle y, \mathbf{1} \rangle = \langle x^*, \mathbf{1} \rangle = k$. Note, for any vector x, $\langle x, x^* \rangle = \frac{k}{n}\langle x, \mathbf{1} \rangle$. Now,

$$\begin{aligned} \langle y, y \rangle - \langle x^*, x^* \rangle &= \langle y, y \rangle - \langle x^*, x^* \rangle + 2\frac{k}{n}\langle x^*, \mathbf{1} \rangle - 2\frac{k}{n}\langle y, \mathbf{1} \rangle \\ &= \langle y, y \rangle + \langle x^*, x^* \rangle - 2\langle y, x^* \rangle = \langle y - x^*, y - x^* \rangle \geq 0 \end{aligned}$$

Moreover, $\langle y - x^*, y - x^* \rangle = 0$ if and only if $y = x^*$. Thus, x^* is the unique minimum. ■

Lemma 6 *Let $n \in \mathbb{N}$ and $k \in \mathbb{R}_{\geq 0}$. Let B be the $2n \times 2n$-matrix where*

$$b_{i,j} = \begin{cases} 0 & \text{if } i = j \\ -2 & \text{if } j = i+1 \text{ and } i = 2k+1 \text{ for some } k \\ -2 & \text{if } i = j+1 \text{ and } i = 2k \text{ for some } k \\ 1 & \text{otherwise} \end{cases}$$

In other words, B has the form

$$\begin{pmatrix} 0 & -2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -2 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & -2 & \cdots & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & \cdots & 1 & 1 & 1 \\ \vdots & & & & \ddots & & & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -2 & 0 \end{pmatrix}$$

Consider the mapping $f : \mathbb{R}^{2n}_{\geq 0} \to \mathbb{R}, f(x) = x^T B x$. Then, $x^* = (x_1^*, x_2^*, \ldots, x_{2n}^*) \in \mathbb{R}^{2n}_{\geq 0}$ is a global maximum of f subject to $\sum_{i=1}^{2n} x_i = k$ if and only if it satisfies $x^*_{2i+1} + x^*_{2i+2} = \frac{k}{n}$ and $x^*_{2i+1} x^*_{2i+2} = 0$ for all $0 \leq i < n$. In that case, $f(x^*) = \frac{n-1}{n} k^2$.

Proof. Note that

$$f(x) = \sum_{i=0}^{n-1} \left(\sum_{\substack{j=0 \\ j \neq i}}^{n-1} (x_{2i+1} + x_{2i+2})(x_{2j+1} + x_{2j+2}) - 4 x_{2i+1} x_{2i+2} \right)$$

Suppose, for contradiction, that x^* is a global maximum but that for some $i \in \{0, \ldots, n-1\}$, $x^*_{2i+1} > 0$ and $x^*_{2i+2} > 0$. Let x' be identical to x^* except that $x'_{2i+1} = x^*_{2i+1} + x^*_{2i+2}$, and $x'_{2i+2} = 0$. Note that x' satisfies the constraints $\sum_{j=1}^{2n} x'_j = k$, and $x' \geq 0$. However, $f(x') > f(x^*)$, because $(x'_{2j+1} + x'_{2j+2}) = (x^*_{2j+1} + x^*_{2j+1})$ for all $j = 0, \ldots, n-1$, but $4 x^*_{2i+1} x^*_{2i+1} > 4 x'_{2i+1} x'_{2i+2} = 0$. Contradiction. Therefore at any global maximum x^*, $x^*_{2i+1} x^*_{2i+2} = 0$, for all $i = 0, \ldots, n-1$.

Consider such a vector x^*. Let I be the set of indices such that for each $i = 0, \ldots, n-1$, exactly one of $2i+1$ and $2i+2$ is in I and such that $x^*_j = 0$ for every index j that is not in I. Note that for any such x^*, $f(x^*) = (x^*)^T B x^* = (x^*)^T_I B_{I,I} x^*_I$. Note that $B_{I,I}$ has exactly the form of matrix A of Lemma 5, and that $\langle x^*_I, \mathbf{1} \rangle = k$. Therefore, by Lemma 5 the unique maximum of $(x)^T B_{I,I} x$, subject to $\langle x, \mathbf{1} \rangle = k$, is $x^*_I = (\frac{k}{n}, \ldots, \frac{k}{n})^T$. From this the statement of Lemma 6 follows. ■

Definition 4 *Let Φ be a Boolean formula in conjunctive normal form. Let V be the set of its variables ($|V| = n$), L the set of literals over V, and $C \subseteq 2^L$ the set of clauses of Φ. The function $v : L \to V$ gives the variable corresponding to a literal, e.g. $v(x_1) = v(\neg x_1) = x_1$. Define the game $\Gamma(\Phi) = (S, u_1)$ where:*

- *$S = L \cup C$ are the strategies for player 1 and 2 and*
- *the utilities are given by*
  - *$u_1(l_1, l_2) = 1$ for all $l_1, l_2 \in L$ with $v(l_1) \neq v(l_2)$.*
  - *$u_1(l, l) = 0$ for all $l \in L$.*
  - *$u_1(l, \neg l) = -2$ for all $l \in L$.*
  - *$u_1(l, c) = -1$ for all $l \in L$ and all $c \in C$.*
  - *$u_1(c, l) = \frac{n-1}{n}$ for all $c \in C$ and all $l \in L$ with $l \notin c$.*
  - *$u_1(c, l) = -1$ for all $c \in C$ and all $l \in L$ with $l \in c$.*
  - *$u_1(c_1, c_2) = -1$ for all $c_1, c_2 \in C$.*

Theorem 7 *Let Φ be a Boolean formula in conjunctive normal form with n variables. If $(l_1, \dots, l_n)$ is an assignment of literals satisfying Φ, then the mixed strategy s with $s(l_i) = \frac{1}{n}$ for $1 \leq i \leq n$ and $s(x) = 0$ for $x \in S - \{l_1, \dots, l_n\}$ is an ESS for the game $\Gamma(\Phi)$. Conversely, if s is an ESS for $\Gamma(\Phi)$, then s is of the above form and $(l_1, \dots, l_n)$ is a satisfying assignment of Φ.*

Proof. Let Φ be a Boolean formula in conjunctive normal form with n variables. We consider the strategic game $\Gamma(\Phi)$.

Let s be an ESS. First, we show that $\operatorname{supp}(s) \cap C = \emptyset$. Assume not. Then, there is a clause $c \in C$ such that $s(c) > 0$. If $s(c) = 1$, then any literal l of c is a best response to s since $U_1(l, s) = u_1(l, c) = -1 = u_1(c, c) = U_1(s, s)$, but $U_1(l, l) = u_1(l, l) = 0 > -1 = u_1(c, l) = U_1(c, l)$, a contradiction to s being an ESS. So suppose $0 < s(c) < 1$. Since s is a NE, we know that $c \neq s$ is a best response to s and

$$U_1(s, c) = \sum_{x \in S} s(x) \underbrace{u_1(x, c)}_{=-1} = -1 = u_1(c, c) = U_1(c, c)$$

contradicting s being an ESS.

Next, we show that $v(\operatorname{supp}(s)) = V$, i.e. for each variable at least one corresponding literal is played. Assume not. Then, there is a literal $l \in L$ such that $s(l) = 0$ and $s(\neg l) = 0$. Enumerating the literals in such a way that $l^{2i+1} = x_i$ and $l^{2i+2} = \neg x_i$ for all $0 \leq i < n$, let $B = (b_{i,j})_{1 \leq i,j \leq 2n}$ be the $2n \times 2n$-matrix where $b_{i,j} = u_1(l^i, l^j)$ and $s' = (s(l^1), \dots, s(l^{2n}))^T$. Note that B is the same matrix B as in Lemma 6. So we can apply Lemma 6 to see that $\frac{n-1}{n} \geq s'^T B s' = \sum_{i,j=1}^{2n} s'_i b_{i,j} s'_j = \sum_{i,j=1}^{2n} s(l^i) s(l^j) u_1(l^i, l^j) = U_1(s, s)$. But, $U_1(l, s) = 1 \cdot \sum_{l \in \operatorname{supp}(s)} s(l) = 1 > \frac{n-1}{n} \geq U_1(s, s)$, so s is not a Nash equilibrium.

Next, we show that if s is an ESS, then there are n pairwise different literals $(l_1, \dots, l_n)$ such that $s(l_i) = \frac{1}{n}$ and $l_i \neq \neg l_j$ for $1 \leq i, j \leq n$. Suppose not. Since $v(\operatorname{supp}(s)) = V$, we can pick n pairwise different literals $(l'_1, \dots, l'_n)$ such that $l'_i \in \operatorname{supp}(s)$ and $l'_i \neq \neg l'_j$ for $1 \leq i, j \leq n$. Set $t(l'_i) = \frac{1}{n}$ for $1 \leq i \leq n$

and $t(i) = 0$ for all $i \in S - \{l'_1, \ldots, l'_n\}$. Since (s,s) is a Nash equilibrium, every $l \in \text{supp}(s)$ is a best response to s, i.e. $U_1(l,s) = U_1(s,s)$. Hence $U_1(t,s) = \sum_{i=1}^n t(l'_i) \sum_{j \in S} s(j) u_1(l'_i, j) = \frac{1}{n} \sum_{i=1}^n U_1(l'_i, s) = U_1(s,s)$, so t is a best response to s. Then

$$\begin{aligned} U_1(s,t) = &\sum_{l,l' \in L} s(l)t(l')u_1(l,l') = \sum_{l',l \in L} t(l')s(l)u_1(l',l) = U_1(t,s) \\ =&U_1(s,s) \le \frac{n-1}{n} \end{aligned} \tag{1}$$

and

$$\begin{aligned} U_1(t,t) =& \sum_{i=1}^n t(l'_i) \left(t(l'_i)u_1(l'_i,l'_i) + \sum_{\substack{j=1 \\ j \neq i}}^n t(l'_j)u_1(l'_i,l'_j) \right) \\ =& \sum_{i=1}^n \frac{1}{n} \left(\frac{1}{n} \cdot 0 + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{n} \cdot 1 \right) = \frac{1}{n} \cdot \frac{1}{n} \cdot n(n-1) = \frac{n-1}{n} \end{aligned} \tag{2}$$

so $U_1(s,t) \le U_1(t,t)$, contradicting s being an ESS.

What remains to be shown is that if s is a mixed strategy such that $s(l_i) = \frac{1}{n}$ for n different $l_i \in L$ with $l_i \neq \neg l_j$ for all $1 \le i,j \le n$ then s is an ESS if and only if $(l_1, \ldots, l_n)$ is a satisfying assignment for Φ.

Suppose s is such a mixed strategy. First, we show that (s,s) is a symmetric Nash equilibrium. We know from equation 2 that $U_1(s,s) = \frac{n-1}{n}$. Let $L^* = \{l_1, \ldots, l_n\}$. Playing any of the $l \in L^*$ gives utility $U_1(l,s) = \frac{1}{n} \cdot 0 + \frac{1}{n}(n-1) \cdot 1 = \frac{n-1}{n}$. Playing any of the $l \in L - L^*$ gives utility $U_1(l,s) = \frac{1}{n} \cdot (-2) + \frac{1}{n}(n-1) \cdot 1 < \frac{n-1}{n}$. Playing any of the $c \in C$ gives utility at most $U_1(c,s) = \frac{1}{n} n \cdot \frac{n-1}{n} = \frac{n-1}{n}$. Therefore, (s,s) is a symmetric Nash equilibrium.

Suppose $(l_1, \ldots, l_n)$ is not a satisfying assignment. Then, there is a clause c such that none of its literals is played. Therefore, $U_1(c,s) = \sum_{l \in L^*} s(l) u_1(c,l) = \sum_{l \in L^*} s(l) \frac{n-1}{n} = \frac{n-1}{n}$. So c is a best response to s and $c \neq s$. Then $U_1(c,c) = -1 = \sum_{l \in L^*} s(l) u_1(l,c) = U_1(s,c)$, so s is not an ESS.

Conversely, suppose $(l_1, \ldots, l_n)$ is a satisfying assignment. Then, every clause contains a literal that is played. Hence,

$$U_1(c,s) = \sum_{l \in L^*} s(l)u_1(c,l) = (-1) \sum_{l \in L^* \cap c} s(l) + \frac{n-1}{n} \sum_{l \in L^* - c} s(l) < \frac{n-1}{n} = U_1(s,s)$$

for all $c \in C$. So, suppose t is a best response to s. Then $\text{supp}(t) \subseteq \text{ext-supp}(s) = L^*$. Like in equation 1, we get $U_1(s,t) = U_1(t,s) = U_1(s,s) = \frac{n-1}{n}$. Let $A = (a_{i,j})_{1 \le i,j \le n}$ be the $n \times n$-matrix where $a_{i,j} = u_1(l_i, l_j)$ and let $t' = (t(l_1), \ldots, t(l_n))^T$. Note that A is the same matrix as A in Lemma 5, so we can apply Lemma 5 to see that

$$\frac{n-1}{n} \ge t'^T A t' = \sum_{i,j=1}^n t'_i b_{i,j} t'_j = \sum_{i,j=1}^n t(l_i)t(l_j)u_1(l_i,l_j) = U_1(t,t)$$

with equality holding only if $t(l_i) = \frac{1}{n}$ for all $1 \le i \le n$. Hence, we have that $t \neq s$ implies $U_1(s,t) > U_1(t,t)$. Therefore, s is an ESS. ∎

Corollary 8 ESS *is* **NP**-*hard.*

Proof. Theorem 7 shows a reduction from SAT to this problem: the game $\Gamma(\Phi)$ has an ESS if and only if Φ has a satisfying assignment. Clearly, $\Gamma(\Phi)$ can be constructed from Φ in polynomial time. Therefore ESS is **NP**-hard. ■

Corollary 9 REG-ESS *is* **NP**-*hard.*

Proof. In the proof of Theorem 7 we have shown that $(l_1, \ldots, l_n)$ is an assignment satisfying a boolean CNF-formula Φ if and only if s with $s(l_i) = \frac{1}{n}$ for $1 \leq i \leq n$ and $s(x) = 0$ for $x \in S - \{l_1, \ldots, l_n\}$ is an ESS for $\Gamma(\Phi)$. In fact, it can easily be seen from the definition of $\Gamma(\Phi)$ that $\text{supp}(s) = \text{ext-supp}(s)$, and hence s is then a regular ESS. Therefore, REG-ESS is **NP**-hard. ■

Corollary 10 #ESS *and* #REG-ESS *are* **#P**-*hard.*

Proof. The number of (regular) ESS in the game $\Gamma(\Phi)$ is the number of assignments to the variables satisfying Φ. Counting the number of satisfying assignments of a CNF boolean formula is **#P**-hard. ■

4 Upper bounds

4.1 REG-ESS is in NP

In [Hai75], Haigh claimed to show that a strategy s is an ESS for $\Gamma = (S, u_1)$ if and only if (s, s) is a Nash equilibrium and the $(k-1) \times (k-1)$-matrix $C = (c_{i,j})$ is negative definite, where $k = |\,\text{ext-supp}(s)|$ and $c_{i,j} = u_1(i,j) + u_1(k,k) - u_1(i,k) - u_1(k,j)$ for $i, j \in \text{ext-supp}(s) - \{k\}$ (where, w.l.o.g., $\text{ext-supp}(s) = \{1, \ldots, k\}$).

In [Aba80], Abakuks pointed out that there is an error in the "only if" part of Haigh's claim. Namely, Abakuks showed that the existence of an ESS only implies the negative definiteness of the matrix C if in addition $s(k) > 0$ and $|\,\text{ext-supp}(s)| - |\,\text{supp}(s)| \leq 1$.

As we will see, the Haigh-Abakuks criteria can fairly easily be used to show that REG-ESS is in **NP**. By a suitable modification of these criteria, we can obtain necessary and sufficient conditions for the existence of arbitrary ESSs which will allow us to show that ESS is in $\mathbf{\Sigma_2^P}$.

Essentially identical conditions, based on "copositivity" of matrices over a cone, were developed by Bomze and used by him in an algorithm for enumerating all ESSs of a game (compare Theorem 14 below with Theorem 3.2 of [Bom92], whose proof depends also on the development in [BP89]). Bomze's enumeration algorithm uses a recursive elimination procedure that involves some complications including possible numerical issues. In particular, we could not make certain that iterating the procedure outlined in Theorem 3.3 of [Bom92] will not cause an exponential blow-up in numerical values. We were thus unable to ascertain our desired complexity upper bounds for ESS directly from Bomze's algorithms.

We will instead give here a self-contained and elementary development of the criteria we shall use, based directly on the work of [Hai75] and [Aba80], and we will then (in the case of ESS) rely on a well known result by Vavasis about the complexity of the quadratic programming decision problem ([Vav90]) to obtain our upper bounds.

Lemma 1 in [Aba80] says the following: if $Y = \left\{ y \in \mathbb{R}^k_{\geq 0} : \sum_{i=1}^k y_i = 1 \right\}$ and $Z = \left\{ z \in \mathbb{R}^k : z \neq 0, \sum_{i=1}^k z_i = 0 \right\}$ and B is a real $k \times k$-matrix, then

- $z^T B z$ for all $z \in Z$ implies that $(x-y)^T B(x-y) < 0$ for all $y \in Y$ with $y \neq x$ and
- if at most one component of x is zero then $(x-y)^T B(x-y) < 0$ for all $y \in Y$ with $y \neq x$ implies that $z^T B z < 0$ for all $z \in Z$.

In the following, we provide a variation of Abakuks' Lemma 1:

Lemma 11 *Let $k \in \mathbb{N}$ and let $x \in \mathbb{R}^k_{\geq 0}$ such that $\sum_{i=1}^k x_i = 1$. Let*

$$Y_x = \left\{ y \in \mathbb{R}^k_{\geq 0} : \sum_{i=1}^k y_i = 1 \right\} - \{x\}$$

and

$$Z_x = \left\{ z \in \mathbb{R}^k : \sum_{i=1}^k z_i = 0, (\forall i \in \{1, \ldots, k\} : x_i = 0 \Rightarrow z_i \geq 0) \right\} - \{\mathbf{0}\}$$

and let B be a $k \times k$-matrix. Then the following statements are equivalent:

- $z^T B z < 0$ *for all* $z \in Z_x$.
- $(y-x)^T B(y-x) < 0$ *for all* $y \in Y_x$.

Proof. Suppose $z^T B z < 0$ for all $z \in Z_x$. Let $y \in Y_x$. Then $y - x \neq \mathbf{0}$, $\sum_{i=1}^k (y_i - x_i) = \sum_{i=1}^k y_i - \sum_{i=1}^k x_i = 1 - 1 = 0$ and for all $1 \leq i \leq k$ with $x_i = 0$ we get $y_i - x_i = y_i \geq 0$, hence $y - x \in Z_x$ and so $(y-x)^T B(y-x) < 0$.

Conversely, suppose $(y-x)^T B(y-x) < 0$ for all $y \in Y_x$. Let $z \in Z_x$. Set

$$\lambda = \min \left\{ \frac{x_i}{|z_i|} : 1 \leq i \leq k, x_i > 0, z_i \neq 0 \right\}$$

Then $\lambda > 0$. Choose $y = x + \lambda z \neq x$. Then $y \geq 0$ because $x \geq 0$ and if $z_i < 0$ then $x_i > 0$ and $y_i = x_i - \lambda |z_i| \geq x_i - \frac{x_i}{|z_i|}|z_i| = 0$ for $1 \leq i \leq k$. Note that

$$\sum_{i=1}^k y_i = \sum_{i=1}^k x_i + \lambda \sum_{i=1}^k z_i = 1 + \lambda \cdot 0 = 1$$

Hence $y \in Y_x$ and thus $z^T B z = \left(\frac{1}{\lambda}(y-x)\right)^T B \left(\frac{1}{\lambda}(y-x)\right) = \frac{1}{\lambda^2}(y-x)^T B(y-x) < \frac{1}{\lambda^2} 0 = 0$. ∎

Lemma 12 *([Hai75]) Let $C = (c_{i,j})_{i,j}$ be a real $m \times m$-matrix, $m \geq 2$. Let $D = (d_{i,j})_{i,j}$ be the $(m-1)\times(m-1)$-matrix given by $d_{i,j} = c_{i,j} + c_{m,m} - c_{i,m} - c_{m,j}$. Let $x \in \mathbb{R}^m$ such that $\sum_{i=1}^m x_i = 0$ and set $x' = x_{\{1,\dots,m-1\}}$. Then $x^T C x = x'^T D x'$.*

Proof. A proof for this and the next lemma was given by Haigh in [Hai75]. For completeness, we provide it here. Let $x \in \mathbb{R}^m$ such that $\sum_{i=1}^m x_i = 0$, i. e. $x_m = -\sum_{i=1}^{m-1} x_i$. Then

$$\begin{aligned}
x^T C x &= \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} x_i c_{i,j} x_j + x_m \left(\sum_{j=1}^{m-1} c_{m,j} x_j + \sum_{i=1}^{m-1} x_i c_{i,m}\right) + x_m^2 c_{m,m} \\
&= \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} \left(x_i c_{i,j} x_j + (-x_i) c_{m,j} x_j + x_i c_{i,m}(-x_j) + (-x_i) c_{m,m}(-x_j)\right) \\
&= \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} x_i \left(c_{ij} + c_{mm} - c_{mj} - c_{im}\right) x_j = x'^T D x'
\end{aligned}$$

■

Lemma 13 *([Hai75]) Let (s, s) be a symmetric Nash equilibrium for the game $\Gamma = (S, u_1)$ and let $M = \text{ext-supp}(s)$. Let $x = s_M$ and $C = (A_\Gamma)_{M,M}$. Let Y_x be defined as in Lemma 11. s is an ESS if and only if $(y - x)^T C(y - x) < 0$ for all $y \in Y_x$.*

Proof. Let t be any best response to s. Consider

$$\begin{aligned}
U_1(t,t) - U_1(s,t) =& t^T A_\Gamma t - s^T A_\Gamma t = (t-s)^T A_\Gamma (t - s + s) \\
=& (t-s)^T A_\Gamma (t-s) + \underbrace{t^T A_\Gamma s - s^T A_\Gamma s}_{=U_1(t,s)-U_1(s,s)=0} = (t-s)^T A_\Gamma (t-s)
\end{aligned}$$

Note that $t(z) = s(z) = 0$ for $z \in S - M$. Let $y = t_M$. Then $(t-s)^T A_\Gamma (t-s) = (y-x)^T C(y-x)$.

Suppose s is an ESS. Then for any $y' \in Y_x$, let $t' \in X$ with $t'(z) = y'(z)$ for $z \in M$ and $t'(z) = 0$ for $z \in S - M$. Then $t' \neq s$ is a best response to s, because supp(t) is contained in $M = \text{ext-supp}(s)$. Thus, $(y'-x)^T C(y'-x) = (t'-s)^T A_\Gamma (t'-s) = U_1(t', t') - U_1(s, t') < 0$.

Conversely, suppose $(y - x)^T C(y - x) < 0$ for all $y \in Y_x$. For any best response $t \neq s$ to s, set $y'' = t_M$. Then $y'' \in Y_x$ and so $U_1(t,t) - U_1(s,t) = (y''-x)^T C(y''-x) < 0$. ■

Theorem 14 *(cf. [Bom92], Theorem 3.2) Let (s, s) be a symmetric Nash equilibrium for the game $\Gamma = (S, u_1)$ with $|\text{ext-supp}(s)| = m$, and $m \geq 2$. Let $M = \text{ext-supp}(s)$ and identify M with $\{1, \dots, m\}$ such that $s(m) > 0$ and let $x = s_M$. Let $C = (A_\Gamma)_{M,M}$ and let D be defined as in Lemma 12. Let*

$$W_x = \left\{w \in \mathbb{R}^{m-1} : (\forall 1 \leq i \leq m-1 : x_i = 0 \Rightarrow w_i \geq 0)\right\} - \{\mathbf{0}\}$$

Then s is an ESS if and only if $w^T D w < 0$ for all $w \in W_x$.

Proof. Let Y_x and Z_x be defined as in Lemma 11. From Lemma 13 we know that $U_1(t,t) - U_1(s,t) < 0$ for all best responses $t \neq s$ to s is equivalent to $(y-x)^T C(y-x) < 0$ for $y \in Y_x$ which itself is equivalent to $z^T Cz < 0$ for all $z \in Z_x$ by Lemma 11.

Now suppose that $w^T Dw < 0$ for all $w \in W_x$. Let $z \in Z_x$. Set $w' = z_{\{1,\ldots,m-1\}}$. Then $w' \in W_x$ and so with Lemma 12, we get $z^T Cz = w'^T Dw' < 0$.

Conversely, suppose that $z^T Cz < 0$ for all $z \in Z_x$. Let $w \in W_x$. Set $z'_i = w_i$ for $1 \leq i \leq m-1$ and $z'_m = -\sum_{i=1}^{m-1} w_i$. Then $z' = (z'_1, \ldots, z'_m)^T \in Z_x$ because $s(m) > 0$ and so with Lemma 12, we get $w^T Dw = z'^T Cz' < 0$. ■

Lemma 15 *Let s be an ESS for $\Gamma = (S, u_1)$. Then (s,s) is the only symmetric Nash equilibrium (t,t) with* $\text{supp}(t) \subseteq \text{ext-supp}(s)$.

Proof. Suppose there was a symmetric Nash equilibrium (t,t) other than (s,s) with $\text{supp}(t) \subseteq \text{ext-supp}(s)$. Then $t \neq s$ is a best response to s and since (t,t) is a Nash equilibrium, $U_1(s,t) \leq U_1(t,t)$. Contradiction to s being an ESS. ■

Theorem 16 REG-ESS *is in* **NP**.

Proof. Given a game $\Gamma = (S, u_1)$ ($n = |S|$) with rational utilities, guess the extended support set $M \subseteq S$ of a (purported) regular ESS s for Γ and let $m = |M|$. Identify S with $\{1,\ldots,n\}$ such that $M = \{1,\ldots,m\}$. Find a symmetric Nash equilibrium (s,s) of Γ with $\text{supp}(s) \subseteq M$ by solving the following linear system of constraints in variables $s_1, \ldots, s_n, w$, where $s = (s_1, \ldots, s_n)^T$:

- $U_1(i,s) = w$ for all $i \in M$.
- $U_1(i,s) \leq w$ for all $i \in S - M$.
- $\sum_{i=1}^n s_i = 1$.
- $s_i \geq 0$ for all $i \in M$.
- $s_i = 0$ for all $i \in S - M$.

A solution to this system can be found in polynomial time, via linear programming. Let s be an arbitrary solution. By Lemma 15 if s is an ESS then s is the only solution to the system above. Thus it doesn't matter what solution we find (if we don't find any solution, then there is no Nash equilibrium and hence no ESS with support set M). Check that $\text{supp}(s) = \text{ext-supp}(s) = M$. This check can be done easily in polynomial time, by trying each pure strategy outside $\text{supp}(s)$ against s.

Note that if $|M| = 1$ then the pure strategy s is the only best response to itself, and thus s is a regular ESS. Suppose $|M| \geq 2$, and let $x = s_M$. Let D and W_x be defined as in Theorem 14. By Theorem 14, s is an ESS if and only if $w^T(-D)w > 0$ for all $w \in W_x$. Set $D' = \frac{1}{2}(D + D^T)$. D' is a symmetric matrix, and note that $w^T D' w = w^T Dw$ for all w.

Note that $W_x = \mathbb{R}^{m-1} - \{\mathbf{0}\}$ because $\text{supp}(s) = M$. Hence s is an ESS if and only if $(-D')$ is positive definite. Positive definiteness of a symmetric matrix can be checked in polynomial time via the determinant criterion (see section 2). Therefore checking whether there is an ESS s for Γ with $\text{supp}(s) = \text{ext-supp}(s) = M$ for the guessed set M can be done in polynomial time. Thus REG-ESS is in **NP**. ∎

Corollary 17 #REG-ESS *is in* **#P**.

Proof. In the proof of Theorem 16 we give a non-deterministic polynomial-time algorithm for deciding whether a game has a regular ESS. Each accepting computation yields a different support set and thus a different regular ESS. Therefore, #REG-ESS is in **#P**. ∎

4.2 ESS is in $\Sigma_2^{\mathbf{P}}$

Definition 5 *Let* QP *denote the following decision version of the quadratic programming problem: Given a $n \times n$-matrix H and a $m \times n$-matrix A, both with integer coefficients, $K \in \mathbb{Q}$, $c \in \mathbb{Z}^m$, and $b \in \mathbb{Z}^m$, is there a $x \in \mathbb{R}^n$ with $Ax \geq b$ such that $x^T Hx + c^T x \leq K$?*

Vavasis [Vav90] proved that the quadratic programming decision problem is in **NP** (see also, e.g., [MK87]).

Theorem 18 *([Vav90])* QP *is in* **NP**.

Theorem 19 ESS *is in* $\Sigma_2^{\mathbf{P}}$.

Proof. Given a game $\Gamma = (S, u_1)$ $(n = |S|)$ with rational utilities, guess the extended support set $M \subseteq S$ for an ESS s for Γ and set $m = |M|$. As in the proof of Theorem 16, compute a symmetric Nash equilibrium (s, s) with $\text{supp}(s) \subseteq M$. Check that $\text{ext-supp}(s) = M$ (this again, can be done easily in polynomial time). Set $l = m - |\text{supp}(s)|$. If $l = 0$ then proceed as in the algorithm in the proof of Theorem 16.

Suppose $l > 0$, and thus $m \geq 2$. Let $x = s_M$. Let D and W_x be defined as in Theorem 14. By Theorem 14, s is an ESS if and only if $w^T(-D)\,w > 0$ for all $w \in W_x$. In other words, s is not an ESS iff there exists $w \in W_x$ such that $w^T(-D)\,w \leq 0$. This is the case iff there exists $w \neq \mathbf{0}$ such that $w_i \geq 0$ for all i such that $x_i = 0$. This in turn, we claim, is the case iff

> there exists a w such that $w_i \geq 0$ for all i where $x_i = 0$, and such that for some $j \in \{1, \ldots, m-1\}$, $w_j \geq 1$ or $-w_j \geq 1$.

To see the last claim, note that if $w^T(-D)w \leq 0$, then for any constant $c > 0$, $(cw)^T(-D)(cw) = c^2 w^T(-D)w \leq 0$. Thus, for $w \neq \mathbf{0}$ where $w^T(-D)w \leq 0$, we can choose a constant $c > 0$ large enough so that either for some positive coefficient w_j, $cw_j \geq 1$ or for some negative coefficient w_j, $-(cw_j) \geq 1$. Thus the vector (cw) will satisfy the desired conditions.

Now, it is easy to check these conditions by solving a sequence of $2(m-1)$ quadratic programming decision problems. Namely, we check for all $1 \leq j \leq m-1$ and for each $\sigma \in \{+1, -1\}$, whether there exists a $w \in \mathbb{R}^{m-1}$ satisfying $w^T(-D)w \leq 0$, and satisfying the linear constraints: $w_i \geq 0$ for each i such that $x_i \geq 0$, and $\sigma w_j \geq 1$.

As described, the matrix $(-D)$ is not necessarily an integer matrix but rational, and the QP decision problem was formulated in terms of integer matrices. However, we can easily "clear denominators" in $(-D)$, finding the least common multiple $\lambda > 0$ of the denominators of all entries of D and setting $H = -\lambda D$ (this can be done easily in P-time). Then H is a $(m-1) \times (m-1)$-matrix with integer entries, and $w^T H w \leq 0$ if and only if $w^T(-D)w \leq 0$, for any $w \in \mathbb{R}^{m-1}$.

Thus, checking that s is not an ESS can be done in NP. Thus, to determine the existence of an ESS involves existentially guessing a support set M, finding s with support set M such that (s, s) is a Nash equilibrium (using linear programming), and then checking that s is an ESS in **coNP**, by checking (in **NP**) that s is not an ESS. This concludes the proof that ESS is in $\mathbf{\Sigma_2^P}$. ■

In summary, we have shown:

Corollary 20 REG-ESS *is* **NP**-*complete and* #REG-ESS *is* **#P**-*complete.*

Proof. This follows immediately from Corollary 9, Theorem 16 and Corollaries 10 and 17. ■

Corollary 21 ESS *is not in* **NP** *unless* **NP** = **coNP**.

Proof. By corollaries 4 and 8, ESS is **NP**-hard and **coNP**-hard. Thus, if ESS in **NP**, then **NP** = **coNP**. ■

5 Concluding remarks

Our results leave open whether the general ESS problem is $\mathbf{\Sigma_2^P}$-complete or belongs to some "intermediate" class above **NP** and **coNP** but below $\mathbf{\Sigma_2^P}$.

An issue not addressed directly by what we have said so far is whether an ESS, if one exists, can be "approximated" efficiently. Here one has to be careful about what it means to approximate an ESS, since indeed none may exist. One formulation would be that there is a polynomial time algorithm that, given $\epsilon > 0$ and the game as input, outputs a mixed strategy s such that *if* there exists a (regular) ESS, then there exists a (regular) ESS s^* such that $\|s^* - s\| < \epsilon$, under some vector norm $\|\cdot\|$. For concreteness, let $\|s\| = \Sigma_{i=1}^{n}|s_i|$ be the L_1 norm (other norms like L_∞ would work just as well). Let us call this a polynomial time ϵ-approximation of (regular) ESSs.

Based on this definition, we can easily conclude the following inappoximability statement from the results in Section 3.2.

Corollary 22 *There is no polynomial time $\frac{1}{m}$-approximation algorithm for finding an ESS in a game $\Gamma = (S, u_1)$ nor for finding a regular ESS in Γ, where $m = |S|$, unless* $\mathbf{P}$ = **NP**.

Proof. Suppose there was such an algorithm. For a boolean formula Φ, we run that algorithm on the game $\Gamma(\Phi) = (S, u_1)$ with $|S| = m = 2|V| + |C|$, where $|V| = n$ is the number of variables of Φ, and $|C|$ is the number of clauses. This would yield a strategy s such that if there exists a (regular) ESS in $\Gamma(\Phi)$, then there exists s^* with $\|s^* - s\| < \frac{1}{m}$. Thus $|s_i^* - s_i| < \frac{1}{m}$ for all $1 \leq i \leq m$. Note however that by Theorem 7, the only candidate (regular) ESSs s^* in that game has, in every coordinate, either probability $\frac{1}{|V|} = \frac{1}{n} > \frac{2}{m}$ or probability 0. Thus if $s_i > \frac{1}{m}$, then the only possible candidate for s_i^* is $s_i^* = \frac{1}{n}$, and if $s_i < \frac{1}{m}$, then the only possible candidate is $s_i^* = 0$. If $s_i = \frac{1}{m}$, then neither is a candidate and hence s is not within distance $< \frac{1}{m}$ of any ESS, therefore no ESS exists.

So, we can build the candidate s^*, check that the probabilities in it sum to 1, and that it corresponds to a truth assignment to variables, meaning exactly one of the two pure strategies corresponding to the two literals for each variable has non-zero probability, and no strategy corresponding to a clause has non-zero probability. We then check whether this is actually a satisfying assignment of Φ. If so, Φ is satisfiable, otherwise Φ is not. Thus we would have solved SAT in polynomial time using our purported approximation algorithm. ■

Other notions of approximation may be preferable. As described in the introduction, a speculative motivation for considering inapproximability of ESSs arises from the dynamical system view of evolutionary stability. Suppose the biological system that our game intends to model does actually converge to an evolutionarily stable strategy by some kind of dynamic process. Then one way to interpret a hardness result for approximating an ESS is as a statement that it must take a "long time", starting from an arbitrary initial state, for the system to converge to an ESS, under any dynamic process that is "locally simple", meaning each "iteration" is easy to compute. Since convergence may only happen in the limit, inapproximability seems more appropriate here than hardness of exact computation.

References

[Aba80] A. Abakuks. Conditions for evolutionarily stable strategies. *J. Appl. Prob.*, 17:559–562, 1980.

[Bom86] I. Bomze. Non-cooperative 2-person games in biology: a classification. *Int. Journal of Game Theory*, 15:31–59, 1986.

[Bom92] I. Bomze. Detecting all evolutionarily stable strategies. *Journal of Optimization Theory and Applications*, 1992.

[Bom02] I. Bomze. Regularity versus degeneracy in dynamics, games, and optimization: a unified approach to different aspects. *SIAM Review*, 44(3):394–414, 2002.

[BP89] I. Bomze and B. M. Pötscher. *Game theoretic foundations of evolutionary stability.* Springer, 1989.

[CS03] V. Conitzer and T. Sandholm. Complexity results about nash equilibria. In *18th Int. Joint Conf. on Artificial Intelligence (IJCAI)*, pages 765–771, 2003.

[GZ89] I. Gilboa and E. Zemel. Nash and correlated equilibria: some complexity considerations. *Games and Economic Behavior*, 1:80–93, 1989.

[Hai75] J. Haigh. Game theory and evolution. *Adv. Appl. Prob.*, 7:8–11, 1975.

[Har73a] J. Harsanyi. Games with randomly distributed payoffs: a new rational for mixed strategy equilibrium points. *Int. J. of Game Theory*, 2:1–23, 1973.

[Har73b] J. Harsanyi. Oddness of the number of equilibrium points: a new proof. *Int. J. of Game Theory*, 2:235–250, 1973.

[HS88] J. Hofbauer and K. Sigmund. *The theory of evolution and dynamical systems*. Cambridge University Press, 1988.

[HS94] P. Hammerstein and R. Selten. Game theory and evolutionary biology (chapter 2). In R. J. Aumann and S. Hart, editors, *Handbook of Game Theory*, volume 2, chapter 2, pages 929–993. Elsevier, 1994.

[LT85] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*. Academic Press, 2nd edition, 1985.

[MK87] K. G. Murty and S. N. Kabadi. Some NP-complete problems in quadratic and non-linear programming. *Mathematical Programming*, 39:117–129, 1987.

[MS65] T. S. Motzkin and E. G. Straus. Maxima for graphs and a new proof of a theorem of Turan. *Canadian Journal of Mathematics*, 17:533–540, 1965.

[MWC+97] J. M. McNamara, J. N. Webb, E. J. Collins, T. Szekely, and A. I. Houston. A general technique for computing evolutionarily stable strategies based on dynamics. *Journal of Theoretical Biology*, 189:211–225, 1997.

[Nas51] J. Nash. Non-cooperative games. *Annals of Mathematics*, 1951.

[OR94] M. J. Osborne and A. Rubinstein. *A course in game theory*. MIT press, 1994.

[Pap94] C. Papadimitriou. *Computational Complexity*. Addison-Wesley, 1994.

[Pap01] C. Papadimitriou. Algorithms, games and the internet. In *Proc. of ACM Symp. on Theory of Computing*, pages 249–253, 2001.

[Sel83] R. Selten. Evolutionary stability in extensive 2-person games. *Mathematical Social Sciences*, 5:269–363, 1983.

[Smi82] J. Maynard Smith. *Evolution and the theory of games*. Cambridge University Press, 1982.

[SP73] J. Maynard Smith and G. R. Price. The logic of animal conflict. *Nature, London*, 246:15–18, 1973.

[Vav90] S. A. Vavasis. Quadratic programming is in NP. *Inf. Process. Lett.*, 36(2):73–77, 1990.

[vD91] E. van Damme. *Stability and perfection of Nash Equilibria.* Springer-Verlag, 2nd edition, 1991.

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