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Resource Bounded Immunity and Simplicity *

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Abstract: Revisiting the thirty years-old notions of resource-bounded immunity and simplicity, we investigate the structural characteristics of various immunity notions: strong immunity, almost immunity, and hyperimmunity as well as their corresponding simplicity notions. We also study limited immunity and simplicity, called k-immunity and feasible k-immunity, and their simplicity notions. Finally, we propose the k-immune hypothesis as a working hypothesis that guarantees the existence of simple sets in NP.

Key Words: immune set, simple set, complete set, weak forcing, generic oracle, random oracle

1 Prologue

The twentieth century opened its curtain with a keynote speech of Hilbert on the list of open problems that should be challenged in the coming century. One problem of his relates to a fundamental question on the computability. In his speech, Hilbert raised the question, known as Hilbert's tenth problem, of whether there exists a "procedure" to calculate integer solutions of polynomial equations. This question signifies the importance of "algorithmic procedure." Subsequently, the twentieth century had delivered the new theory of computation and computability. The notion of computability was first introduced by Gödel [18] in his epochal paper on the incompleteness of any consistent theory. Matijasevich's solution to Hilbert's tenth problem is a remarkable success of classical recursion theory.

In the last half century, a new breed of computation theory—computational complexity theory—hatched from classical recursion theory as electronic computing device has been materialized. In particular, the classes P and NP are defined analogously to the classes of recursive sets and recursively enumerable (r.e., in short) sets, respectively. By the pioneering work of Cook, Karp, and Levin, the theory of NP-completeness has boosted the progress of complexity theory. Since the 1970s, the complexity class NP has been widely recognized as an important complexity class that includes many natural problems in computer science, and structural complexity theory has evolved revolving around this class NP. A glossary of NP-complete problems are found in, for instance, the textbook of Gary and Johnson [17]. In early 1970s, Meyer and Stockmeyer [28] further defined the *polynomial-time hierarchy*, which is built over NP by a way of relativization, and this notion was refined later by Stockmeyer [40] and Wrathall [46]. It is strongly believed by a number of complexity theoreticians that NP differs from P and, moreover, the polynomial-time hierarchy indeed forms an infinite hierarchy. If NP properly includes P, as is believed, then how hard NP would be? Assuming that $\mathrm{P} \neq \mathrm{NP}$, NP is known to have an infinite number of layers of equivalence classes between P and the class of NP-complete sets. Is there any common characteristic among NP sets that do not fall into P? It is thus of great importance to study the structural characteristics of the sets sitting in the difference $\mathrm{NP} - \mathrm{P}$.

Back in the 1940s, Post [35] asked whether there exists an intermediate r.e. set that is neither recursive nor Turing-complete. As an attempt to answer this question, he constructed a non-recursive set, called a *simple* set, which turns out not to be bounded truth-table complete (more strongly, not disjunctively complete). Resource-bounded immune and simple sets were explicitly discussed by Flajolet and Steyaert [16] in a general

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framework under the terms "$\mathcal{C}$-immune sets" and "$\mathcal{C}$-simple sets" for any time-bounded complexity class $\mathcal{C}$. Briefly speaking, a $\mathcal{C}$-immune set is an infinite set that has no infinite subset in $\mathcal{C}$ whereas a $\mathcal{C}$-simple set is a set in $\mathcal{C}$ whose complement is $\mathcal{C}$-immune. The initial research on resource-bounded immunity and simplicity was deeply rooted in recursion theory.

Recently, there have been a surge of renewed interests in resource-bounded immunity and simplicity and we have made a significant progress in understanding the structure of NP through these notions. In this paper, taking a conventional approach toward the structural properties of the polynomial-time hierarchy, we extensively study the notions of resource-bounded immunity and simplicity. The purpose of this paper is to give a comprehensive guidance toward our understanding of these notions developed in computational complexity theory. We wish to present a broad spectrum of consequences obtained in the course of our study on resource-bounded immunity and simplicity. Of all complexity classes, we focus only on the classes lying within the polynomial-time hierarchy. Other classes, such as $\mathrm{C_{=}P}$ and EXP, have been studied in, e.g., [36, 38]. The goals of our investigation are to (i) analyze the behaviors of variants of immunity and simplicity notions, (ii) study the relationships between immunity and other complexity-theoretical notions, and (iii) explore new directions for better understandings of the polynomial-time hierarchy.

In an early work of Ko and Moore [27], a number of structural properties of P-immune sets were obtained; for instance, a P-immune set exists even in the complexity class E. Since then, there have been known close connections between immune sets and various other notions, such as instance complexity [33] and complexity core [30]. An additional notion of P*-bi-immunity* was considered by Balcázar and Schöning [5], following the previous notions of *almost everywhere* complexity and *polynomial approximation* algorithms. Departing from recursion theory, structural complexity theory has developed its own immunity notions using resource-bounded computations. Balcázar and Schöning [5] introduced the notion of *strongly* P*-bi-immune* sets and showed the existence of such sets within E. An *almost* P*-immune* set was introduced by Orponen [32] and Orponen, Russo, and Schöning [34]. The complementary set of any almost P-immune set is particularly called a P-levelable set, due to Orponen, et al. [34] (where the term "levelable" was suggested by Ko [26]). Recently, Yamakami [48] and Schaefer and Fenner [38] studied resource-bounded hyperimmune sets.

The existence of an NP-simple set is, unfortunately, unknown since the unproven consequence $\mathrm{NP} \neq \text{co-NP}$ immediately follows. We introduce the weaker notion of almost NP-simple sets and show that an NP-simple set exists if and only if an almost NP-simple set exists in P. This notion also has a connection to Uspenski's notion of pseudosimple sets [43]. By contrast, relativized results on the existence of simple sets abound. Homer and Maass [24] and Balcázar [3] exhibited relativized worlds where NP-simple sets actually exist. Homer and Maass also showed that the statement $\mathrm{P} \neq \mathrm{NP}$ is not sufficient for the existence of an NP-simple set. The recursive oracle result of Balcázar was expanded by Vereshchagin [44] to the existence of NP-simple sets relative to a random oracle. For each level k of the polynomial-time hierarchy, Bruschi [11] constructed various relativized worlds where, for example, a Σ_k^{P}-simple set exists. We further prove that, at each level k of the polynomial-time hierarchy, a strongly Σ_k^{P}-simple set exists relative to a certain recursive oracle. In addition, we show that, relative to a generic oracle, a strongly NP-simple set exists. This immediately implies that an NP-simple set exists relative to a generic oracle. Recently, Schaefer and Fenner [38] showed that even honestly NP-hypersimple sets exist relative to a generic oracle. By contrast, we show the existence of a honestly NP-hypersimple set relative to a random oracle. We also construct a relativized world for each k in which a Σ_k^{P}-hypersimple set truly exists.

The relationship between the simplicity notions and the reducibilities has been a central focal point in the recent literature. We can view reducibility as an algorithmic way to encode the essential information of a given set into another set. It is known that, under an appropriate assumption, stronger reducibility makes NP-simple sets "incomplete" for NP; that is, simple sets cannot be complete under such stronger reductions. With regard to non-honest reductions, Hartmanis et al. [20] first showed that if $\mathrm{NP} \cap \text{co-NP} \nsubseteq \bigcap_{\epsilon>0} \mathrm{DTIME}(2^{n^{\epsilon}})$ then there is no P-m-complete NP-simple set for NP. Recently, Schaefer and Fenner [38] demonstrated that no NP-simple set can be P-1tt-complete for NP unless $\mathrm{UP} \subseteq \bigcap_{\epsilon>0} \mathrm{DTIME}(2^{n^{\epsilon}})$, where UP is the unambiguous polynomial-time complexity class. In this paper, we improve these results by showing that no Σ_k^{P}-simple set can be Δ_j^{P}-1tt-complete unless $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}) \subseteq \mathrm{SUB}\Delta_{\max\{j,k\}}^{\mathrm{EXP}}$ for any positive integers i, j, k, where $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$ is the unambiguous version of $\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}$ introduced in [47]. In addition, we show without any unproven assumption that no Σ_k^{P}-hyperimmune set can be P-T-complete for Σ_k^{P}.

Since the existence of an NP-simple set is not known, Homer [23] looked into much weaker notions of NP-immunity and NP-simplicity by defining k*-immune* sets and k*-simple* sets within NP using $O(n^k)$-time bounded nondeterministic computations. He demonstrated the existence of a k-simple set for each positive integer k.

We further demonstrate the existence of a k-simple set that is not feasibly k-simple, where a *feasibly k-simple* set is an analogue of an effectively simple set. Employing a diagonalization argument, we can construct a feasibly k-immune set in Δ_2^{P}. In connection to Homer's result, we propose a working hypothesis, so-called the *k-immune hypothesis*: every infinite NP set has an infinite subset recognized by $O(n^k)$-time nondeterministic Turing machines. The existence of such a number k yields the existence of NP-simple sets. This hypothesis may propel the study of NP-simple sets. We then show that, relative to a generic oracle, the k-immune hypothesis fails.

Organization of the Paper. The rest of this paper is organized in the following fashion: we begin with the $\mathcal{C}$-immunity and $\mathcal{C}$-simplicity notions in Section 3 and then formulate the notions of strong $\mathcal{C}$-immunity and strong $\mathcal{C}$-simplicity in Section 4, almost $\mathcal{C}$-immunity and almost $\mathcal{C}$-simplicity in Section 5, and $\mathcal{C}$-hyperimmunity and $\mathcal{C}$-hypersimplicity in Section 6. Moreover, the k-immunity and k-simplicity and their variants are discussed in Section 8. In the same section, the k-immune hypothesis is studied. In addition, we discuss the relationships between completeness notions and simplicity in Section 7.

2 Foundations of Notions and Notation

Throughout this paper, we use standard notions and notation found in the most introductory textbooks of recursion theory (e.g., [31]) and computational complexity theory (e.g., [15]). For simplicity, we set our alphabet Σ to be $\{0,1\}$. This restriction does not affect the results of this paper. The reader who is already familiar with computational complexity theory and recursion theory may skip the most of this basic section.

Numbers and Strings. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{N}$ (or ω) be the set of all natural numbers (i.e., nonnegative integers). Let $\mathbb{N}^+ = \mathbb{N}\setminus\{0\}$. For any finite set A, the notation $|A|$ denotes the *cardinality* of A. For any two integers m,n with $m<n$, we write $[m,n]_{\mathbb{Z}}$ to denote the set $\{m,m+1,m+2,\ldots,n\}$. All *logarithms* are taken to base 2 and a *polynomial* means a multivariate polynomial with integer coefficients. For convenience, we set $\log 0 = 0$. The *tower of* $2s$ is defined as follows. Let $2_0 = 1$ and let $2_{n+1} = 2^{2_n}$ for any integer $n\in\mathbb{N}$. We set $Tower = \{2_n \mid n\in\mathbb{N}\}$. Later, we will introduce its variant.

Throughout this paper, we set our alphabet Σ to be $\{0,1\}$ unless otherwise stated. A *string over* Σ is a finite sequence of symbols drawn from Σ. In particular, the *empty string* is denoted λ. The notation $|s|$ represents the *length* of a string s; that is, the number of symbols in s. The notation Σ^* denotes the collection of all strings over Σ. Similarly, for any fixed single symbol a, $\{a\}^*$ denotes the set $\{a^i \mid i\in\omega\}$, where a^i is a shorthand for the i repetitions of a. Moreover, for any subset A of Σ^*, the notation aA denotes the set $\{ax \mid x\in A\}$. We often identify a nonnegative integer n with the $(n+1)$th string in the standard lexicographic order on Σ^*: $\lambda < 0 < 1 < 00 < 01 < 11 < \cdots$ (sorted first by length and then lexicographically). For example, 0 denotes λ, 3 is 00, and 7 is 000. In this order, for any string x, x^- (x^+, resp.) represents the *predecessor* (*successor*, resp.) of x if one exists. Conventionally, we identify a set A with its *characteristic function*, which is defined by $A(x)=1$ if $x\in A$ and $A(x)=0$ otherwise.

A subset of Σ^* is called a *language* or simply a *set*. For such a set A, $\Sigma^*\setminus A$ is the *complement* of A and is denoted $\overline{A}$. For each number $n\in\mathbb{N}$, we write Σ^n, $\Sigma^{\leq n}$, and $\Sigma^{<n}$ to denote the collections of all strings of length n, length $\leq n$, and length $<n$, respectively. For any sets A and B, the notation $A\oplus B$ stands for the set $\{u0 \mid u\in A\}\cup\{v1 \mid v\in B\}$, the *disjoint union* (or *join*) of A and B. Let $A\triangle B = (A\setminus B)\cup(B\setminus A)$, and the notation $A =^* B$ means $A\triangle B$ is finite. We say that a complexity class $\mathcal{C}$ is *closed under finite variations* if, for every sets A and B, $A=^* B$ and $A\in\mathcal{C}$ imply $B\in\mathcal{C}$. The *census function* of A, denoted $cens_A$, is the function such that $cens_A(n) = |A\cap\Sigma^{\leq n}|$ for all $n\in\mathbb{N}$. A set S is *polynomially sparse* (*sparse*, for short) if there exists a polynomial p such that $cens_A(n)\leq p(n)$ for all $n\in\mathbb{N}$. A *tally* set is a subset of either $\{0\}^*$ or $\{1\}^*$.

Turing Machines, Complexity Classes, $INDEX_{(k)}$, and $\mathrm{NP}_{(k)}$. In this paper, we use a standard *multi-tape off-line Turing machine* (*TM*, in short) as a mathematical model of computation throughout this paper. A TM is used as an *acceptor* (which recognizes a language) or a *transducer* (which computes a function). We assume the reader's familiarity with various types of TMs (see [4, 15] for their definitions and properties). Of particular interests are deterministic, nondeterministic, and alternating TMs. In particular, we say that an alternating TM has k *alternations* if every computation path on every input has at most k alternating states of $\forall$ and $\exists$, starting with $\exists$-state. A Σ_k*-machine* refers to an alternating TM with k alternations starting with an $\exists$-state. In particular, a Σ_1-machine is a nondeterministic Turing machine. For convenience, we define a Σ_0-machine to be just a deterministic Turing machine. Similarly, a Δ_k-machine is a deterministic TM with access to an oracle which is recognized by a certain $O(n)$-time bounded Σ_{k-1}-machine. Moreover, we use an *unambiguous TM*, which always has at most one accepting path. For convenience, let $\mathcal{C}$ be a complexity class defined by a certain type of a TM. Generally, we will use the term "$\mathcal{C}$-machines" to mean a TM that is used to define a set in $\mathcal{C}$. For instance, an "NP-machine" refers to a polynomial-time nondeterministic TM, where NP is the class of all languages that can be recognized by polynomial-time nondeterministic Turing machines. Conventionally, we identify the acceptance and rejection of a TM with 1 and 0, respectively. For any acceptor M and any input x, we often write $M(x) = 0$ ($M(x) = 1$, resp.) to mean that M accepts (rejects, resp.) input x.

For any TM M and any function t from $\mathbb{N}$ to $\mathbb{N}$, the notation $M(x)_t$ denotes the outcome of $M(x)$ if M halts on input x in at most $t(|x|)$ steps and outputs either 0 (rejection) or 1 (acceptance); otherwise, $M(x)_t$ is undefined. For any set A, we use the notation $M(x) \simeq A(x)$ to mean that either M halts on input x and outputs $A(x)$ or M does not halt on input x. We say that M is *consistent with* A or shortly A-*consistent* (denoted $M \simeq A$) if $M(x) \simeq A(x)$ for all but finitely many strings x. Moreover, M is A-*consistent within time* $t(n)$ (denoted $M \simeq_t A$) if $M(x)_t \simeq A(x)$ for almost all x.

For any oracle TM M, any set A, and any string x, the notation $Q(M, A, x)$ denotes the set of all words queried by M on input x to oracle A and $L(M, A)$ denotes the language recognized by M with oracle A. The notation $\mathrm{DTIME}^A(t(n))$ denotes the collection of all languages $L(M, A)$ for a constant $c > 0$ and a certain deterministic oracle TM M running within time $ct(n) + c$, where n is the length of inputs. Similarly, we define $\mathrm{NTIME}^A(t(n))$ using nondeterministic TMs. In particular, when $A = \emptyset$, we omit the superscript A and simply write $\mathrm{DTIME}(t(n))$ and $\mathrm{NTIME}(t(n))$.

This paper focuses only on complexity classes whose computational resources are limited to polynomial time or exponential time. The basic complexity classes of our interests are given as follows. For an arbitrary oracle A, let P^A and NP^A be respectively $\bigcup_{k\in\mathbb{N}} \mathrm{DTIME}^A(n^k)$ and $\bigcup_{k\in\mathbb{N}} \mathrm{NTIME}^A(n^k)$. As before, when $A = \emptyset$, we omit superscript A. Similarly, let E and NE be respectively $\bigcup_{c\in\mathbb{N}} \mathrm{DTIME}(2^{cn})$ and $\bigcup_{c\in\mathbb{N}} \mathrm{NTIME}(2^{cn})$. The class UP is the subset of NP defined by polynomial-time unambiguous TMs. Later, we introduce a more general unambiguous complexity class $\mathrm{U}(\mathcal{C})$. The *relativized polynomial-time hierarchy* relative to oracle A consists of the following complexity classes: $\Delta_0^{\mathrm{P}}(A) = \Sigma_0^{\mathrm{P}}(A) = \Pi_0^{\mathrm{P}}(A) = \mathrm{P}^A$, $\Sigma_{k+1}^{\mathrm{P}}(A) = \mathrm{NP}^{\Sigma_k^{\mathrm{P}}(A)}$, and $\Pi_{k+1}^{\mathrm{P}}(A) = \text{co-}\Sigma_{k+1}^{\mathrm{P}}(A)$ for each $k \in \mathbb{N}$. The notation PH^A denotes $\bigcup_{k\in\mathbb{N}}(\Sigma_k^{\mathrm{P}}(A) \cup \Pi_k^{\mathrm{P}}(A))$. If $A = \emptyset$, then we obtain the (unrelativized) polynomial-time hierarchy $\{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}}, \Pi_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$ by dropping superscript A. In this paper, we define only the Δ-levels of the *exponential-time hierarchy* and the *subexponential-time hierarchy*: $\Delta_0^{\mathrm{EXP}} = \mathrm{EXP}$, $\Delta_{k+1}^{\mathrm{EXP}} = \mathrm{EXP}^{\Sigma_k^{\mathrm{P}}}$, $\mathrm{SUB}\Delta_0^{\mathrm{EXP}} = \mathrm{SUBEXP}$, and $\mathrm{SUB}\Delta_{k+1}^{\mathrm{EXP}} = \mathrm{SUBEXP}^{\Sigma_k^{\mathrm{P}}}$ for every $k \in \mathbb{N}$, where EXP^A and SUBEXP^A denote respectively $\bigcup_{k\in\mathbb{N}} \mathrm{DTIME}^A(2^{n^k})$ and $\bigcap_{\epsilon>0} \mathrm{DTIME}^A(2^{n^\epsilon})$ for any oracle A.

We assume a standard effective enumeration $\{\varphi_s\}_{s\in\Sigma^*}$ of all nondeterministic TMs φ_s's (with repetitions). Each index s of such a machine φ_s is conventionally called the *Gödel number* of the machine φ_s. For each index s, define the set $W_s = \{x \mid \varphi_s(x){\downarrow}= 1\}$, where "$\varphi_s(x){\downarrow}$" means that φ_s eventually halts on input x. Fix $k \in \mathbb{N}$. Let $\mathrm{NP}_{(k)}$ be the collection of all sets W_s such that, for any string $x \in W_s$, the running time of φ_s on input x is at most $|s| \cdot |x|^k + |s|$. Note that $\mathrm{NP} = \bigcup_{k\in\mathbb{N}} \mathrm{NP}_{(k)}$. We set $INDEX_{(k)} = \{s \mid W_s \in \mathrm{NP}_{(k)}\}$. For any string $x \in \Sigma^*$ and any set $A \subseteq \Sigma^*$, let $C^A(x)$ denote the *relativized Kolmogorov complexity of* x *relative to* A; that is, the minimal length of indices e such that $M_e^A(\lambda) = x$, where M_e^A is the eth deterministic TM (without any specific time bound).

We often express a finite sequence of strings by a single string. Let $\Sigma^{<\omega}$ denote the set of all finite sequences of strings. We assume a bijection $\langle\ \rangle$ from $\Sigma^{<\omega}$ to Σ^* that is polynomial-time computable and polynomial-time invertible. For such an encoded sequence $s = \langle x_1, x_2, \ldots, x_k\rangle$, $set(s)$ denotes the corresponding set $\{x_1, x_2, \ldots, x_k\}$.

Partial Functions, $\mathrm{F}\Delta_k^{\mathrm{P}}$, $\Sigma_k^{\mathrm{P}}\mathrm{SV}$, and $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$. In this paper, we mainly use "partial" functions and all functions are assumed to be single-valued although, in other literature, more general "multi-valued" functions are discussed. Note that total functions are also partial functions. For any partial function f, the notations $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ denote the *domain* of f and the *range* of f, respectively. Let f be any partial function from Σ^* to Σ^*. We say that f is *lexicographically increasing* if, for any string $x \in \mathrm{dom}(f)$, $f(x)$ is greater than x in the standard lexicographical order on Σ^*. Moreover, f is *polynomially bounded* if there exists a polynomial p such that $|f(x)| \le p(|x|)$ for any $x \in \mathrm{dom}(f)$. By contrast, f is called *polynomially honest* (*honest*, in short) if there exists a polynomial p such that $|x| \leq p(|f(x)|)$ for any $x \in \mathrm{dom}(f)$. Whenever it is clear from the context that $f(x)$ is of the form $\langle y_1, \ldots, y_k \rangle$ for every $x \in \mathrm{dom}(f)$ (where k may depend on x), we say that f is *componentiwise honest* if there exists a polynomial p such that $|x| \leq p(|y_j|)$ for any string $x \in \mathrm{dom}(f)$ and any number $j \in [1,k]_{\mathbb{Z}}$ with $k \geq 1$.

To avoid notational mess, we define standard function classes as collections of *partial* functions and, whenever we need total functions, we explicitly indicate the *totality* of functions but use the same notation. Let FP be the set of all partial functions computable deterministically in polynomial time. Now, fix $k \in \mathbb{N}^+$. For convenience, we use the notation $\mathrm{F}\Delta_k^{\mathrm{P}}$ for the collection of all sigle-valued partial functions f such that there exist a set $B \in \Sigma_{k-1}^{\mathrm{P}}$ and a polynomial-time oracle TM M satisfying the following condition: for every x, if $x \in \mathrm{dom}(f)$ then $M^B(x)$ halts with an accepting state and outputs $f(x)$ and, otherwise, $M^B(x)$ halts in a rejecting state (in this case, $f(x)$ is *undefined*). Clearly, $\mathrm{F}\Delta_1^{\mathrm{P}}$ coincides with FP. A single-valued partial function f is in NPSV if there is a polynomial-time nondeterministic TM M such that if $x \notin \mathrm{dom}(f)$, then $M(x)$ terminates with all rejecting configurations, and otherwise, $M(x)$ terminates with at least one accepting configuration and $M(x)$ outputs $f(x)$ along *all* accepting computation paths. The suffix "SV" in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ stands for "single-valued." We expand NPSV to $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ in a machine-independent way as follows. First, set $\Sigma_0^{\mathrm{P}}\mathrm{SV} = \mathrm{FP}$ for convenience. For any partial function f from Σ^* to Σ^*, the *graph* of f is the set $Graph(f) = \{\langle x, f(x) \rangle \mid x \in \mathrm{dom}(f)\}$. For each number $k \in \mathbb{N}^+$, let $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ denote the class of all single-valued partial functions f such that f is polynomially bounded and $Graph(f)$ is in Σ_k^{P}. It is not difficult to prove that $\Sigma_1^{\mathrm{P}}\mathrm{SV}$ indeed coincides with NPSV. To emphasize the total functions, we use the notation $\Sigma_k^{\mathrm{P}}\mathrm{SV}_t$ to denote the collection of all *total* functions in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$.

Let $\mathcal{C}$ be any complexity class of languages. Using the graphs of functions, we define a general *unambiguous complexity class* $\mathrm{U}\mathcal{C}$. A set A is in $\mathrm{U}(\mathcal{C})$ (or simply $\mathrm{U}\mathcal{C}$) if there exists a single-valued partial function f such that (i) f is polynomially bounded, (ii) $Graph(f) \in \mathcal{C}$, and (iii) $A = \mathrm{dom}(f)$ [47]. In particular, we obtain $\mathrm{U}\Delta_k^{\mathrm{P}}$ and $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$ for each $k \in \mathbb{N}$. Notice that $\mathrm{U}\Delta_1^{\mathrm{P}}$ coincides with UP.

Reductions and Quasireductions. Fix $k \in \mathbb{N}$ and let A and B be any subsets of Σ^*. A partial function f from Σ^* to Σ^* is called a *Σ_k^{P}-m-quasireduction* (*Δ_k^{P}-m-quasireduction*, resp.) *from A to B* if (i) f is in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ ($\mathrm{F}\Delta_k^{\mathrm{P}}$, resp.), (ii) $\mathrm{dom}(f)$ is infinite, and (iii) for any string $x \in \mathrm{dom}(f)$, $x \in A \iff f(x) \in B$. If in addition f is *total*, then f is called a *Σ_k^{P}-m-reduction* (*Δ_k^{P}-m-reduction*, resp.) *from A to B* and we say that A is Σ_k^{P}-m-reducible (Δ_k^{P}-m-reducible, resp.) to B. If A is Δ_k^{P}-m-reducible to B via a one-to-one reduction f, we say that A is *Δ_k^{P}-1-reducible* to B via f. In contrast, A is *Δ_k^{P}-T-reducible* to B if there exists an oracle Δ_k^{P}-machine M which recognizes A with access to B as an oracle. Moreover, we define bounded reducibilities. A set A is called *Δ_k^{P}-tt-reducible* to B via (ν, f, α) if (i) ν is a total $\mathrm{F}\Delta_k^{\mathrm{P}}$-function mapping from Σ^* to $\{0\}^*$ (where the number $|\nu(x)|$ is called *norm* of the reduction at x), (ii) f is a total $\mathrm{F}\Delta_k^{\mathrm{P}}$-function from Σ^* to Σ^* such that $f(x) = \langle y_1, y_2, \ldots, y_k \rangle$ for certain strings $y_1, y_2, \ldots, y_k$, where $k = |\nu(x)|$, for every x, (iii) α is a total $\mathrm{F}\Delta_k^{\mathrm{P}}$-function from $\Sigma^* \times \Sigma^*$ to $\{0,1\}$ such that, for every x, $x \in A$ if and only if $\alpha(x, B(f(x))) = 1$, where $B(f(x))$ is an abbreviation of the string $B(y_1)B(y_2)\cdots B(y_k)$ for $k = |\nu(x)|$ and $f(x) = \langle y_1, y_2, \ldots, y_k \rangle$. For any constant $i \in \mathbb{N}^+$, A is *Δ_k^{P}-itt-reducible* to B via (f, α) if A is Δ_k^{P}-tt-reducible to B via (ν, f, α), where $\nu(x) = 0^i$ for all x; in other words, the norm of this reduction is always i at any x. A set A is *Δ_k^{P}-btt-reducible* to B if there exists a number $i \in \mathbb{N}^+$ such that A is Δ_k^{P}-$i$tt-reducible to B. A set A is *Δ_k^{P}-d-reducible to B* via f if (i) f is a total $\mathrm{F}\Delta_k^{\mathrm{P}}$-function and (ii) for every x, $x \in A \iff B \cap set(f(x)) \neq \emptyset$. In contrast, A is *Δ_k^{P}-c-reducible to B* via f if $\overline{A}$ is Δ_k^{P}-d-reducible to $\overline{B}$ via f.

Next, we define honest reductions. For any $r \in \{m, T\}$ ($r \in \{d, c\}$, resp.), A is *h-Δ_k^{P}-r-reducible to B* via reduction f if A is Δ_k^{P}-r-reducible to B via f such that f is honest (componentwise honest, resp.). In case where $r \in \{btt, tt\}$, A is said to be *h-Δ_k^{P}-r-reducible to B* via a reduction triplet (ν, f, α) if A is Δ_k^{P}-r-reducible to B via (ν, f, α) with an extra condition that f is componentwise honest. For Turing reductions, we say that A is *h-Δ_k^{P}-T-reducible* to B via a reduction machine M if A is Δ_k^{P}-T-reducible to B via M such that M makes only honest queries; more precisely, (i) M runs in polynomial time, (ii) for every x, $x \in A \iff M^B(x) = 1$, and (iii) there exists a polynomial p such that $p(|w|) \geq |x|$ for every x and every $w \in Q(M, B, x)$. In this case, we say that the reduction machine M is *honest*.

When $\mathcal{C} \in \{\Delta_k^{\mathrm{P}} \mid k \in \mathbb{N}^+\}$ and $r \in \{1, m, d, c, ktt, btt, tt, T\}$, notationally we write $A \leq_r^{\mathcal{C}} B$ ($A \leq_r^{\text{h-}\mathcal{C}} B$, resp.) to mean that A is $\mathcal{C}$-r-reducible (honestly $\mathcal{C}$-r-reducible, resp.) to B. A set S is called *$\mathcal{C}$-r-hard for $\mathcal{D}$* if every set in $\mathcal{D}$ is $\mathcal{C}$-r-reducible to S. Moreover, a set S is called *$\mathcal{C}$-r-complete for $\mathcal{D}$* if S is in $\mathcal{D}$ and S is $\mathcal{C}$-r-hard for $\mathcal{D}$. For the honest reductions, the notions of *h-$\mathcal{C}$-r-hardness* and *h-$\mathcal{C}$-r-completeness* are defined in a similar fashion.

Finally, we say that a complexity class $\mathcal{C}$ is *closed downward under a reduction $\leq_r$ on infinite sets* if, for any pair of infinite sets A and B, $A \leq_r B$ and $B \in \mathcal{C}$ imply $A \in \mathcal{C}$.

Forcing, Generic Sets, and Random Oracles. A *forcing condition* σ is a partial function from Σ^* to $\{0,1\}$ such that $\mathrm{dom}(\sigma)$ is finite. Such a function σ is identified with the string $\langle\langle v_0, v_1, \ldots, v_{n-1}\rangle, \langle w_0, w_1, \ldots, w_{n-1}\rangle\rangle$, where $\mathrm{dom}(\sigma) = \{v_0, v_1, \ldots, v_{n-1}\}$ with $v_0 < v_1 < \cdots < v_{n-1}$ and $\sigma(v_i) = w_i$ for all $i < n$. By this identification, a set of forcing conditions can be treated as a set of strings. For two forcing conditions σ and τ, we say that τ *extends* σ (denoted $\sigma \subseteq \tau$) if $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\tau)$ and $\sigma(x) = \tau(x)$ for all $x \in \mathrm{dom}(\sigma)$. This notion of "extension" can be generalized even when τ is a total function from Σ^* to $\{0,1\}$. For any subset A of Σ^*, write $\sigma \subseteq A$ to mean that A (viewed as the characteristic function) extends σ. A set S of forcing conditions is *dense along* A if every forcing condition $\sigma \subseteq A$ has an extension τ in S. In particular, we say that S is *dense* if S is dense along every subset A of Σ^*. A set A *meets* a set S of forcing conditions if A extends a certain forcing condition in S. For any complexity class $\mathcal{C}$, a set A is called $\mathcal{C}$*-generic* if A meets every set in $\mathcal{C}$ that is dense along A. In particular, when $\mathcal{C}$ is the class of all arithmetical sets, we use the conventional term "Cohen-Feferman generic" instead. When a generic set is specifically used as an oracle, it is called a *generic oracle*.

For any set $A \subseteq \{0,1\}^*$, we identify A with its characteristic sequence $\alpha_A = A(\lambda)A(0)A(1)A(00)\cdots$; namely, the ith bit of α_A is the value of A on the lexicographically ith string on Σ^*. Such an infinite sequence corresponds to a real number in the unit interval $[0,1]$. Let r_A be the real number whose binary expansion is of the form $0.\alpha_A$. Notice that some real numbers have two equivalent binary expansions (say, 0.01 and $0.00\dot{1}$); however, since the set $\{0.s\dot{1} \mid s \in \Sigma^*\}$ has the Lebesgue measure 0 in $[0,1]$, we can ignore this duality problem for our purpose. In general, let $\varphi(X)$ be any mathematical property with a variable X running over all subsets of $\{0,1\}^*$. We say that $\varphi(X)$ *holds (with probability* 1*) relative to a random oracle* X if the set $\{r_A \mid A \subseteq \{0,1\}^* \wedge \varphi(A) \text{ holds}\}$ has Lebesgue measure 1 in the interval $[0,1]$.

3 Immunity and Simplicity

The original notions of immunity and simplicity date back to the mid 1940s. Post [35] first constructed a simple set for the class of recursively enumerable sets. The new breed of resource-bounded immunity and simplicity waited to be introduced until mid 1970s by an early work of Flajolet and Steyaert [16]. In their seminal paper, Flajolet and Steyaert constructed various recursive sets that, for instance, have no infinite DTIME($t(n)$)-subsets under the term "DTIME($t(n)$)-immune sets." Later, Ko and Moore [27] studied the polynomial-time bounded immunity, which is now preferably called P-immunity. Subsequently, Balcázar and Schöning [5], motivated by Berman's [8] work, considered P-bi-immune sets, which are P-immune sets whose complements are also P-immune. Homer and Maass [24] extensively discussed the cousin of P-immune sets, known as NP-simple sets. The importance of these notions was widely recognized through the 1980s. Since these notions can be easily expanded to any complexity class $\mathcal{C}$, we begin with an introduction of the general notions of $\mathcal{C}$-immune sets, $\mathcal{C}$-bi-immune sets, and $\mathcal{C}$-simple sets. They are further expanded in various manners in later sections.

Definition 3.1 Let $\mathcal{C}$ be any complexity class.

1. A set S is $\mathcal{C}$*-immune* if S is infinite and there is no infinite subset of S in $\mathcal{C}$.
2. A set S is $\mathcal{C}$*-bi-immune* if S and $\overline{S}$ are both $\mathcal{C}$-immune.
3. A set S is $\mathcal{C}$*-simple* if S belongs to $\mathcal{C}$ and $\overline{S}$ is $\mathcal{C}$-immune.

We sometimes use the term "$\mathcal{C}$-coimmune" to mean that the complement of a given set is $\mathcal{C}$-immune. Using this term, a $\mathcal{C}$-simple set is a $\mathcal{C}$-set that is infinite $\mathcal{C}$-coimmune. Clearly, the intersection of any two $\mathcal{C}$-immune sets is either $\mathcal{C}$-immune or finite. Note that the existence of a $\mathcal{C}$-simple set immediately implies $\mathcal{C} \neq \text{co-}\mathcal{C}$; however, the separation $\mathcal{C} \neq \text{co-}\mathcal{C}$ does not necessarily guarantee the existence of $\mathcal{C}$-simple sets. In this paper, we focus only on the complexity classes lying in the polynomial-time hierarchy.

It is well-known that P-immune sets exist even in the class E. In particular, Ko and Moore [27] constructed a P-immune set that is also P-tt-complete for E. Note that no h-P-m-complete set for NP can be P-immune since the image of a P-immune set by a polynomial-time computable reduction is either finite or P-immune. Using a relativization technique, Bennett and Gill [6] demonstrated that a P-immune set exists in NP relative to a random oracle with probability 1. A recursive oracle relative to which NP contains P-immune sets was constructed later by Homer and Maass [24]. Torenvliet and van Emde Boas [42] strengthened their result by demonstrating a relativized world where NP has a P-immune set which is also NP-simple. The difference between sparse immune sets and tally immune sets was discussed by Hemaspaandra and Jha [21], who constructed an oracle relative to which NP has a P-immune sparse set but no P-immune tally sets. By a different approach, Blum and Impagliazzo [7] proved the existence of P-immune sets in NP relative to a generic oracle. As for sets in the Boolean hierarchy over NP, Cai et al. [13] proved that they are neither NP-bi-immune, co-NP-bi-immune, nor BH-immune, where BH is the union of all classes in the Boolean hierarchy within NP.

The notion of $\mathcal{C}$-immunity is related to various other notions, which include complexity cores [30] and instance complexity [33]. Balcázar and Schöning [5] showed that a set S is P-bi-immune exactly when Σ^* is a complexity core for S. A set A is a Σ_k^{P}*-hardcore for* X if, for every Σ_k-machine[†] M recognizing A and every polynomial p, there exists a finite subset S of X such that $M(x)_p$ does not exist for all $x \in X - S$. Similarly, we can define the notion of Δ_k^{P}-hardcore for X using a Δ_k-machines instead of Σ_k-machines. Furthermore, as Orponen et al. [33] showed, instance complexity also characterizes P-immunity. Let $\mathcal{C}$ be any complexity class in the polynomial-time hierarchy. The *t-time bounded $\mathcal{C}$-instance complexity of x with respect to A*, denoted $\mathcal{C}$-$ic^t(x : A)$, is defined to be the minimal length of index e such that the $\mathcal{C}$-machine indexed e, which is A-consistent within time $t(n)$, outputs $A(x)$ on input x in time $t(n)$.

In the following lemma, we give the characterization of $\mathcal{C}$-immunity by (a generalization of) the above notions. The lemma can be easily obtained by modifying the proofs in [5, 33].

Lemma 3.2 *Let $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$ and let S be any recursive subset of Σ^*. The following three statements are equivalent.*

1. *S is $\mathcal{C}$-immune.*
2. *S is a $\mathcal{C}$-hardcore for S.*
3. *For any polynomial p and any constant $c > 0$, the set $\{x \in S \mid \mathcal{C}\text{-}ic^p(x : S) \leq c\}$ is finite.*

Note that recursiveness of S in Lemma 3.2 cannot be removed.

Proof of Lemma 3.2. We show only the case where $\mathcal{C} = \Sigma_k^{\mathrm{P}}$ for a certain integer $k \geq 1$. Let S be any recursive subset of Σ^*. Assume that S is recognized by a deterministic TM N.

3 implies 2) Assume that there exist a Σ_k-machine indexed e and a polynomial p such that M_e recognizes S and $M_e(x)_p$ exists for infinitely-many x in S. Thus, Σ_k^{P}-$ic^p(x : S) \leq e$ for infinitely-many x in S.

2 implies 1) In this proof, we need the computability of S. Assume that S is not Σ_k^{P}-immune; namely, there exist a polynomial p and a Σ_k-machine M running in time $p(n)$ that recognizes a certain infinite subset of S. We define another machine M' as follows:

> On input x, run M on the same input x. Whenever M halts in an accepting state, then accept the input and halt. Otherwise, run N on input x and output whatever N does.

Clearly, M' is a Σ_k-machine and recognizes S. Note that, for infinitely-many x in S, the running time of M' on each input x does not exceed $p(|x|)$. Therefore, S cannot be a Σ_k^{P}-hardcore.

1 implies 3) Assume that there exist an infinite subset A of S, a polynomial p, and an integer $c > 0$ such that Σ_k^{P}-$ic^p(x : S) \leq c$ for all strings x in A. Thus, for every x in A, there is a Σ_k-machine indexed e with $0 \leq |e| \leq c$, say M_e, such that M_e is S-consistent within time p and $M_e(x) = S(x)$. For each e, define $L_e = \{x \in A \mid M_e \simeq_p S \wedge M_e(x)_p = S(x)\}$. Note that A equals $\bigcup_{e:1 \leq |e| \leq c} L_e$. Since A is infinite, L_e is also infinite for a certain index e with $1 \leq |e| \leq c$. Take such an index e and define $L = \{x \mid M_e(x)_p = 1\}$. Clearly, $L_e \subseteq L \subseteq S$ because $M_e \simeq_p S$. Since L_e is infinite, L is also infinite. Moreover, L belongs to Σ_k^{P} since M_e is a Σ_k-machine. Hence, S cannot be Σ_k^{P}-immune. □

Balcázar and Schöning [5] built a bridge between P-bi-immune sets and finite-to-one reductions, which led them further to introduce the notion of strongly P-bi-immune sets. Expanding their argument, we give below a characterization of $\mathcal{C}$-bi-immunity as well as $\mathcal{C}$-immunity. For any partial function f and any element b, let $f^{-1}(b) = \{x \in \mathrm{dom}(f) \mid f(x) = b\}$. Note that $f^{-1}(b) = \varnothing$ if $b \notin \mathrm{ran}(f)$.

Lemma 3.3 *Let $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$ and $S \subseteq \Sigma^*$.*

1. *S is $\mathcal{C}$-immune if and only if (i) S is infinite and (ii) for every set B, every $\mathcal{C}$-m-quasireduction f from S to B, and every u in B, the inverse image $f^{-1}(u)$ is a finite set.*

[†] Book and Du [10] gave a machine-independent definition of hardcores.

2. *S is $\mathcal{C}$-bi-immune if and only if (i) S is infinite and (ii) for every set B, every $\mathcal{C}$-m-quasireduction f from S to B, and every u, the inverse image $f^{-1}(u)$ is finite.*

Proof. We show only the case where $\mathcal{C} = \Sigma_k^{\rm P}$ for a certain $k \in \mathbb{N}^+$. Since the second claim immediately follows from the first one, we hereafter give the proof of the first claim.

Assume that S is not $\Sigma_k^{\rm P}$-immune. There exists an infinite $\Sigma_k^{\rm P}$-subset A of S. Take a fixed element a_0 in A. Define $f(x) = a_0$ if $x \in A$, and $f(x)$ is undefined otherwise. It is obvious that $f \in \Sigma_k^{\rm P}{\rm SV}$. Conversely, assume that there exists a set B and a $\Sigma_k^{\rm P}$-m-quasireduction f from S to B such that $f^{-1}(u)$ is infinite for a certain string $u \in B$. Clearly, $f^{-1}(u) \subseteq S$ and $f^{-1}(u) \in \Sigma_k^{\rm P}$ since ${\rm dom}(f) \in \Sigma_k^{\rm P}$ and $Graph(f) \in \Sigma_k^{\rm P}$. Therefore, S is not $\Sigma_k^{\rm P}$-immune. □

The characterizations of $\mathcal{C}$-immunity given in Lemmas 3.2 and 3.3 demonstrate a significant role of the immunity in complexity theory. Balcázar and Schöning [5] used their characterization to introduce a stronger notion of P-bi-immunity: strong P-bi-immunity. A more general notion, called strongly $\mathcal{C}$-immunity, will be discussed in Section 4.

Whether an NP-simple set exists is one of the long-standing open problems because such a set separates NP from co-NP. Nonetheless, NP-simple sets are known to exist in various relativized worlds. In early 1980s, Homer and Maass [24] and Balcázar [3] constructed relativized worlds where an NP-simple set exists. Later, Vereshchagin [44] proved that, relative to a random oracle, an NP-simple set exists with probability 1. From Theorem 4.9 in Section 4, for instance, it immediately follows that an NP-simple set exists relative to a Cohen-Feferman generic oracle. Torenvliet [41] built an oracle relative to which a $\Sigma_2^{\rm P}$-simple set exists. For a much higher level k of the polynomial-time hierarchy, Bruschi [11] constructed an oracle relative to which $\Sigma_k^{\rm P}$-simple sets exist using the size lower bounds of certain non-uniform constant-depth circuits. In addition, sets being both simple and immune were studied in, e.g., [12, 42].

In the rest of this section, we focus on closure properties of the class of all $\Sigma_k^{\rm P}$-immune sets because no such closure property has been systematically studied in the literature. We claim that this class is closed downward under h-$\Delta_k^{\rm P}$-c-reductions on infinite sets; however, we cannot replace this conjunctive reducibility by disjunctive reducibility.

Theorem 3.4 *Let $k \in \mathbb{N}^+$.*

1. *The class of all $\Sigma_k^{\rm P}$-immune sets is closed downward under h-$\Delta_k^{\rm P}$-c-reductions on infinite sets.*
2. *The class of all* NP*-immune sets is not closed under h-*P*-d-reductions or h-*P*-2tt-reductions on infinite sets.*

Proof. 1) Let A and B be any infinite sets and let f be any h-$\Delta_k^{\rm P}$-c-reduction f from A to B. Assume that B is $\Sigma_k^{\rm P}$-immune. We want to show that A is also $\Sigma_k^{\rm P}$-immune. Assume to the contrary that A contains an infinite $\Sigma_k^{\rm P}$-subset C. There exists a polynomial p such that $|x| \leq p(|y|)$ for all x and for all $y \in set(f(x))$. Note that the set $\bigcup_{x\in C} set(f(x))$ is infinite since f is componentwise honest. Let $D = \{y \mid \exists x \in C[|x| \leq p(|y|) \wedge y \in set(f(x))]\}$. Note that D belongs to $\Sigma_k^{\rm P}$ since f is in ${\rm F}\Delta_k^{\rm P}$ and C is in $\Sigma_k^{\rm P}$. Moreover, D is an infinite subset of B. This contradicts our assumption. Hence, A is $\Sigma_k^{\rm P}$-immune.

2) Define $A = \{0\}^*$ and let f be the function defined as follows: $f(x) = \langle 0x, 1x\rangle$ for every string x. We construct an NP-immune set B to which f h-P-d-reduces A together with an auxiliary set C. Let $\{N_j\}_{j\in\omega}$ be an effective enumeration of nondeterministic oracle TMs whose running time is bounded above by polynomials independent of the choice of oracles.

The desired sets $B = \bigcup_{m\in\mathbb{N}} B_m$ and $C = \bigcup_{m\in\mathbb{N}} C_m$ are constructed by stages.

> Stage 0: Let $B_0 = \emptyset$ and $C_0 = \emptyset$.
>
> Stage $m+1$: At this stage, we wish to define B_{m+1} and C_{m+1}. Find the minimal natural number j such that the following three conditions hold: (i) $L(N_j) \cap \Sigma^{\leq m} \subseteq B_m$, (ii) either $N_j(0^{m+1}) = 1$ or $N_j(10^m) = 1$, and (iii) $j \notin C_m$. For this j, define C_{m+1} as $C_m \cup \{j\}$. The set B_{m+1} is defined by the following three cases. In case where $N_j(0^{m+1}) = N_j(10^m) = 1$, let B_{m+1} be $B_m \cup \{0^{m+1}\}$. If $N_j(0^{m+1}) = 1$ and $N_j(10^m) = 0$, then let $B_{m+1} = B_m \cup \{10^m\}$. If $N_j(0^{m+1}) = 0$ and $N_j(10^m) = 1$, then let $B_{m+1} = B_m \cup \{0^{m+1}\}$. Note that $B_{m+1} \subseteq \Sigma^{\leq m+1}$.

By the above construction, clearly B is infinite and, for every string x, x is of the form 0^k for some natural number k if and only if at least one of $0x$ and $1x$ belongs to B. Hence, f h-P-d-reduces A to B. Note that this reduction is also an h-P-2tt-reduction.

To conclude the proof, it suffices to show that B has no infinite NP-subset. To lead to a contradiction, we assume that $L(N_j)$ is an infinite subset of B for a certain index j. In case where $j \in C$, j must be in C_{m+1} at a certain stage m. If we take minimal such m, then j is used at stage $m+1$, and hence $L(N_j)$ is not a subset of B, a contradiction. Therefore, $j \notin C$. Since $L(N_j)$ is infinite, there is a natural number m of the following property: (i) N_j accepts a string of length $m+1$, and (ii) every index $i < j$ such that $i \in C$ is entered in C before stage $m+1$. Since $L(N_j)$ is a subset of B, by (i), N_j accepts at least one of 0^{m+1} and 10^m. By (ii), j is used at stage $m+1$. This implies that $L(N_j)$ is not a subset of B, a contradiction. Therefore, B has no infinite NP-subset. □

How complex are Σ_k^{P}-simple sets? Intuitively, $\mathcal{C}$-simple sets are "thin" and thus cannot be the "complete" for the class $\mathcal{C}$. As an immediate consequence of Theorem 3.4(1), we obtain the following corollary.

Corollary 3.5 *Let $k \in \mathbb{N}^+$. No Σ_k^{P}-simple set is h-Δ_k^{P}-d-complete for Σ_k^{P}.*

Proof. Assume that B is Σ_k^{P}-simple and h-Δ_k^{P}-d-complete for Σ_k^{P}. Notice that $\overline{B}$ is Σ_k^{P}-immune. Fix any infinite set A in P. Note that $\overline{A}$ is h-Δ_k^{P}-c-reducible to $\overline{B}$ because of the completeness of B. By Proposition 3.4(1), $\overline{A}$ is Σ_k^{P}-immune. By the immunity condition, $\overline{A}$ is not in Σ_k^{P}, a contradiction. Thus, B is not Σ_k^{P}-simple. □

Recently, Agrawal (cited in [38]) showed, using the NP-levelability of $\overline{\mathrm{SAT}}$ (assuming SAT $\notin$ P), that no NP-simple set is h-P-btt-complete for NP, where SAT is the set of all satisfiable Boolean formulas. His argument will be generalized in Section 6.

4 Strong Immunity and Strong Simplicity

Following the introduction of P-bi-immunity, Balcázar and Schöning [5] stepped forward to introduce the notion of strongly P-bi-immunity, which comes from the quasireducibility-characterization of P-bi-immunity given in Lemma 3.3(2). While P-bi-immunity requires its quasireductions to be finite-to-one, strong P-bi-immunity requires the quasireductions to be almost one-to-one, where a quasireduction f is called *almost one-to-one on* a set S if the *collision set* $\{(x,y) \in (\mathrm{dom}(f) \cap S)^2 \mid x < y \wedge f(x) = f(y)\}$ is finite. Such strongly P-bi-immune sets are known to exist even in the class E [5]. Resource-bounded genericity also implies strong immunity as shown in Proposition 4.3.

Generalizing the notion of P-bi-immunity, we can introduce strong $\mathcal{C}$-bi-immunity for any complexity class $\mathcal{C}$ lying in the polynomial-time hierarchy. Moreover, we newly introduce the notions of strong $\mathcal{C}$-immunity and strong $\mathcal{C}$-simplicity. Recall that Σ_k^{P}-m-quasireductions are all single-valued functions in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ for each $k \in \mathbb{N}^+$.

Definition 4.1 Let $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$.

1. A set S is *strongly $\mathcal{C}$-immune* if (i) S is infinite and (ii) for every set B and every $\mathcal{C}$-m-quasireduction f from S to B, f is almost one-to-one on S.
2. A set S is *strongly $\mathcal{C}$-bi-immune* if S and $\overline{S}$ are both strongly $\mathcal{C}$-immune.
3. A set S is *strongly $\mathcal{C}$-simple* if S is in $\mathcal{C}$ and $\overline{S}$ is strongly $\mathcal{C}$-immune.

In other words, a set S is strongly $\mathcal{C}$-immune if and only if S is infinite and the set $\{x \in \mathrm{dom}(f) \mid f(x) = f(z)\}$ is a singleton for every set B, every $\mathcal{C}$-m-quasireduction f from S to B, and for all but finitely-many strings $z \in \mathrm{dom}(f) \cap S$. In particular, when $\mathcal{C} = \mathrm{P}$, Definition 4.1(2) coincides with the notion of P-bi-immune sets given in [5].

It follows from Lemma 3.3(1) that strong $\mathcal{C}$-immunity and strong $\mathcal{C}$-simplicity are a restriction of $\mathcal{C}$-immunity and $\mathcal{C}$-simplicity, respectively. We state this fact as the following lemma.

Lemma 4.2 *For any complexity class $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$, every strongly $\mathcal{C}$-immune set is $\mathcal{C}$-immune and every strongly $\mathcal{C}$-simple set is $\mathcal{C}$-simple.*

A major difference between $\mathcal{C}$-immunity and strong $\mathcal{C}$-immunity is shown in the following example. For any NP-immune set A, the disjoint union $A \oplus A$ is also NP-immune; on the contrary, it is not strongly NP-immune because $A \oplus A$ can be reduced to A by the following almost two-to-one function f defined by $f(\lambda) = \lambda$ and $f(xb) = x$ for any $b \in \{0,1\}$, where λ is the empty string. Therefore, the class of all strongly NP-immune sets is not closed under the disjoint-union operator. Historically, using the structural difference between these two notions, Balcázar and Schöning [5] constructed a set in E which is P-bi-immune but not strongly P-bi-immune.

What kind of sets become strongly $\mathcal{C}$-immune? Using strongly self-bi-immune sets, Balcázar and Mayordomo characterized the resource-bounded generic sets of Ambos-Spies et al. [1, 2]. Along a similar line of the study of genericity, we prove the following proposition.

Proposition 4.3 *Let $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$. Any $\mathcal{C}$-generic set is strongly $\mathcal{C}$-bi-immune.*

Proof. We show only the case where $\mathcal{C} = \Delta_k^{\mathrm{P}}$ for a certain number $k \in \mathbb{N}^+$. Assume that A is Δ_k^{P}-generic but not strongly Δ_k^{P}-immune. Since A is not strongly Δ_k^{P}-immune, there exist a set B and a Δ_k^{P}-m-quasireduction f from A to B such that the collision set $D = \{(x,y) \in \mathrm{dom}(f)^2 \mid x < y \wedge f(x) = f(y)\}$ is infinite. We denote by S the collection of all forcing conditions σ such that there exist at least two elements $x, y \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(f)$ satisfying that $\sigma(x) \neq \sigma(y)$ and $f(x) = f(y)$. Since $f \in \mathrm{F}\Delta_k^{\mathrm{P}}$, S belongs to Δ_k^{P}.

Next, we show that S is dense. Let σ be any forcing condition. Take a pair (x,y) of distinct strings from the difference $\mathrm{dom}(f) \setminus \mathrm{dom}(\sigma)$ that satisfy $f(x) = f(y)$. Such a pair clearly exists because $\mathrm{dom}(\sigma)$ is finite and D is infinite. We define τ as the unique forcing condition that satisfies the following: $\mathrm{dom}(\tau) = \mathrm{dom}(\sigma) \cup \{x, y\}$, $\sigma \subseteq \tau$, and $\tau(x) \neq \tau(y)$. Obviously, τ belongs to S. Hence, S is indeed dense. Since A is Δ_k^{P}-generic, A must meet S; namely, there exists a forcing condition ρ in S that is extended to A. It thus follows that $f(x) = f(y)$ and $A(x) \neq A(y)$ for a certain pair $(x,y) \in \mathrm{dom}(f)^2$. This contradicts our assumption that f is Δ_k^{P}-m-reduces A to B. Therefore, A is strongly Δ_k^{P}-immune. In a similar way, we can also prove that $\overline{A}$ is strongly Δ_k^{P}-immune. □

Now, we consider a closure property of the class of all strongly Σ_k^{P}-immune sets. We prove that this class is closed under h-Δ_k^{P}-1-reductions. Nevertheless, we cannot replace h-Δ_k^{P}-1-reductions by h-Δ_k^{P}-m-reductions because the quasireductions that define strong immunity are almost one-to-one.

Proposition 4.4 *Let k be any number in $\mathbb{N}^+$.*

1. *The class of all strongly Σ_k^{P}-immune sets is closed downward under h-Δ_k^{P}-1-reductions on infinite sets.*
2. *The class of all strongly* NP*-immune sets is not closed downward under h-*P*-m-reductions on infinite sets.*

Proof. 2) This comes from the fact that the disjoint union $A \oplus A$ cannot be strongly NP-immune for any NP-immune set A.

1) Let A be any infinite set and assume that A is not strongly Σ_k^{P}-immune. Assume also that f is an h-Δ_k^{P}-1-reduction from A to a set B and p is a polynomial that witnesses the honesty of f. We wish to show that B is not strongly Σ_k^{P}-immune.

Since A is not strongly Σ_k^{P}-immune, there exists a set C and a Σ_k^{P}-m-quasireduction g from A to C such that, for infinitely-many strings u in $\mathrm{dom}(g) \cap A$, the set $\{x \in \mathrm{dom}(g) \mid g(x) = g(u)\}$ has at least two elements. For readability, write D for the domain of g. Hence, D is an infinite Σ_k^{P}-set. By the definition of g, it follows that, for any string $x \in D$, $x \in A \iff g(x) \in C$. Since f is honest, $f(D)$ is an infinite set. Moreover, $f(D)$ belongs to Σ_k^{P} since a string y belongs to $f(D)$ if and only if there exists an x such that $|x| \leq p(|y|)$, $x \in D$, and $f(x) = y$.

Next, define the partial function h from B to C as follows. Let the domain of h be exactly $f(D)$. For each $y \in \mathrm{dom}(h)$, define $h(y)$ to be $g(x)$, where x is the string in D satisfying that $|x| \leq p(|y|)$ and $f(x) = y$. Since f is one-to-one, such x uniquely exists. Clearly, h is a Σ_k^{P}-m-quasireduction from B to C. Notice that, for infinitely many strings v in $\mathrm{dom}(h) \cap B$, the set $\{y \in \mathrm{dom}(h) \mid h(y) = h(v)\}$ also has at least two elements. In other words, B is not strongly Σ_k^{P}-immune, as required. □

As an immediate consequence of Proposition 4.4(1), we can show that no strongly Σ_k^{P}-simple set can be h-Δ_k^{P}-1-complete for Σ_k^{P}, where $k \in \mathbb{N}^+$. The proof of this claim is similar to that of Corollary 3.5 and is left to the avid reader.

Corollary 4.5 *For each $k \in \mathbb{N}^+$, no strongly Σ_k^{P}-simple set is h-Δ_k^{P}-1-complete for Σ_k^{P}.*

We turn our interest to the relativization of strongly Σ_k^{P}-immune sets. Before giving our main result, we describe a useful lemma that connects Σ_k^{P}-immunity to strongly Σ_k^{P}-immunity using the new tower of 2s defined as follows. Let $\hat{2}_0 = 1$ and let $\hat{2}_n$ be the tower of $2n$ 2's, that is, $\hat{2}_{n+1} = 2^{2^{\hat{2}_n}}$ for each $n \in \mathbb{N}^+$. In other words, $\hat{2}_n = \log\log \hat{2}_{n+1}$. Let $\hat{T} = \{\hat{2}_n \mid n \in \mathbb{N}\}$.

Lemma 4.6 *Let* $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}^+\}$ *and let* A *be any set in* EXP. *If* A *is* $\mathcal{C}$*-immune and* $A \subseteq \{1^n \mid n \in \hat{T}\}$, *then* A *is also strongly* $\mathcal{C}$*-immune.*

Note that Lemma 4.6 relativizes.

Proof of Lemma 4.6. We show the lemma only for the case where $\mathcal{C} = \Sigma_k^{\mathrm{P}}$ for a certain number $k \in \mathbb{N}^+$. Let A be any subset of $\{1^n \mid n \in \hat{T}\}$ in EXP. Assume that A is not strongly Σ_k^{P}-immune; namely, there exist a Σ_k^{P}-m-quasireduction f from A to a certain set B such that f is not almost one-to-one on A. Since $A \in \mathrm{EXP}$, we can take a deterministic TM M that recognizes A in time at most 2^{n^c+c} for a certain constant $c > 0$, where n is the length of an input. Let d be the minimal positive integer satisfying $(\log\log n)^c \leq d \cdot \log n + d$ for any number $n \geq 0$. We want to show that A is not Σ_k^{P}-immune.

Fix an element b in B arbitrarily. We define the partial function g in the following fashion. On input x, if $x \notin \{1^n \mid \hat{T}\}$, then let $g(x) = f(x)$. Now, assume that $x = 1^{\hat{2}_i}$ for a certain number $i \in \mathbb{N}$. If $f(x) = f(1^{\hat{2}_j})$ and $M(1^{\hat{2}_j}) = 1$ for a certain natural number $j < i$, let $g(x) = b$ and otherwise, let $g(x) = f(x)$. Since $|1^{\hat{2}_j}| = \hat{2}_j \leq \log\log|x|$, the running time of M on input $1^{\hat{2}_j}$ is at most $2^{(\log\log|x|)^c+c}$, which is further bounded above by $2^{c+d}|x|^d$ for any nonempty string x. Hence, g is in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$. Consider the set $g^{-1}(b)$. Since f is not almost one-to-one, g maps infinitely-many strings in A into b. Thus, $g^{-1}(b)$ must be infinite. Obviously, g Σ_k^{P}-m-quasireduces A to B. By Lemma 3.3(1), A is not Σ_k^{P}-immune. □

We show in Proposition 4.7 a relativized world where strongly Σ_k^{P}-simple sets actually exist. Our proof of this proposition heavily relies on Bruschi's [11] construction of a recursive oracle relative to which a Σ_k^{P}-simple set exists.

Proposition 4.7 *For each* $k \in \mathbb{N}^+$, *there exists a strongly* $\Sigma_k^{\mathrm{P}}(A)$*-simple set relative to a certain recursive oracle* A.

Proof. Let $k \in \mathbb{N}^+$. For each oracle A, we define the oracle-dependent set $L_k^A = \{0^n \mid n \in \hat{T} \wedge \forall y_1 \in \Sigma^n \exists y_2 \in \Sigma^n \cdots Q_k y_k \in \Sigma^n [0^n y_1 y_2 \cdots y_k \notin A]\}$, where Q_k is $\exists$ if k is even and Q_k is $\forall$ otherwise. Obviously, L_k^A is in $\Pi_k^{\mathrm{P}}(A)$ for any oracle A. The key to our proof is the existence of a recursive oracle A that makes L_k^A $\Sigma_k^{\mathrm{P}}(A)$-immune. In his proof of [11, Theorem 5.4], Bruschi employed a well-studied circuit lower bound technique in the course of the construction of an immune set. We do not attempt to recreate his proof here; however, we note that his construction works on any sufficiently large string input, in particular, of the form 0^m. Therefore, we can build in a recursive fashion an oracle A for which L_k^A is $\Sigma_k^{\mathrm{P}}(A)$-immune. Since obviously L_k^A is in EXP^A, Lemma 4.6 ensures that L_k^A is strongly $\Sigma_k^{\mathrm{P}}(A)$-immune. □

Concerning a random oracle, Vereshchagin [44, 45] earlier demonstrated the existence of an NP-simple set relative to a random oracle with probability 1. By analyzing his construction in [45], we can prove the existence of honestly NP-hypersimple sets relative to a random oracle.

Proposition 4.8 *Relative to a random oracle, a strongly* NP*-simple set exists with probability* 1.

Proof. In [44, Theorem 3], Vereshchagin proved that, with probability 1, the oracle-dependent tally set $L^X = \{1^n \mid n \in \mathit{Tower} \wedge \forall w \in \Sigma^n \exists v \in \Sigma^{\lfloor \log n \rfloor}[wv \in A]\}$ has no NP^X-subsets relative to a random oracle X. For our purpose, we further define $K^A = \{1^n \mid n \in \hat{T} \wedge 1^n \in L^A\}$. Clearly, K^A is in co-NP^A for any oracle A. Since $K^A \subseteq \{1^n \mid n \in \hat{T}\}$ and $K^A \in \mathrm{EXP}^A$, by Lemma 4.6, if K^A is NP^A-immune then it is also strongly NP^A-immune. Similar to Vereshchagin's proof, we can show that, with probability 1, K^X is NP^X-immune relative to a random oracle X. Therefore, K^X is strongly NP^X-immune relative to a random oracle X with probability 1. □

Finally, we present in Theorem 4.9 that a strongly NP^G-simple set exists relative to a Cohen-Feferman generic oracle G. This immediately implies the existence of an NP^G-simple set relative to the same generic oracle G.

Theorem 4.9 *A strongly* NP^G*-simple set exists relative to a Cohen-Feferman generic oracle* G.

The proof of Theorem 4.9 uses *weak forcing* instead of Feferman's original finite forcing. Let $\varphi(X)$ be any arithmetical statement including variable X, which runs over all subsets of Σ^*. We say that a forcing condition σ *forces* φ (notationally, $\sigma \Vdash \varphi(X)$) if $\varphi(G)$ is true for every Cohen-Feferman generic set G that extends σ.

This forcing relation $\Vdash$ satisfies the following five properties:

1. $\sigma \Vdash \neg\varphi \iff$ no extension of σ forces φ,
2. $\sigma \Vdash (\varphi \wedge \psi) \iff \sigma \Vdash \varphi$ and $\sigma \Vdash \psi$,
3. $\sigma \Vdash (\varphi \vee \psi) \Rightarrow \sigma$ has an extension ρ such that $\rho \Vdash \varphi$ or $\rho \Vdash \psi$,
4. $\sigma \Vdash \forall x\, \varphi(x) \iff \sigma \Vdash \varphi(n)$ for all numbers n, and
5. $\sigma \Vdash \exists x\, \varphi(x) \Rightarrow \sigma$ has an extension ρ such that $\rho \Vdash \varphi(n)$ for a certain number n.

In properties 4 and 5, the symbol n represents either a natural number or a string. Note that properties 3 and 5 have only one-way implications. See the standard textbook, e.g., [31] for more details on weak forcing and its connection to Feferman's finite forcing. In the textbook [31], Cohen-Feferman generic sets are called "ω-generic" sets. Note that, in our setting, the domain of a forcing condition may be any finite set of strings; thus, a forcing condition is not necessarily an initial segment of an oracle. In the proof of Theorem 4.9, a forcing relation refers to *weak forcing*.

For simplicity, we encode a computation path of a nondeterministic oracle TM into a binary string and identify such a path with its encoding. Using this encoding, we can enumerate lexicographically all the computation paths of the machine. For any such machine N, any oracle B, and any string x, we define the useful set $Q_{\boxplus}(N,B,x)$ as follows. When N^B rejects input x, $Q_{\boxplus}(N,B,x)$ is set to be empty. Assume otherwise. Take the lexicographically first accepting computation path γ of N^B on input x and let $Q_{\boxplus}(N,B,x)$ be the set of all strings queried along the computation path γ.

Proof of Theorem 4.9. In this proof, we use the oracle-dependent set $L^X = \{x \mid \forall y \in \Sigma^{|x|}[xy \notin X]\}$, where X is any oracle. Letting G be any Cohen-Feferman generic set, we wish to show that L^G is strongly NP^G-immune. This implies that $\overline{L^G}$ is strongly NP^G-simple since $\overline{L^G}$ is in NP^G.

Let k be any number in $\mathbb{N}^+$. Let N be any nondeterministic oracle TM with an output tape running within time $n^k + k$ independent of the choice of an oracle, where k is a certain fixed natural number. Henceforth, we use the symbol X to denote a variable running over all subsets of Σ^*. We define f^X to be the function computed by N^X. On any input x, N^X always enters an accepting state or a rejecting state. Note that, whenever N^X produces no accepting computation path on input x, $f^X(x)$ is undefined. Now, assume that f^G is an NP^G-m-quasireduction mapping from L^G to a certain set. Consider the collision set

$$K^X = \{(x,y) \in \mathrm{dom}(f^X)^2 \mid x \in L^X \ \wedge\ x < y \ \wedge\ f^X(x) = f^X(y)\}.$$

Since f^G is a NP^G-m-quasireduction from L^G, the set K^G coincides with the original collision set defined in the beginning of this section. It therefore suffices to show that K^G is finite. Because of the definition of weak forcing, we can assume without loss of generality that the following three statements are forced by the empty forcing condition (in other words, they are forced by every forcing condition).

1) For any string x, $x \notin \mathrm{dom}(f^X) \iff N^X$ on input x enters no accepting state,
2) For any string $x \in \mathrm{dom}(f^X)$, N^X outputs $f^X(x)$ in *all* accepting computation paths,
3) For any $n \in \mathbb{N}$, N^X runs within time $n^k + k$ on every input of length n.

Now, consider the following arithmetical statement, which says that every element in K^X is bounded above by a certain constant n.

- $\varphi_0(X) :\equiv \exists (u,v) \in (\Sigma^*)^2\, [f^X(u) = f^X(v) \ \wedge\ u \in L^X \ \wedge\ v \notin L^X]$, and
- $\varphi_1(X) :\equiv \exists n \in \omega\, \forall (u,v) \in (\Sigma^*)^2\, [u,v \notin \mathrm{dom}(f^X) \ \vee\ u \notin L^X \ \vee\ u \geq v \ \vee\ f^X(u) \neq f^X(v) \ \vee\ |v| \leq n]$.

For simplicity, write $\varphi(X)$ for $(\varphi_0(X) \ \vee\ \varphi_1(X))$. Notice that, in general, f^A might not NP^A-m-quasireduce L^A for a certain oracle A. Our goal is to prove that $\varphi(G)$ is true. This clearly implies that K^G is finite since $\varphi_0(G)$ is obviously false and thus $\varphi_1(G)$ must be true. To achieve our goal, we define $\mathcal{D} = \{\sigma \mid \sigma \Vdash \varphi(X)\}$ and then claim that $\mathcal{D}$ is dense. Assuming that $\mathcal{D}$ is dense, the genericity of G guarantees the existence of a forcing condition σ such that $\sigma \Vdash \varphi(X)$ and $\sigma \subseteq G$. By the definition of weak forcing, $\varphi(G)$ must be true. This will complete the proof.

Now, we need to prove that $\mathcal{D}$ is dense. Let σ be any forcing condition. We want to show that there exists an extension τ of σ in $\mathcal{D}$. Assume otherwise that no extension of σ forces $\varphi(X)$. From property 1 of weak forcing, σ forces $\neg\varphi(X)$, which implies $\sigma \Vdash \neg\varphi_0(X)$ and $\sigma \Vdash \neg\varphi_1(X)$. Since $\sigma \Vdash \neg\varphi_1(X)$, σ forces

"$\forall n \in \omega\, \exists (u,v) \in (\Sigma^*)^2 [u,v \in \mathrm{dom}(f^X) \wedge u \in L^X \wedge u < v \wedge f^X(u) = f^X(v) \wedge |v| > n]$." Take any natural number n that is greater than $|\mathrm{dom}(\sigma)|$. By properties 4 and 5, there exist a forcing condition ρ extending σ and a pair (u,v) of strings such that $|v| > n$, $u < v$, and ρ forces "$u,v \in \mathrm{dom}(f^X) \wedge u \in L^X \wedge f^X(u) = f^X(v)$." In what follows, since the aforementioned statements 1), 2), and 3) are forced by ρ, we can assume that the domain of ρ consists only of the following four sets: (i) $\mathrm{dom}(\sigma)$, (ii) $u\Sigma^{|u|}$ (to force "$u \in L^X$"), (iii) $Q_{\boxplus}(N,\rho,u)$ (to decide the value of $f^X(u)$), and (iv) $Q_{\boxplus}(N,\rho,v)$ (to decide the value of $f^X(v)$). Note that $|Q_{\boxplus}(N,\rho,u)| \leq |u|^k + k$ and $|Q_{\boxplus}(N,\rho,v)| \leq |v|^k + k$. Note also that $u\Sigma^{|u|}$ and $v\Sigma^{|v|}$ are disjoint and $|v| \geq \max\{|\mathrm{dom}(\sigma)|, |u|\}$. Since $u < v$ and $|v| > n$, the cardinality $|v\Sigma^{|v|} \cap \mathrm{dom}(\rho)|$ is at most $2(|v|^k + k) + |v|$. Therefore, there exists at least one string w of length $|v|$ such that $vw \notin \mathrm{dom}(\rho)$. With this w, we can extend ρ to another forcing condition τ that forces "$u \in L^X \wedge v \notin L^X \wedge f^X(u) = f^X(v)$." This contradicts the assumption that $\sigma \Vdash \neg \varphi_0(X)$. Consequently, $\mathcal{D}$ is dense. This completes the proof of the theorem. □

5 Almost Immunity and Almost Simplicity

We have shown in the previous section that strong $\mathcal{C}$-immunity and its simplicity strengthen the ordinary notion of $\mathcal{C}$-immunity and $\mathcal{C}$-simplicity. In contrast to these notions, Orponen [32] and Orponen, Russo, and Schöning [34] expanded P-immunity to the new notion of almost P-immunity. The complementary notion of almost P-immunity under the term P-levelability (a more general term "levelable sets" was first used by Ko [26] in a resource-bounded setting) was extensively discussed by Orponen et al. [34]. Naturally, we can generalize these notions to almost $\mathcal{C}$-immunity and $\mathcal{C}$-levelability for any complexity class $\mathcal{C}$. Furthermore, we newly introduce the notions of almost $\mathcal{C}$-bi-immune sets and almost $\mathcal{C}$-simple sets.

Definition 5.1 Let $\mathcal{C}$ be any complexity class.

1. A set S is *almost $\mathcal{C}$-immune* if S is the union of a $\mathcal{C}$-immune set and a set in $\mathcal{C}$.
2. An infinite set is *$\mathcal{C}$-levelable* if it is not almost $\mathcal{C}$-immune.
3. A set S is *almost $\mathcal{C}$-bi-immune* if S and $\overline{S}$ are both almost $\mathcal{C}$-immune.
4. A set S is *almost $\mathcal{C}$-simple* if S is an infinite set in $\mathcal{C}$ and $\overline{S}$ is the union of a set A in $\mathcal{C}$ and a $\mathcal{C}$-immune set B, where $B \setminus A$ is infinite.

It follows from Definition 5.1(1) that every almost $\mathcal{C}$-immune set is infinite since so is every $\mathcal{C}$-immune set. The definition of almost $\mathcal{C}$-simplicity in Definition 5.1(4) is slightly different from other simplicity definitions because the infinity condition of the difference $B \setminus A$ is necessary to guarantee $\mathcal{C} \neq \text{co-}\mathcal{C}$, provided that an almost $\mathcal{C}$-simple set exists. This is shown in the following lemma.

Lemma 5.2 *Let $\mathcal{C}$ be any complexity class closed under finite variations, finite union and finite intersection. If an almost $\mathcal{C}$-simple set exists, then $\mathcal{C} \neq \text{co-}\mathcal{C}$.*

Proof. Assume that $\mathcal{C} = \text{co-}\mathcal{C}$ and let S be any almost $\mathcal{C}$-simple set. There exist a set $A \in \mathcal{C}$ and a $\mathcal{C}$-immune set B such that $\overline{S} = A \cup B$ and $B \setminus A$ is infinite. Since $S, A \in \mathcal{C} \cap \text{co-}\mathcal{C}$, $\overline{S} - A$ is in $\mathcal{C} \cap \text{co-}\mathcal{C}$. Note that $B \setminus A \subseteq B$ and $B \setminus A = \overline{S} \setminus A \in \mathcal{C}$. Since B is $\mathcal{C}$-immune, $B \setminus A$ must be finite. This contradicts the almost $\mathcal{C}$-immunity of S. Therefore, the lemma holds. □

The following lemma is immediate from Definition 5.1.

Lemma 5.3 *For any complexity class $\mathcal{C}$, every $\mathcal{C}$-immune set is almost $\mathcal{C}$-immune and every $\mathcal{C}$-simple set is almost $\mathcal{C}$-simple.*

Several characterizations of almost P-immunity and P-levelability are shown in [34] in terms of maximal P-subsets and P-to-finite reductions. We can naturally expand these characterizations to almost Δ_k^{P}-immunity and Δ_k^{P}-levelability (but not to the Σ-level classes of the polynomial-tim hierarchy).

To understand the characteristics of almost $\mathcal{C}$-immunity, we begin with a simple observation. We say that a set S is *polynomially paddable* (*paddable*, in short) if there is an one-to-one polynomial-time computable function *pad* (called the *padding function*) from Σ^* to Σ^* such that, for all pairs $x, y \in \Sigma^*$, $x \in S \iff pad(\langle x, y\rangle) \in S$. A set S is *honestly paddable* if it is paddable with a padding function that is componentwise honest. It is known in [34] that any honestly paddable set not in P is P-levelable. As observed in [37], the essence of this assertion is that if $A \notin \mathrm{P}$ and A is length-increasing P-m-selfreducible then A is P-levelable, where A is *length-increasing*

$\mathcal{C}$*-m-selfreducible* if A is $\mathcal{C}$-m-reducible to A via a certain length-increasing reduction. This observation can be generalized to Δ_k^{P}-levelable sets in the following lemma. The second part of the lemma will be used in Section 6.

Lemma 5.4 *Let $k \in \mathbb{N}^+$ and $A \subseteq \Sigma^*$. Assuming that $A \notin \Delta_k^{\mathrm{P}}$, if A is length-increasing Δ_k^{P}-m-selfreducible, then A and $\overline{A}$ are both Δ_k^{P}-levelable. Thus, if $\Delta_k^{\mathrm{P}} \neq \Sigma_k^{\mathrm{P}}$ then Σ_k^{P} as well as Π_k^{P} has a Δ_k^{P}-levelable set.*

Although our proof is fundamentally the same as that of [37], we include it for completeness. For any function f and any number $k \in \mathbb{N}^+$, the notation $f^{(k)}$ denotes the k-fold composition of f. For convenience, define $f^{(0)}$ to be the identity function.

Proof of Lemma 5.4. Assume to the contrary that A is almost Δ_k^{P}-immune and A is Δ_k^{P}-m-selfreducible via a length-increasing reduction f. Since $\overline{A}$ is also Δ_k^{P}-m-selfreducible via f, it suffices to prove the lemma only for A. Assume also that A is outside of Δ_k^{P}. Since A is almost Δ_k^{P}-immune, A is expressed as of the form $B \cup C$, where B is a set in Δ_k^{P} and C is a Δ_k^{P}-immune set. Note that the difference $C \setminus B$ is infinite since, otherwise, A falls into Δ_k^{P}. Now, define $D = \{x \mid x \notin B \wedge f(x) \in B\}$. Clearly, $D \subseteq C$.

To lead to a contradiction, it suffices to show that D is an infinite set in Δ_k^{P}. Since f is in $\mathrm{F}\Delta_k^{\mathrm{P}}$, D is in Δ_k^{P}. If D is finite, then the set $C \setminus (B \cup D)$ must be infinite. In this case, let z_0 be the lexicographically largest element in D. Take the minimal string x in $C \setminus (B \cup D)$ such that $|x| > |z_0|$. The set $F = \{f^{(i)}(x) \mid i \in \mathbb{N}\}$ must include an element in B because, otherwise, F becomes an infinite Δ_k^{P}-subset of C, a contradiction. Hence, there exists a number $k \in \mathbb{N}$ such that $f^k(x)$ falls in D. This implies that $|f^k(x)| \leq |z_0| < |x|$, which contradicts the length-increasing property of f. Therefore, D is infinite, as required.

The second part of the lemma follows from the fact that, under the assumption $\Delta_k^{\mathrm{P}} \neq \Sigma_k^{\mathrm{P}}$, the class Σ_k^{P} as well as Π_k^{P} contains honestly paddable sets not in Δ_k^{P}, which satisfy the premise of the first part of the lemma. □

Most known NP-m-complete sets are known to be honestly paddable and thus, by Lemma 5.4, the complements of these sets are P-levelable sets, which are also NP-levelable unless P = NP. Therefore, most known NP-m-complete sets cannot be almost NP-simple. This result can be compared with Proposition 5.7.

Earlier, Ko and Moore [27] considered the resource-bounded notion of "productive sets." Another formulation based on $\mathrm{NP}_{(k)}$ was later given by Joseph and Young [25], who use the terminology of k-creative sets, where k is any number in $\mathbb{N}^+$. Now, fix $k \in \mathbb{N}^+$. A set S in NP is called *k-creative* if there exists a function $f \in \mathrm{FP}$ such that, for all $x \in INDEX_{(k)}$, $f(x) \in S \Longleftrightarrow f(x) \in W_x$. This function f is called the *productive function* for S. If in addition f is honest, S is called *honestly k-creative*. Joseph and Young [25] showed that every k-creative set is P-m-complete for NP. Orponen et al. [34] showed that, unless P = NP, every honestly k-creative set is P-levelable by demonstrating that any honestly k-creative set is length-increasing P-m-selfreducible. From Lemma 5.4, it follows that if P $\neq$ NP then any honestly k-creative set and its complement are both P-levelable. Consequently, we obtain the following result.

Corollary 5.5 *For any $k \in \mathbb{N}^+$, no honestly k-creative set is almost* NP*-simple.*

Our notion of almost $\mathcal{C}$-simplicity is similar to what Uspenskii [43] discussed under the term "pseudosimplicity." Here, we give a resource-bounded version of his pseudosimplicity. A set S is called $\mathcal{C}$*-pseudosimple* if there is an infinite $\mathcal{C}$-subset A of $\overline{S}$ such that $S \cup A$ is $\mathcal{C}$-simple. Although $\mathcal{C}$-simple sets cannot be $\mathcal{C}$-pseudosimple by our definition, any infinite $\mathcal{C}$-pseudosimple set is almost $\mathcal{C}$-simple. The latter claim is shown as follows. Suppose that S is an infinite $\mathcal{C}$-pseudosimple set and A is a $\mathcal{C}$-subset of $\overline{S}$ for which $S \cup A$ is $\mathcal{C}$-simple. This means that $\overline{S} \setminus A$ is $\mathcal{C}$-immune. Therefore, S is almost $\mathcal{C}$-simple.

The following theorem shows a close connection among simplicity, almost simplicity, and pseudosimplicity. This theorem signifies the importance of almost simple sets.

Theorem 5.6 *For each $k \in \mathbb{N}^+$, the following three statements are equivalent.*

1. *There exists a Σ_k^{P}-simple set.*
2. *There exists an infinite Σ_k^{P}-pseudosimple set in* P.
3. *There exists an almost Σ_k^{P}-simple set in* P.

Proof. Let k be any number in $\mathbb{N}^+$.

2 implies 3) This implication holds because any infinite Σ_k^{P}-pseudosimple set is indeed almost Σ_k^{P}-simple as mentioned before.

3 implies 1) Assume that S is an almost Σ_k^{P}-simple set in P. By the definition, there exist a set A in Σ_k^{P} and a Σ_k^{P}-immune set B such that $\overline{S} = A \cup B$ and $B \setminus A$ is infinite. Define $C = B \setminus A$. We show that $\overline{C}$ is the desired Σ_k^{P}-simple set. First, since $\overline{C} = S \cup A$ and $A \in \mathrm{P}$, $\overline{C}$ belongs to Σ_k^{P}. Second, since $C \subseteq B$, C is Σ_k^{P}-immune. This yields the Σ_k^{P}-simplicity of $\overline{C}$.

1 implies 2) Suppose that there exists a Σ_k^{P}-simple set S. Under this assumption, we want to claim that both $0\Sigma^*$ and $1\Sigma^*$ are Σ_k^{P}-pseudosimple. This is shown as follows. For each bit $a \in \{0,1\}$, let $A_a = aS$ and $B_a = a\overline{S}$. The immunity of $\overline{S}$ implies that B_0 and B_1 are both infinite. Consider the case for $0\Sigma^*$. Note that A_1 is a Σ_k^{P}-subset of $1\Sigma^*$. Since $\overline{S}$ is Σ_k^{P}-immune and $B_1 \subseteq 1\overline{S}$, it follows that B_1 is Σ_k^{P}-immune. Observe that $B_1 = 1\Sigma^* \cap \overline{A_1}$. Hence, its complement $0\Sigma^* \cup A_1$ is a Σ_k^{P}-simple set. This concludes that $0\Sigma^*$ is Σ_k^{P}-pseudosimple. By a similar argument, $1\Sigma^*$ is Σ_k^{P}-pseudosimple. □

Theorem 5.6 indicates the importance of the structure of P in the course of the study of Σ_k^{P}-simplicity. In a relativized world where a Σ_k^{P}-simple set exists [11], since Theorem 5.6 relativizes, there exists an almost Σ_k^{P}-simple set within P.

We note the relativization of almost NP-simple sets. In [45], Vereshchagin actually proved that there are two partitions L_0^X and L_1^X of $\{1^n \mid n \in \mathit{Tower}\}$ such that L_0^X is NP^X-immune and in co-NP^X and L_1^X is co-NP^X-immune and in NP^X relative to a random oracle with probability 1. This implies that the set L_1^X is an almost NP^X-simple set that is also co-NP^X-immune relative to a random oracle.

Finally, we briefly discuss a closure property of the class of all almost Σ_k^{P}-immune sets under polynomial-time reductions. For each number k in $\mathbb{N}^+$, the class of all Σ_k^{P}-immune sets is closed under h-Δ_k^{P}-d-reductions on infinite sets whereas the class of all almost Σ_k^{P}-immune sets is closed under h-Δ_k^{P}-m-reductions on infinite sets. The latter claim is proven as follows. Let A be any infinite set. Assume that f is an h-P-m-reduction from A to an almost Σ_k^{P}-immune set B. This means that B is the union of two subsets B_1 and B_2, where B_1 is Σ_k^{P}-immune and $B_2 \in \Sigma_k^{\mathrm{P}}$. Therefore, A should be the union of the two subsets $f^{-1}(B_1)$ and $f^{-1}(B_2)$. Since B_1 is infinite, $f^{-1}(B_1)$ is also infinite. Hence, by Proposition 3.4, $f^{-1}(B_1)$ is Σ_k^{P}-immune. Since f is honest, it follows that $f^{-1}(B_2) \in \Sigma_k^{\mathrm{P}}$. Hence, A is almost Σ_k^{P}-immune. This immediately implies the following consequence.

Proposition 5.7 *For each $k \in \mathbb{N}^+$, no almost Σ_k^{P}-simple set is h-Δ_k^{P}-m-complete for Σ_k^{P}.*

6 Hyperimmunity and Hypersimplicity

Since Post [35] constructed a so-called *hypersimple* set, the notions of hyperimmunity and hypersimplicity have played a significant role in the progress of classical recursion theory. A resource-bounded version of these notions was first considered by Yamakami [48] and studied extensively by Schaefer and Fenner [38]. The definition of Schaefer and Fenner is based on the notion of "honest NP-arrays", which differs from the notion of "strong arrays" in recursion theory (see, e.g., [31]), where a strong array is a series of pairwise disjoint finite sets. For our formalization, we demand only "eventually disjointness" for sets in an array rather than "pairwise disjointness."

A binary string x is said to *represent* the finite set $\{a_1, a_2, \ldots, a_k\}$ if and only if $x = \langle a_1, a_2, \ldots, a_k \rangle$ and $a_1 < a_2 < \cdots < a_k$ in the standard lexicographic order on Σ^*. For convenience, we say that a set A *surpasses* another set B if there exists a string $z \in A$ satisfying that $z > x$ (lexicographically) for all strings $x \in B$.

Definition 6.1 Let $k \in \mathbb{N}^+$, $A \subseteq \Sigma^*$, and $\mathcal{C} \in \{\Sigma_k^{\mathrm{P}}, \Delta_k^{\mathrm{P}}\}$.

1. An infinite sequence $\mathcal{D} = \{D_s\}_{s\in\Sigma^*}$ of finite sets is called a Σ_k^{P}-*array* (Δ_k^{P}-*array*, resp.) if there exists a *partial* function f in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ ($\mathrm{F}\Delta_k^{\mathrm{P}}$, resp.) such that (i) $\mathrm{dom}(f)$ is infinite, (ii) $D_s \neq \emptyset$ and $f(s)$ represents D_s for any string $s \in \mathrm{dom}(f)$, and (iii) $D_s = \emptyset$ for any string $s \notin \mathrm{dom}(f)$. This f is called a *supporting function* of $\mathcal{D}$ and the set $\bigcup_{s\in\mathrm{dom}(f)} D_s$ is called the *support* of $\mathcal{D}$. The *width* of $\mathcal{D}$ is the supremum of the cardinality $|D_s|$ over all indices $s \in \mathrm{dom}(f)$.
2. A $\mathcal{C}$-array $\mathcal{D}$ has an *infinite support* if the support of $\mathcal{D}$ is infinite.
3. A $\mathcal{C}$-array $\{D_s\}_{s\in\Sigma^*}$ via f is *polynomially honest* (*honest*, in short) if f is componentwise honest; namely, there exists a polynomial p such that $|s| \leq p(|x|)$ for any $s \in \mathrm{dom}(f)$ and any $x \in D_s$.
4. A $\mathcal{C}$-array $\{D_s\}_{s\in\Sigma^*}$ via f is *eventually disjoint* if, for every string $x \in \mathrm{dom}(f)$, there exists a string y in $\mathrm{dom}(f)$ such that $y \geq x$ (lexicographically), $D_x \cap D_y = \emptyset$, and D_y surpasses D_x.
5. A $\mathcal{C}$-array $\{D_s\}_{s\in\Sigma^*}$ via f *intersects* A if $D_s \cap A \neq \emptyset$ for any s in $\mathrm{dom}(f)$.

The honesty condition of an $\mathcal{C}$-array guarantees that the array is eventually disjoint. In addition, any eventually disjoint $\mathcal{C}$-array has an infinite support because, for any element D in the array, we can always find another disjoint element D'.

A simple relationship between Σ_k^{P}-simplicity and a honest Σ_k^{P}-array is given in the following lemma, which was implicitly proven by Yamakami [48] and later explicitly stated in [38] for the case where $k = 1$.

Lemma 6.2 *Let $k \in \mathbb{N}^+$ and let A be any Σ_k^{P}-simple set. For every number $\ell \in \mathbb{N}^+$, there is no honest Σ_k^{P}-array $\mathcal{D}$ such that (i) the width of $\mathcal{D}$ is at most ℓ and (ii) $\mathcal{D}$ intersects $\overline{A}$.*

Proof. Let A be any Σ_k^{P}-simple set and let ℓ be any number in $\mathbb{N}^+$. To lead to a contradiction, we assume that there exists a honest Σ_k^{P}-array $\mathcal{D} = \{D_x\}_{x\in\Sigma^*}$ via f such that the width of $\mathcal{D}$ is $\leq \ell$ and $\mathcal{D}$ intersects $\overline{A}$. For brevity, write D for the support of $\mathcal{D}$. Obviously, D is infinite and in Σ_k^{P}. Since f is componentwise honest, we can take a polynomial p satisfying $|x| \leq p(|y|)$ for all $x \in \mathrm{dom}(f)$ and all $y \in D_x$. Now, define ℓ_{max} to be the maximal value i such that $|D_x \cap A| = i$ for infinitely many strings x in $\mathrm{dom}(f)$. Take any sufficiently large number $n_0 \in \mathbb{N}$ that guarantees $|D_x \cap A| \leq \ell_{max}$ for all strings x of length $\geq n_0$.

Define the set $B = \{y \mid \exists x[n_0 \leq |x| \leq p(|y|) \ \wedge\ y \in D_x \ \wedge\ |(D_x \setminus \{y\}) \cap A| = \ell_{max}]\}$, which is clearly in Σ_k^{P} since so is A and D. This B is infinite because the set $\{x \in \mathrm{dom}(f) \mid |x| \geq n_0 \ \wedge\ |D_x \cap A| = \ell_{max}\}$ and D are both infinite and $\mathcal{D}$ intersects $\overline{A}$. Now, we show that $B \subseteq \overline{A}$. This is true because, if $y \in B \setminus \overline{A}$, then $\ell_{max} = |(D_x \setminus \{y\}) \cap A| < |D_x \cap A| \leq \ell_{max}$, a contradiction. Therefore, B is an infinite subset of $\overline{A}$. Since B is in Σ_k^{P}, $\overline{A}$ cannot be Σ_k^{P}-immune. This contradicts our assumption that A is Σ_k^{P}-simple. □

We introduce below the notions of $\mathcal{C}$-hyperimmunity and honest $\mathcal{C}$-hyperimmunity.

Definition 6.3 Let $\mathcal{C} \in \{\Delta_k^{\mathrm{P}}, \Sigma_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$.

1. A set S is *(honestly) $\mathcal{C}$-hyperimmune* if S is infinite and there is no (honest) $\mathcal{C}$-array $\mathcal{D}$ such that $\mathcal{D}$ is eventually disjoint and intersects S.
2. A set S is *(honestly) $\mathcal{C}$-bi-hyperimmune* if S and $\overline{S}$ are both (honestly) $\mathcal{C}$-hyperimmune.
3. A set S is *(honestly) $\mathcal{C}$-hypersimple* if S is in $\mathcal{C}$ and $\overline{S}$ is (honestly) $\mathcal{C}$-hyperimmune.

Note that "NP-hyperimmunity" defined by Schaefer and Fenner [38] coincides with our honest NP-hyperimmunity. Any honestly $\mathcal{C}$-hyperimmune set is $\mathcal{C}$-immune because, assuming that S is not $\mathcal{C}$-immune, we can choose an infinite subset A of S in $\mathcal{C}$ and define $D_s = \{s\}$ if $s \in A$ and $D_s = \emptyset$ otherwise, which shows that A is not honestly $\mathcal{C}$-hyperimmune.

Lemma 6.4 *For any complexity class $\mathcal{C} \in \{\Sigma_k^{\mathrm{P}}, \Delta_k^{\mathrm{P}} \mid k \in \mathbb{N}\}$, every honestly $\mathcal{C}$-hyperimmune set is $\mathcal{C}$-immune and every honestly $\mathcal{C}$-hypersimple set is $\mathcal{C}$-simple.*

In Proposition 4.3, we have seen that $\mathcal{C}$-generic sets are strongly $\mathcal{C}$-bi-immune. Similarly, $\mathcal{C}$-generic sets are examples of $\mathcal{C}$-hyperimmune sets.

Proposition 6.5 *Let $k \in \mathbb{N}^+$. All Σ_k^{P}-generic sets are honestly Σ_k^{P}-hyperimmune.*

The following proof works only for Σ_k^{P}-genericity but not for Δ_k^{P}-genericity.

Proof of Proposition 6.5. Fix $k \in \mathbb{N}^+$ and let A be any Σ_k^{P}-generic set. Assume that A is not honestly Σ_k^{P}-hyperimmune; that is, there exists a Σ_k^{P}-array $\mathcal{D} = \{D_s\}_{s\in\Sigma^*}$ via f such that $\mathrm{dom}(f)$ is infinite, $\mathcal{D}$ is honest, and $\mathcal{D}$ intersects A. Since f is componentwise honest, take an increasing polynomial p such that $|x| \leq p(|y|)$ for any $x \in \mathrm{dom}(f)$ and any $y \in D_x$.

First, we define S to be the collection of all nonempty forcing conditions σ such that there exists an element $x \in \mathrm{dom}(f)$ satisfying that $D_x \subseteq \mathrm{dom}(\sigma)$ and $\sigma(y) = 0$ for all $y \in D_x$. Note that $x \in \mathrm{dom}(f)$ implies $|x| \leq p(|\sigma|)$ since σ is defined on all the strings in D_x. Thus, S belongs to Σ_k^{P}. Next, we want to show that S is dense. Let σ be any forcing condition. Take any string x such that $D_x \cap \mathrm{dom}(\sigma) = \emptyset$ and $D_x \neq \emptyset$. Such an x exists because $\mathcal{D}$ is eventually disjoint. For such an x, define τ as the unique forcing condition satisfying the following: $\sigma \subseteq \tau$, $\mathrm{dom}(\tau) = \mathrm{dom}(\sigma) \cup D_x$, and $\tau(y) = 0$ for all $y \in D_x$. Clearly, τ is in S. This implies that S is dense.

Since A is Σ_k^{P}-generic, we obtain $\sigma \subseteq A$ for a certain σ in S. By the definition of S, there exists a string $x \in \mathrm{dom}(f)$ satisfying that $A(y) = 0$ for all $y \in D_x$, which implies $D_x \cap A = \emptyset$, a contradiction. Therefore, A

is honestly Σ_k^{P}-hyperimmune. $\square$

In late 1970s, Selman [39] introduced the notion of P-*selective* sets, which are analogues of semi-recursive sets in recursion theory. These sets connect P-immunity to P-hyperimmunity. In general, for any class $\mathcal{F}$ of *total* functions, we say that a set S is $\mathcal{F}$-*selective* if there exists a function (called the *selector*) f in $\mathcal{F}$ such that, for all pairs $(x,y) \in \Sigma^* \times \Sigma^*$, (i) $f(x,y) \in \{x,y\}$ and (ii) $\{x,y\} \cap S \neq \emptyset$ implies $f(x,y) \in S$. Recall the total function class $\Sigma_k^{\mathrm{P}}\mathrm{SV}_t$.

Lemma 6.6 *Let $k \in \mathbb{N}^+$. Every Σ_k^{P}-immune $\Sigma_k^{\mathrm{P}}\mathrm{SV}_t$-selective set is honestly Σ_k^{P}-hyperimmune.*

Observe that the complement of a $\Sigma_k^{\mathrm{P}}\mathrm{SV}_t$-selective set S is also $\Sigma_k^{\mathrm{P}}\mathrm{SV}_t$-selective because the exchange of the output string of any selector for S gives rise to a selector for $\overline{S}$. Note also that Lemma 6.6 relativizes.

Proof of Lemma 6.6. Let $k \geq 1$ and assume that S is $\Sigma_k^{\mathrm{P}}\mathrm{SV}_t$-selective but not honestly Σ_k^{P}-hyperimmune. We want to show that S has an infinite Σ_k^{P}-subset. Let f be a selector for S and let $\mathcal{D} = \{D_s\}_{s\in\Sigma^*}$ be an honest Σ_k^{P}-array intersecting S via g. Define h as follows. Let $y \in \mathrm{dom}(g)$. Assume that $D_y = \{x_1, x_2, \cdots, x_m\}$ with $x_1 < x_2 < \cdots < x_m$. Let $y_1 = x_1$ and $y_{i+1} = f(y_i, x_{i+1})$ for every $i \in [1, m-1]_{\mathbb{Z}}$ and then define $h(y) = y_m$. Clearly, h is in $\Sigma_k^{\mathrm{P}}\mathrm{SV}$ since $Graph(h)$ is in Σ_k^{P}. For any string $y \in \mathrm{dom}(g)$, $h(y)$ belongs to S since D_y intersects S. Note that h is honest since so is $\mathcal{D}$. Let p be any polynomial such that $|y| \leq p(|h(y)|)$ for any string y in $\mathrm{dom}(h)$. Define $B = \{x \mid \exists y \in \Sigma^{\leq p(|x|)}[y \in \mathrm{dom}(h) \wedge h(y) = x]\}$, which is in Σ_k^{P}. Clearly, B is a subset of S and is infinite since $\mathcal{D}$ is honest. $\square$

It follows from Lemma 6.6 that every NP-simple P-selective set is honestly NP-hypersimple since the complement of any P-selective set is also P-selective.

Next, we show that strong P-immunity does not imply honest P-hyperimmunity within the class E. Earlier, Balcázar and Schöning [5] created a strongly P-bi-immune set S in E with the density $|S \cap \Sigma^{\leq n}| = 2^{n+1} - n - 1$ for any $n \in \mathbb{N}$. For each x, let D_x consist of the first $|x|+1$ elements of $\Sigma^{|x|}$. Clearly, D_x intersects S. This implies that S is not honestly P-hyperimmune. Therefore, we obtain the following proposition.

Proposition 6.7 *There exists a strongly* P*-bi-immune set in* E *that is not honestly* P*-hyperimmune.*

As a main theorem, we show the P-T-incompleteness of Σ_k^{P}-hypersimple sets.

Theorem 6.8 *Let $k \in \mathbb{N}^+$.*

1. *No Σ_k^{P}-hypersimple set is* P*-T-complete for Σ_k^{P}.*
2. *No honestly Σ_k^{P}-hypersimple set is h-*P*-T-complete for Σ_k^{P}.*

Note that it is not clear whether we can replace the P-T-completeness in Theorem 6.8 by the Δ_k^{P}-T-completeness. Now, we want to prove Theorem 6.8. Our proof utilizes Lemma 6.9.

Lemma 6.9 *Let k be any number in $\mathbb{N}^+$ and let A be any infinite set in Σ_k^{P}.*

1. *If $A \leq_T^{\mathrm{P}} B$ and B is Σ_k^{P}-hyperimmune and in* EXP*, then $\overline{A}$ is almost Δ_k^{P}-immune.*
2. *If $A \leq_T^{h\text{-}\mathrm{P}} B$ and B is honestly Σ_k^{P}-hyperimmune, then $\overline{A}$ is almost Δ_k^{P}-immune.*

We postpone the proof of Lemma 6.9 and instead give the proof of Theorem 6.8.

Proof of Theorem 6.8. We prove only the first claim since the second claim follows similarly. Now, assume that B is a Σ_k^{P}-hypersimple set that is P-T-complete for Σ_k^{P}. This means that $\overline{B}$ is Σ_k^{P}-hyperimmune and is in Π_k^{P}. The existence of a Σ_k^{P}-hypersimple set implies $\Delta_k^{\mathrm{P}} \neq \Sigma_k^{\mathrm{P}}$. Note that B is lying in EXP. Since B is P-T-complete for Σ_k^{P}, every Σ_k^{P}-set A is P-T-reducible to B. Lemma 6.9 shows that every Π_k^{P}-set is almost Δ_k^{P}-immune. This contradicts Lemma 5.4, in which Π_k^{P} has a Δ_k^{P}-levelable set. Therefore, B cannot be Σ_k^{P}-hypersimple. $\square$

We still need to prove Lemma 6.9, which requires a key idea of Agrawal (mentioned earlier), who showed that no NP-simple set is h-P-btt-complete for NP. We extend his core argument to Lemma 6.10. For convenience, we say that a complexity class $\mathcal{C}$ is *closed under intersection with Δ_k^{P}-sets* if, for any set A in $\mathcal{C}$ and any set B in Δ_k^{P}, the intersection $A \cap B$ belongs to $\mathcal{C}$.

Lemma 6.10 *Let $\mathcal{C}$ be any complexity class containing Δ_k^{P} such that $\mathcal{C}$ is closed under intersection with Δ_k^{P}-sets. Let A be any set in $\mathcal{C}$ whose complement is Δ_k^{P}-levelable. If A is Δ_k^{P}-T-reducible to B via a reduction machine M, then there exists an infinite set C in $\mathcal{C}$ such that $Q(M,B,x)\cap B\neq\emptyset$ for all $x\in C$.*

Proof. Assume that A is Δ_k^{P}-T-reducible to B via a certain reduction machine M; namely, $A=\{x\mid M^B(x)=1\}$. For convenience, introduce the set $E=\{x\mid M^{\emptyset}(x)=0\}$, which is obviously in Δ_k^{P}.

First, we consider the case where $|E\cap A|$ is infinite. In this case, for every $x\in E\cap A$, $M^B(x)=1$ but $M^{\emptyset}(x)=0$. Hence, M on input x queries a string in B, which implies $Q(M,B,x)\cap B\neq\emptyset$. Let $C=E\cap A$. Clearly, C is infinite and is in $\mathcal{C}$ since A is in $\mathcal{C}$ and $\mathcal{C}$ is closed under intersection with Δ_k^{P}-sets.

Second, we consider the other case where $|E\cap A|$ is finite. Let $E'=E\setminus A$. Since E' differs from E on finitely-many elements, E' is in Δ_k^{P}. Note that $E'\subseteq\overline{A}$ by its definition. Using the assumption that $\overline{A}$ is Δ_k^{P}-levelable, there exists an infinite set $Q'\in\Delta_k^{\mathrm{P}}$ such that $Q'\subseteq\overline{A}$ and $Q'\cap E'=\emptyset$. Let x be any string in Q'. Since $x\in\overline{A}$, $M^B(x)$ outputs 0. Nonetheless, from $x\notin E$, $M^{\emptyset}(x)$ equals 1. Thus, $Q(M,B,x)\cap B\neq\emptyset$. Write C for Q'. Obviously, C is in $\mathcal{C}$ (because $\Delta_k^{\mathrm{P}}\subseteq\mathcal{C}$) and is clearly infinite. □

At last, we complete the proof of Lemma 6.9.

Proof of Lemma 6.9. 2) Let A be any infinite Σ_k^{P}-set and let B be a honestly Σ_k^{P}-hyperimmune set. Assume that A is h-P-T-reducible to B via a certain reduction machine M. We want to show that $\overline{A}$ is almost Δ_k^{P}-immune. Assume to the contrary that $\overline{A}$ is a Δ_k^{P}-levelable set in Π_k^{P}. By Lemma 6.10, there exists an infinite set C in Σ_k^{P} such that $Q(M,B,x)\cap B\neq\emptyset$ for all $x\in C$.

Claim. For any string x, $Q(M,B,x)\cap B\neq\emptyset$ if and only if $Q(M,\emptyset,x)\cap B\neq\emptyset$.

The proof of the above claim proceeds as follows. Assume that $Q(M,B,x)\cap B=\emptyset$. This means that all queries of M on input x with oracle B are answered NO. Thus, $Q(M,B,x)=Q(M,\emptyset,x)$, which implies $Q(M,\emptyset,x)\cap B=\emptyset$. The other direction is similar.

The partial function h is defined as a map from Σ^* to Σ^* as follows. The set C is the domain of h. Choose any string x in C and consider the set $Q(M,\emptyset,x)$. Note that $|Q(M,\emptyset,x)|$ is polynomially bounded. Moreover, $Q(M,\emptyset,x)$ cannot be empty by the above claim. Let $h(x)$ be the unique string that represents $Q(M,\emptyset,x)$. Now, define D_x to be $set(h(x))$ if $x\in C$ and $D_x=\emptyset$ otherwise.

The sequence $\mathcal{D}=\{D_x\}_{x\in\Sigma^*}$ satisfies $D_x\cap B\neq\emptyset$ for all $x\in\mathrm{dom}(h)$. Hence, $\mathcal{D}$ is a honest Σ_k^{P}-array since M is honest and C is in Σ_k^{P}. This contradicts our assumption that B is honestly Σ_k^{P}-hyperimmune.

1) Different from 2), we need to prove that the array $\mathcal{D}=\{D_x\}_{x\in\Sigma^*}$ defined in 1) is eventually disjoint. First, assume that M runs within time n^d+d for all inputs of length n and any oracle, where d is an absolute positive constant. Unfortunately, since M is not guaranteed to be honest, we cannot prove the eventual-disjointness of $\mathcal{D}$. To overcome this problem, we modify the reduction machine M as follows.

Since B is in EXP, there exists a exponential-time deterministic TM N which recognizes B. Without losing generality, we can assume that M runs within time 2^{n^d+d} for any input of length n. We modify the reduction machine M as follows.

> On input x, simulate M on input x. If M makes a query y, then check if $|y|^d>\log|x|$. If so, make a query y to oracle. Otherwise, run N on input x and make its outcome an oracle answer.

Let M_+ be the new oracle TM defined above. Note that M_+ is an oracle P-machine because its running time is, for a certain absolute constant c, bounded above by $c(|x|^d+d+2^{\log|x|+d})$, which is $O(|x|^d)$. It is thus clear that M_+ P-T-reduces A to B. Note that M_+ on input x cannot query any string shorter than length $\log|x|$. Therefore, M_+ satisfies the following condition: for every x, there exists a string y with $y\geq x$ such that (i) $Q(M_+,B,x)\cap Q(M_+,B,y)=\emptyset$ and (ii) if $Q(M_+,B,y)\neq\emptyset$, then $Q(M_+,B,y)$ surpasses $Q(M_+,B,x)$.

Now, consider the array $\{D_x\}_{x\in\Sigma^*}$ obtained from M_+ similar to 1). Since $D_x=Q(M_+,\emptyset,x)$ for all $x\in\mathrm{dom}(h)$, it follows that $\mathcal{D}$ is eventually disjoint. □

Although the existence of a Σ_k^{P}-simple set is unknown, as Schaefer and Fenner [38] demonstrated, it is relatively easy to prove the existence of an honest NP^G-hypersimple set relative to a Cohen-Feferman generic oracle G. For a higher level k of the polynomial-time hierarchy, we build in the following proposition a recursive oracle relative to which a Σ_k^{P}-hypersimple set exists.

Proposition 6.11 *For each $k\in\mathbb{N}^+$, there exists a recursive oracle A such that a $\Sigma_k^{\mathrm{P}}(A)$-hypersimple set* *exists.*

We prove Proposition 6.11 using Bruschi's [11] result and Lemma 6.6. To use Lemma 6.6, we need the following supplemental lemma. Recall the set $\hat{T}$ introduced in Section 4.

Lemma 6.12 *Let A be any set in* EXP. *If $A \subseteq \{1^n \mid n \in \hat{T}\}$, then A is* P*-selective.*

Proof. Since $A \in \mathrm{EXP}$, there exists a number $c \in \mathbb{N}$ and a deterministic TM M that recognizes A in at most 2^{n^c+c} steps, where n is the length of an input. Consider the following function f. On input $(x, y) \in \Sigma^* \times \Sigma^*$, if $|x| \notin T$ and $|y| \notin T$, then f outputs the lexicographically minimal string between x and y. Otherwise, there are three possibilities: $|y| \leq \log\log|x|$, $|x| \leq \log\log|y|$, or $x = y$. If $x = y$, then output x. Assume that $|y| \leq \log\log|x|$. In this case, f outputs y if $M(y) = 1$ and f outputs x otherwise. Note that the running time of M on input y is bounded above by $2^{|y|^c+c}$, which is at most $|x|^d + d$ for a certain constant $d \geq 1$ depending only on c. For the last case, f outputs x if $M(x) = 1$ and y otherwise. Obviously, f is deterministically computed in polynomial time and thus, f belongs to FP. It is easy to verify that f satisfies the selectivity condition. □

Note that Lemma 6.12 relativizes. Using this lemma, we can prove Proposition 6.11.

Proof of Proposition 6.11. This proof is similar to that of Proposition 4.7. For each $k \in \mathbb{N}^+$, consider the set $L_k^A = \{0^n \mid n \in \hat{T} \wedge \forall y_1 \in \Sigma^n \exists . y_2 \in \Sigma^n \cdots Q_k y_k \in \Sigma^n [0^n y_1 y_2 \cdots y_k \notin A]\}$, where Q_k is $\exists$ if k is even and Q_k is $\forall$ otherwise. Note that, for any oracle A, L_k^A belongs to $\Pi_k^{\mathrm{P}}(A)$ and thus to EXP^A. Applying Lemma 6.12 to L_k^A, we obtain the P^A-selectivity of L_k^A. As noted in the proof of Proposition 4.7, we can build a recursive oracle A such that L_k^A is $\Sigma_k^{\mathrm{P}}(A)$-immune. From Lemma 6.6, it immediately follows that L_k^A is honestly $\Sigma_k^{\mathrm{P}}(A)$-hyperimmune. □

We also show a random oracle result of the existence of honestly NP-simple set using Lemmas 6.6 and 6.12.

Proposition 6.13 *With probability 1, an honestly* NP^X*-hypersimple set exists relative to a random oracle X.*

Proof. Recall from the proof of Proposition 4.8 that the set $K^A = \{1^n \mid n \in \hat{T} \wedge \forall w \in \Sigma^n \exists v \in \Sigma^{\lfloor \log n \rfloor}[wv \in A]\}$ is in co-NP^A for any oracle A. By Lemma 6.12, K^A is P^A-selective for any oracle A. We already noted in the proof of Proposition 4.8 that, with probability 1, K^X is NP^X-immune relative to a random oracle X. Since Lemma 6.6 relativizes, K^X is honestly NP^X-hyperimmune relative to a random oracle X. □

An important open problem is to prove that, at each level k of the polynomial-time hierarchy, an honest Σ_k^{P}-hypersimple set exists relative to a random oracle with probability 1.

7 Completeness under Non-Honest Reductions

Immunity has a deep connection to various completeness notions. For example, there is a simple, tt-complete set; however, no simple set is btt-complete (see, e.g., [31]). In the previous sections, we have shown that various types of resource-bounded simple sets cannot be complete under certain polynomial-time honest reductions. This section instead focuses on the incompleteness of simple sets under non-honest reductions.

To remove the honesty condition from reductions, we often need to make extra assumptions for similar incompleteness results. In mid 1980s, Hartmanis, Li, and Yesha [20] proved that (i) no NP-immune set in EXP is P-m-hard for NP if NP $\nsubseteq$ SUBEXP and (ii) no NP-simple set is P-m-complete if NP $\cap$ co-NP $\nsubseteq$ SUBEXP. These results can be expanded to any Δ-level of the polynomial-time hierarchy and of the subexponential-time hierarchy. We also improve the latter result, which follows from our main theorem on the Δ_k^{P}-1tt-completeness.

Proposition 7.1 *Let j and k be any nonnegative integers.*

1. *No Σ_k^{P}-immune set in Δ_j^{EXP} is Δ_k^{P}-m-hard for Σ_k^{P} if $\Sigma_k^{\mathrm{P}} \nsubseteq \mathrm{SUB}\Delta_{\max\{j,k\}}^{\mathrm{EXP}}$.*
2. *No Σ_k^{P}-simple set is Δ_k^{P}-m-complete for Σ_k^{P} if $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}) \nsubseteq \mathrm{SUB}\Delta_k^{\mathrm{EXP}}$.*

Note that Proposition 7.1(1) follows from Lemma 7.2(1) and Proposition 7.1(2) is a direct consequence of Theorem 7.4(2) since every Δ_k^{P}-m-reduction is also a Δ_k^{P}-1tt-reduction. We will prove Theorem 7.4(2) later.

The key idea of Hartmanis et al. [20] is to find a set that can be honestly reducible. The following lemma is a generalization of a technical part of [20]. For any reduction F and any set A, the *restriction of F on A* is the function defined as F on any input in A and "undefined" on input not in A.

Lemma 7.2 *Let $j,k \in \mathbb{N}^+$. Assume that $A \notin \mathrm{SUB}\Delta^{\mathrm{EXP}}_{\max\{j,k\}}$ and $B \in \Delta^{\mathrm{EXP}}_{j}$.*

1. *If A is Δ^{P}_{k}-m-reducible to B via f, then there exists a set C in Δ^{P}_{k} such that the restriction of f on C is honest and $A \cap C$ and $\overline{A} \cap C$ are infinite and coinfinite.*
2. *If A is Δ^{P}_{k}-1tt-reducible to B, then there exists a set C in Δ^{P}_{k} such that $A \cap C$ is h-Δ^{P}_{k}-1tt-reducible to B and $A \cap C$ and $\overline{A} \cap C$ are infinite and coinfinite.*

Before proving Lemma 7.2, we present a small lemma. For any function α from $\Sigma^* \times \{0,1\}$ to $\{0,1\}$, α_x denotes the function from $\{0,1\}$ to $\{0,1\}$ defined by $\alpha_x(y) = \alpha(x,y)$ for all $y \in \{0,1\}$.

Lemma 7.3 *Let $A, B \subseteq \Sigma^*$ and $j \in \mathbb{N}^+$. Let $r \in \{ktt, btt, tt, T\}$. The following three statements are equivalent.*

1. *A is h-Δ^{P}_{j}-r-reducible to B.*
2. *$A \leq^{\Delta^{\mathrm{P}}_{j}}_{r} B$ via a certain reduction whose restriction on A is componentwise honest.*
3. *$A \leq^{\Delta^{\mathrm{P}}_{j}}_{r} B$ via a certain reduction whose restriction on $\overline{A}$ is componentwise honest.*

Proof. Obviously, (1) implies both (2) and (3). It thus suffices to show that (2) implies (1) since we can show the implication of (3) to (1) in a similar way. We show only the case where $r = ktt$. Now, assume that A is Δ^{P}_{j}-$k$tt-reducible to B via a certain reduction pair (f,α) with the condition that the restriction of f on A is componentwise honest. Let $f_1, f_2, \ldots, f_k$ be the k $\mathrm{F}\Delta^{\mathrm{P}}_{j}$-functions satisfying that $f(x) = \langle f_1(x), f_2(x), \ldots, f_k(x)\rangle$ for every $x \in \mathrm{dom}(f)$. Let p be any polynomial such that $|x| \leq p(|f_i(x)|)$ for all $x \in A$ and for each $i \in [1,k]_{\mathbb{Z}}$. For simplicity, assume that $p(n) \geq n$ for any $n \in \mathbb{N}$. To show (1), we define the new reduction pair (g,β) as follows. Take an arbitrary string x and let $n = |x|$. If $n \leq p(|f_i(x)|)$ for all $i \in [1,k]_{\mathbb{Z}}$, then define $g(x)$ to be $f(x)$ and β_x to be α_x. Otherwise, define $g(x)$ to be the k-tuple $\langle x, \ldots, x\rangle$ (i.e., k x's) and β_x to be the constant 0 function (that is, $\beta_x(y) = 0$ for all $y \in \{0,1\}$). It therefore follows that (g,β) Δ^{P}_{j}-$k$tt-reduces A to B. □

Proof of Lemma 7.2. We show only the second claim. The proof for the first claim is similar except for the use of Lemma 7.3. For any index $i \in \mathbb{N}$, let K_i be any fixed P-m-complete set for Σ^{P}_{i}. For simplicity, let $i = \max\{j,k\}$. Assume that A is Δ^{P}_{k}-1tt-reducible to B via a reduction pair (f,α). Since f is a 1tt-reduction, we can define $f'(x)$ as $set(f(x))$ for each input x. For each number $\ell \in \mathbb{N}^+$, define the set $C_\ell = \{x \mid |x| \leq |f'(x)|^\ell\}$. Clearly, C_ℓ belongs to Δ^{P}_{k} since f is in $\mathrm{F}\Delta^{\mathrm{P}}_{k}$. Note that A is infinite and coinfinite because $A \notin \mathrm{SUB}\Delta^{\mathrm{EXP}}_{i}$.

Since $f' \in \mathrm{F}\Delta^{\mathrm{P}}_{k}$, there exists a polynomial-time deterministic oracle TM N which computes f with K_{k-1} as an oracle. Similarly, we have an exponential-time deterministic oracle TM M recognizing B with K_{j-1} as an oracle. Take a large enough number d and assume that the running time of N is at most $n^d + d$ and that of M is at most 2^{n^d+d} for all inputs of length n and all oracles. Now, we claim the following.

Claim. The sets $A \cap C_\ell$ and $\overline{A} \cap C_\ell$ are both infinite and coinfinite for all but finitely many numbers ℓ.

The desired set C is defined as C_ℓ for any fixed ℓ that satisfies the above claim. The claim guarantees that the restriction of f' on $A \cap C$ is (componentwise) honest. By Lemma 7.3, $A \cap C_\ell$ is h-Δ^{P}_{k}-1tt-reducible to B.

It still remains to prove the claim. We prove only the claim for $A \cap C_\ell$. First, note that $A \cap C_\ell$ is always coinfinite since $\overline{A \cap C_\ell}$ contains $\overline{A}$. Next, we show that $A \cap C_\ell$ is infinite for all but finitely many numbers ℓ. Assume to the contrary that there are infinitely-many numbers $\ell > 0$ for which $|x|^{1/\ell} > |f'(x)|$ for all but finitely-many strings x in A. Let L be the collection of all such ℓ's. Take any ℓ from L and fix an number n_ℓ satisfying that, for all strings $x \in \Sigma^{\geq n_\ell}$, $|x| \leq |f'(x)|^\ell$ implies $x \notin A$. Consider the following algorithm $\mathcal{A}$:

> On input x, let $n = |x|$. If $|x| \leq n_\ell$, then output $A(x)$. Otherwise, first compute $f'(x)$ by running $N^{K_{k-1}}$ on input x. If $|x| \leq |f'(x)|^\ell$, then reject x. If $|x| > |f'(x)|^\ell$, then compute $z = M^{K_{j-1}}(f'(x))$ and output the value $\alpha(z)$.

The running time of $\mathcal{A}$ on input x is at most $c(|x|^d + d + 2^{|x|^{d/\ell}+d})$ for a certain constant c independent of input. It thus follows that $\mathcal{A}$ recognizes A in time $O(2^{n^{d/\ell}})$ with access to oracle K_{i-1}. This means that A is in $\mathrm{DTIME}^{K_{i-1}}(2^{n^{d/\ell}})$. Since ℓ is arbitary in L, A belongs to $\bigcap_{\ell \in L} \mathrm{DTIME}^{K_{i-1}}(2^{n^{d/\ell}})$, which actually equals

$\mathrm{SUB}\Delta_i^{\mathrm{EXP}}$ since d is independent of ℓ and L is infinite. This contradicts our assumption that $A \notin \mathrm{SUB}\Delta_i^{\mathrm{EXP}}$. This complete the proof of the claim. □

The original result of Hartmanis et al. refers to the P-m-completeness of NP-simple sets. Recently, Schaefer and Fenner [38] showed a similar result for the P-1tt-completeness. They proved that no NP-simple set is P-1tt-complete for NP if UP $\not\subseteq$ SUBEXP. A key to their proof is the fact[‡] that $Sep(\mathrm{SUBEXP}, \mathrm{NP})$ implies UP $\subseteq$ SUBEXP, where $Sep(\mathcal{C}, \mathcal{D})$ means the separation property in [47] that, for any two disjoint sets $A, B \in \mathcal{D}$, there exists a set S in $\mathcal{C} \cap \text{co-}\mathcal{C}$ satisfying that $A \subseteq S \subseteq \overline{B}$.

The following theorem shows that the assumption "UP $\not\subseteq$ SUBEXP" in [38] can be replaced by "U(NP $\cap$ co-NP) $\not\subseteq$ SUBEXP."

Theorem 7.4 *Let $j, k \in \mathbb{N}^+$.*

1. *No Σ_k^{P}-immune set in Δ_j^{EXP} is Δ_k^{P}-1tt-hard for $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$ if $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}) \not\subseteq \mathrm{SUB}\Delta_{\max\{j,k\}}^{\mathrm{EXP}}$.*
2. *No Σ_k^{P}-simple set is Δ_k^{P}-1tt-complete for Σ_k^{P} if $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}) \not\subseteq \mathrm{SUB}\Delta_k^{\mathrm{EXP}}$.*

We give the proof of Theorem 7.4. Our proof follows from Lemmas 7.2, 7.3, and 7.5.

Lemma 7.5 *Let $k \in \mathbb{N}^+$ and assume that A is h-Δ_k^{P}-1tt-reducible to a Σ_k^{P}-immune set B.*

1. *If A belongs to Σ_k^{P}, then there exists a set $C \in \Delta_k^{\mathrm{P}}$ and a total function $f \in \mathrm{F}\Delta_k^{\mathrm{P}}$ such that $\overline{A} \cap \overline{C}$ is finite, f Δ_k^{P}-m-reduces A to $\overline{B}$, and the restriction of f on C is honest.*
2. *A belongs to $\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}$ if and only if A belongs to Δ_k^{P}.*

Proof. 1) Assume that A is in Σ_k^{P}, B is Σ_k^{P}-immune, and $A \leq_{1tt}^{h\text{-}\Delta_k^{\mathrm{P}}} B$ via a reduction pair (f, α), where f maps Σ^* to Σ^* and α maps $\Sigma^* \times \{0,1\}$ to $\{0,1\}$. Since $f(x)$ is always of the form $\langle y \rangle_1$, for brevity, we identify $\langle y \rangle_1$ with y itself. With this identification, we simply write $y = f(x)$ instead of $\langle y \rangle_1 = f(x)$. It thus follows that $A = \{x \mid \alpha_x(B(f(x))) = 1\}$. For convenience, let ONE and FLIP be the functions from $\{0,1\}$ to $\{0,1\}$ defined as $\mathrm{ONE}(y) = 1$ and $\mathrm{FLIP}(y) = 1 - y$ for all $y \in \{0,1\}$.

First, we want to show the existence of a certain number $n_0 \in \mathbb{N}$ such that, for any string $x \in A$ with $|x| \geq n_0$, α_x becomes either ONE or FLIP. Consider the set $A_0 = \{x \in A \mid \alpha_x(0) = 0\}$. It follows that $f(A_0) \subseteq B$ because $y = f(x)$ and $x \in A_0$ imply $\alpha_x(B(y)) \neq \alpha_x(0)$, which leads to $B(y) = 1$. Since f is honest, the set $f(A_0)$ belongs to Σ_k^{P}. The Σ_k^{P}-immunity of B requires $f(A_0)$ to be finite and thus, A_0 is finite because of the honesty condition of f. The desired number n_0 can be defined as the minimal number such that $|x| < n_0$ for all $x \in A_0$.

Using this n_0, A can be expressed as the union of the following three disjoint sets $A_1 = A \cap \{0,1\}^{<n_0}$, $A_2 = \{x \mid |x| \geq n_0 \wedge \alpha_x = \mathrm{ONE}\}$, and $A_3 = \{x \mid |x| \geq n_0 \wedge \alpha_x = \mathrm{FLIP} \wedge f(x) \notin B\}$. Obviously, A_1 is finite and A_2 is in Δ_k^{P} since α is in $\mathrm{F}\Delta_k^{\mathrm{P}}$. Now, define $C = \Sigma^{\geq n_0} \cap \overline{A_2}$. Note that $\overline{C} = \Sigma^{<n_0} \cup A_2$. Clearly, C is in Δ_k^{P} since so is A_2. Note that $\overline{A} \cap \overline{C}$ is finite because $\overline{A} \cap \overline{C} \subseteq \Sigma^{<n_0}$. Now, we explicitly define the Δ_k^{P}-m-reduction g from A to $\overline{B}$ as follows. Let z_0 and z_1 be any two fixed strings such that $z_0 \notin B$ and $z_1 \in B$. Let $g(x) = z_0$ if $x \in A_1 \cup A_2$; $g(x) = z_1$ if $x \in \overline{A_1} \cap \Sigma^{<n_0}$; $g(x) = f(x)$ if $x \in C$. The restriction g on C is honest because f is honest. It is clear that g Δ_k^{P}-m-reduces A to $\overline{B}$.

2) It suffices to show the "only if" part. Assume that A belongs to $\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}$. Take the number n_0 and the set A_3 defined in the proof of (1) since A is in Σ_k^{P}. Consider the set $C = \{x \mid |x| \geq n_0 \wedge \alpha_x = \mathrm{FLIP} \wedge f(x) \in B\}$. Note that C equals $\{x \mid |x| \geq n_0 \wedge \alpha_x = \mathrm{FLIP} \wedge x \notin A\}$. Since $A \in \Pi_k^{\mathrm{P}}$ and $\alpha \in \mathrm{F}\Delta_k^{\mathrm{P}}$, C belongs to Σ_k^{P}. Moreover, since B is Σ_k^{P}-immune and f is honest, C should be finite. Therefore, for all but finitely many strings x, $\alpha_x = \mathrm{FLIP}$ if and only if $x \in A_3$. Therefore, A_3 belongs to Δ_k^{P} and, as a consequence, A belongs to Δ_k^{P}. □

Finally, we present the proof of Theorem 7.4. Recall from [47] that $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}) \subseteq \Sigma_k^{\mathrm{P}}$ at any level $k \in \mathbb{N}^+$.

Proof of Theorem 7.4. We show only the first claim because the second claim follows from the first one. Now, take arbitrary numbers $j, k \in \mathbb{N}^+$ and let $i = \max\{j, k\}$. Assume that B is Σ_k^{P}-immune and in Δ_j^{EXP} and that B is Δ_k^{P}-1tt-hard for $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$. Moreover, assume that there exists a set A in $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$ but not in $\mathrm{SUB}\Delta_i^{\mathrm{EXP}}$. We want to derive a contradiction from these assumptions.

Since $A \in \mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$, A is of the form $\{x \mid \exists y[\langle x, y \rangle \in \mathit{Graph}(f)]\}$ for a certain polynomially-bounded partial function f whose graph is in $\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}$. Take a polynomial p such that $|f(x)| \leq p(|x|)$ for any $x \in \mathrm{dom}(f)$.

[‡]Actually, the result of Schaefer and Fenner can be strengthened as follows: $Sep(\mathrm{SUBEXP}, \mathrm{U}(\mathrm{NP} \cap \text{co-NP}))$ if and only if $\mathrm{U}(\mathrm{NP} \cap \text{co-NP}) \subseteq \mathrm{SUBEXP}$. This is obtained by analyzing a similar result in [47].

Similar to [47], define $A_1 = \{\langle x,z\rangle \mid \exists y \in \Sigma^{\leq p(|x|)}[z \leq y \wedge \langle x,y\rangle \in \mathit{Graph}(f)]\}$ and $A_2 = \{\langle x,z\rangle \mid \exists y \in \Sigma^{\leq p(|x|)}[z > y \wedge \langle x,y\rangle \in \mathit{Graph}(f)]\}$. Clearly, A_1 and A_2 are mutually disjoint and in $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}) \subseteq \Sigma_k^{\mathrm{P}}$. Notice that $A \leq_1^{\mathrm{P}} A_1$ and $A \leq_1^{\mathrm{P}} A_2$. Hence, neither of A_1 and A_2 belong to $\mathrm{SUB}\Delta_i^{\mathrm{EXP}}$. Because of the Δ_k^{P}-1tt-hardness of B, A_1 is Δ_k^{P}-1tt-reducible to B. By Lemmas 7.3 and 7.2, we can take a set C in Δ_k^{P} such that $A_1 \cap C$ is infinite and coinfinite and $A_1 \cap C$ is h-Δ_k^{P}-1tt-reducible to B. Moreover, since $A_1 \cap C$ is in Σ_k^{P}, by Lemma 7.5(1), there exist a set $D \in \Delta_k^{\mathrm{P}}$ and a total $\mathrm{F}\Delta_k^{\mathrm{P}}$-function g such that $\overline{A_1 \cap C} \cap \overline{D}$ is finite, g Δ_k^{P}-m-reduces $A_1 \cap C$ to $\overline{B}$, and the restriction of g on D is honest. Since $A_2 \cap D$ is a subset of $\overline{A_1 \cap C} \cap D$, the image $g(A_2 \cap D)$ is in B. Moreover, $A_2 \cap D$ is infinite because, otherwise, from the inclusion $A_2 \cap \overline{D} \subseteq \overline{A_1 \cap C} \cap \overline{D}$, A_2 becomes finite. Since g is honest on the domain D, $g(A_2 \cap D)$ must be infinite. Since $g(A_2 \cap D)$ is in Σ_k^{P}, B cannot be Σ_k^{P}-immune, a contradiction. □

A natural open question arising from Theorem 7.4 is to determine whether the assumption $\mathrm{U}(\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}})$ used in the theorem is practically optimal in such a sense that there exists a relativized world in which $\Sigma_k^{\mathrm{P}} \nsubseteq \mathrm{SUB}\Delta_k^{\mathrm{EXP}}$ but a Σ_k^{P}-simple Δ_k^{P}-1tt-complete set for Σ_k^{P} exists.

8 Limited Immunity and Simplicity

Within our current knowledge, we cannot prove or disprove the existence of an NP-simple set. The difficulty comes from the fact that an NP-immune set requires *every* NP-subset to be finite. If we restrict our attention to certain types of NP-subsets, then we may overcome the difficulty. Under the name of k-immune sets, Homer [23] required only $\mathrm{NP}_{(k)}$-subsets, for a fixed number k, to be finite. He then demonstrated how to construct a k-simple set within NP. In this section, we investigate the notions obtained by restricting the conditions of immunity and simplicity. We first review Homer's notions of k-immunity and k-simplicity.

Recall from Section 2 the identification between Σ^* and $\mathbb{N}$. Here, we freely identify binary strings with natural numbers. Hence, the notation $|i|$ for any number $i \in \mathbb{N}$ means the length of the string that is identified with i. Note that $|i| = \lfloor \log(i+1) \rfloor$ for any number $i \in \mathbb{N}$.

Definition 8.1 [23] Let $k \in \mathbb{N}^+$.

1. A set S is *k-immune* if S is infinite and there is no index i in $INDEX_{(k)}$ such that $W_i \subseteq S$ and W_i is infinite.
2. A set S is *k-simple* if S is in NP and $\overline{S}$ is k-immune.

Obviously, any k-immune set becomes k'-immune for any $k' \leq k$ since $INDEX_{(k')} \subseteq INDEX_{(k)}$. Similarly, any k-simple set is k'-simple if $k' \leq k$.

Homer [23] constructed a k-simple set for each $k \in \mathbb{N}^+$ using Ladner's [29] delayed diagonalization technique. He proceeded the diagonalization process slowly to construct a k-simple set A that satisfies the following property: for each number $n \in \mathbb{N}$, the cardinality $|A \cap \Sigma^n|$ is at most $2\log n$. By performing his diagonalization process much more slowly, we can obtain the following lemma.

Lemma 8.2 *For every integer $k \geq 1$, there exists a k-simple set A with the following property: $|A \cap \Sigma^n|$ is at most $2\log\log n$ for each number $n \in \mathbb{N}$.*

An "effective" version of immune and simple sets, called *effectively immune* and *effectively simple sets*, has been studied in recursion theory. Effectively simple sets are known to be T-complete and there also exists an effectively simple tt-complete set. If A is strongly effectively immune, then $\overline{A}$ cannot be immune. See, e.g., [31] for more details. Recently, Ho and Stephen [22] constructed a simple set to which any effectively simple set can be 1-reducible. Analogously, we consider a resource-bounded version of such effectively immune and simple sets.

Definition 8.3 Let $k \in \mathbb{N}^+$.

1. A set S is *feasibly k-immune* if (i) S is infinite and (ii) there exists a polynomial p such that, for all indices $i \in INDEX_{(k)}$, $W_i \subseteq S$ implies $|W_i| \leq 2^{p(i)}$.
2. A set S is *feasibly k-simple* if $S \in \mathrm{NP}$ and $\overline{S}$ is feasibly k-immune.

Obviously, every feasibly k-immune set is k-immune for any number $k \in \mathbb{N}^+$. Using a straightforward diagonalization, we can construct a feasibly k-immune set that falls in Δ_2^{P}.

Proposition 8.4 *Let $k \in \mathbb{N}^+$. There exists a feasibly k-immune set in Δ_2^{P}.*

The desired set A that we will construct in the following proof is not honest P-hyperimmune because A satisfies $|\overline{A} \cap \Sigma^n| \leq n$ for all numbers $n \in \mathbb{N}^+$.

Proof of Proposition 8.4. First, we fix k arbitrarily. For each $i, s \in \mathbb{N}$, let $\varphi_{i,s}$ be the machine obtained from φ_i by restricting its running time as follows: let $\varphi_{i,s}(x) = \varphi_i(x)$ if $\varphi_i(x)$ terminates within step s and let $\varphi_{i,s}(x)$ be undefined otherwise. We wish to construct the desired feasibly k-simple set $A = \bigcup_{i\in\mathbb{N}} A_i$ by stages, where each A_i is a subset of $\Sigma^{\leq n}$.

During our construction process, we intend to meet the following two requirements:

1) $R_i^{(0)}$: $|\overline{A} \cap \Sigma^i| \leq i$.
2) $R_i^{(1)}$: if $|W_i| > 2^{i+1}$ and $i \in INDEX_{(k)}$, then $W_i \cap \overline{A} \neq \emptyset$.

The first requirement makes A infinite. The second requirement implies that, for any index i in $INDEX_{(k)}$, if $W_i \subseteq A$ then $|W_i| \leq 2^{i+1}$. From these two requirements, we therefore obtain the feasible k-immunity of A. We also build the set $MARKED_i$ to mark all used indices at stage i and finally let $MARKED = \bigcup_{i\in\mathbb{N}} MARKED_i$. For any finite subset B of Σ^*, the notation $\max(B)$ simply denotes the lexicographically maximal string in B. Since $B \subseteq \Sigma^{\leq|\max(B)|}$, it holds that $|B| \leq |\Sigma^{\leq|\max(B)|}| \leq 2^{|\max(B)|+1}$. For brevity, write $\ell(i,j,k)$ for $|i|\cdot|j|^k + |i|$. Our construction proceeds as follows:

Stage 0: Let $A_0 = \emptyset$ and $MARKED_0 = \emptyset$.

Stage $n \geq 1$: At this stage, we consider all the strings of length n. Consider the following set:

$$C = \{\langle i,j\rangle \mid i \leq |j| \ \wedge\ \varphi_{i,\ell(i,j,k)}(j)\!\downarrow = 1 \ \wedge\ |\{x < j \mid \varphi_{i,\ell(i,x,k)}(x)\!\downarrow = 1\}| > i\}.$$

For each $i \leq n$, let $j_{n,i}$ be the lexicographically maximal string j in Σ^n such that $\langle i,j\rangle \in C$ if j exists. Otherwise, let $j_{n,i}$ be undefined. Consider the set C_n of all $i \leq n$ for which $j_{n,i}$ exists. Define $MARKED_n$ to be $MARKED_{n-1} \cup C_n$. Finally, define A_n to be $A_{n-1} \cup (\Sigma^n \setminus \{j_{n,i} \mid i \in C_n\})$.

First, we show that C belongs to NP. Obviously, we need an NP-computation to determine whether $\varphi_{i,\ell(i,j,k)}(j)\!\downarrow = 1$. Moreover, we can determine whether the cardinality $|\{x < j \mid \varphi_{i,\ell(i,x,k)}(x)\!\downarrow = 1\}|$ is more than i by nondeterministically guessing $i+1$ x's that satisfy $\varphi_{i,\ell(i,x,k)}(x)\!\downarrow = 1$. Since $i+1 \leq |j|+1$, this process requires only an NP-computation. Therefore, C belongs to NP. To compute each set A_i, we need to determine all the elements in C_i, i.e., all the well-defined $j_{n,i}$'s. This is done by a standard binary search technique using C as an oracle in polynomial time. Therefore, A is in $\mathrm{P}^C \subseteq \mathrm{P}^{\mathrm{NP}}$.

It remains to show that the two requirements $R_i^{(0)}$ and $R_i^{(1)}$ are eventually satisfied for every index i.

Claim. $R_i^{(0)}$ is satisfied at each stage i.

Note that $|C_i| \leq i$ at each stage i. Hence, we obtain $|\overline{A} \cap \Sigma^i| \leq |C_i| \leq i$, which clearly satisfies the requirement $R_i^{(0)}$.

Claim. $R_i^{(1)}$ is satisfied for any number i.

To prove this claim, it suffices to show that, for any index i in $INDEX_{(k)}$, if $|W_i| > 2^{i+1}$ then i is in $MARKED$ because $i \in MARKED$ means that, at a certain stage n, there exists the unique string $j_{n,i}$ in Σ^n such that $j_{n,i} \in W_i$ and $j_{n,i} \in \overline{A}$. Now, suppose that $i \in INDEX_{(k)}$ and $|W_i| > 2^{i+1}$. Recall that $W_i = \{x \mid \varphi_{i,\ell(i,x,k)}(x)\!\downarrow = 1\}$.

(Case 1) Consider the case where W_i is finite and $i > |\max(W_i)|$. This case never happens because $|W_i| \leq 2^{|\max(W_i)|+1} < 2^{i+1}$, a contradiction

(Case 2) Assume that either W_i is ininfite or $i \leq |\max(W_i)|$. We split this case into two subcases.

(Case 2a) Consider the case where W_i is finite and $i \leq |\max(W_i)|$. Define j_0 as $\max(W_i)$. Now, let $n = |j_0|$. We show that $i \in MARKED$. Assume otherwise that $i \notin MARKED$. Since $j_0 \in W_i$, we have $\varphi_{i,\ell(i,j_0,k)}(j_0)\!\downarrow = 1$. Clearly, $i \leq |j_0|$. Recall that $C_n = \{i \mid i \leq n \ \wedge\ j_{n,i} \text{ exists}\}$. Since $i \notin MARKED$, i cannot belong to C_n. This means that $j_{n,i}$ does not exist. Thus, we obtain $\langle i,j\rangle \notin C$ for any $j \in \Sigma^n$. In particular, $\langle i,j_0\rangle \notin C$ since $j_0 \in \Sigma^n$. By the definition of C, we conclude that $|\{x \in W_i \mid x < j_0\}| \leq i$. It follows from

$j_0 = \max(W_i)$ that $|W_i| = |\{x \in W_i \mid x < j_0\}| + 1 \leq i + 1$, which implies $|W_i| \leq 2^{i+1}$, a contradiction. Hence, i is in $MARKED$.

(Case 2b) In the case where W_i is infinite, let j_1 be the lexicographically $2^{i+1} + 1$st string in W_i. Assume that $i \notin MARKED$. Similar to Case 2a, this leads to the conclusion that $|\{x \in W_i \mid x < j_1\}| \leq i$. However, the choice of j_1 implies $|\{x \in W_i \mid x < j_1\}| = 2^{i+1} > i$, a contradiction. Therefore, i is in $MARKED$, as requested. □

From Definition 8.3, it follows that every feasibly k-immune set is k-immune. The converse, however, does not hold since there exists a k-simple set which is not feasibly k-simple for each number $k \in \mathbb{N}^+$. The theorem below is slightly stronger than this claim since any feasibly k-simple set is feasibly 1-simple.

Theorem 8.5 *For each $k \in \mathbb{N}^+$, there exists a k-simple set which is not feasibly 1-simple.*

In what follows, we prove Theorem 8.5. To show this, we use the language $L_u = \{uv \mid \exists m \in \mathbb{N}[|uv| = 2^{2^m}] \wedge \log\log|uv| = |u|\}$, where u is any fixed string. Observe that, for each string u, there exists a Gödel number i of length $O(|u|)$ for which φ_i recognizes L_u. More precisely, the following lemma holds.

Lemma 8.6 *There exists a number a in $\mathbb{N}$ that satisfies the following condition: for every string u with $|u| \geq a$, there exists an index i such that (i) $W_i = L_u$, (ii) $|u| < |i| < a(|u|+1)$, and (iii) the running time of φ_i on input x is at most $|i|(|x|+1)$ for all $x \in W_i$.*

Lemma 8.6 can be shown by directly constructing an appropriate deterministic TM that recognizes L_u. Now, we are ready to give the proof of Theorem 8.5.

Proof of Theorem 8.5. Let $k \geq 1$ and let A be any k-simple set that satisfies Lemma 8.2. We want to show that A is not feasibly 1-simple. Write $L_u = \{uv \mid \exists m \in \mathbb{N}[|uv| = 2^{2^m}] \wedge \log\log|uv| = |u|\}$ for each string $u \in \Sigma^*$.

Take any natural number a for which Lemma 8.6 holds. Consider the following claim.

Claim. For each number $\ell \in \mathbb{N}$, there exists an index i such that the following four conditions hold: (i) $W_i \subseteq \overline{A}$, (ii) $\ell < |i|$, (iii) $\log\log(|W_i|) \geq (i^{1/a})/2 - 1$, and (iv) the running time of φ_i on input x is at most $|i|(|x|+1)$ for any string x in W_i.

Assume for a while that the above claim holds. The claim guarantees that, for every polynomial p, there exists an index $i \in INDEX_{(1)}$ satisfying that $W_i \subseteq \overline{A}$ and $|W_i| > 2^{p(i)}$. Therefore, $\overline{A}$ is not feasibly 1-immune.

Now, we prove the claim. Let ℓ and m be any natural numbers satisfying that $m \geq \max\{a, \ell\}$ and $2m < 2^m$. Define $n = 2^{2^m}$, which is equivalent to $m = \log\log n$. The condition $|A \cap \{0,1\}^n| \leq 2m$ implies that the cardinality of the set $\{uv \mid |uv| = n \wedge |u| = m\}$ is at most $2m$. The set $\{u \in \Sigma^m \mid L_u \cap A \neq \emptyset\}$ therefore has the cardinality at most $2m$. Since $2m < 2^m$, we have at least one string u of length m satisfying that $L_u \subseteq \overline{A}$. Fix such a string u. Clearly, $|L_u| = 2^{n-m} \geq 2^{n/2}$ since $2m < n$. By Lemma 8.6, a certain index i satisfies the following three conditions: $W_i = L_u$, $m < |i| < a(m+1)$, and the running time of φ_i on input x is at most $|i|(|x|+1)$ for any $x \in W_i$. Hence, it follows that $(2\log n)^a = 2^{a(m+1)} \geq 2^{|i|+1} \geq i$, which implies $\log n \geq \frac{1}{2} i^{1/a}$. Therefore, $\log\log(|W_i|) \geq \log(n/2)$, which is at least $\frac{i^{1/a}}{2} - 1$. This completes the proof of the claim. □

We return to the old question of whether NP-simple sets exist. Unfortunately, there seems no strong evidence that suggests the existence of such a set. Only relativization provides a world where NP-simple sets exist. At the same time, we can construct another world where these sets do not exist. These relativization results clearly indicate that the question of whether NP-simple sets exist needs unrelativizable proof techniques.

In the past few decades, the *Berman-Hartmanis isomorphism conjecture* [9] has served as a working hypothesis in connection to NP-complete problems. By clear contrast, there has been no "natural" working hypothesis that yields the existence of NP-simple sets. For example, the hypothesis P $\neq$ NP does not suffice since Homer and Maass [24] showed a relativized world where the assumption P $\neq$ NP does not imply the existence of an NP-simple set. Motivated by Homer's k-simplicity result, we propose the following working hypothesis:

> The k-immune hypothesis: There exists a positive integer k such that every infinite NP set has an infinite $\mathrm{NP}_{(k)}$-subset.

Under this hypothesis, we can derive the desired consequence: the existence of NP-simple sets.

Lemma 8.7 *If the k-immune hypothesis holds, then there exists an* NP*-simple set.*

Proof. Assume that the k-immune hypothesis is true. There exists a positive integer k such that every infinite NP-set has an infinite $\mathrm{NP}_{(k)}$-subset. Consider any k-simple set A. We claim that A is NP-simple. If A is not NP-simple, then $\overline{A}$ has an infinite NP-subset B. By our hypothesis, B contains an infinite $\mathrm{NP}_{(k)}$-subset. Hence, A cannot be k-simple, a contradiction. Therefore, A is NP-simple. □

To close this section, we claim that the k-immune hypothesis fails relative to a Cohen-Feferman generic oracle.

Proposition 8.8 *The k-immune hypothesis fails relative to a Cohen-Feferman generic oracle.*

In the following proof, we use *weak forcing* as in the proof of Theorem 4.9. We also assume that, for each number $k \in \mathbb{N}^+$, there exists a list $\langle N_{(k),i} \mid i \in \mathbb{N} \rangle$ of nondeterministic oracle Turing machines such that $\mathrm{NP}^A_{(k)} = \{L(N_{(k),i}, A) \mid i \in \mathbb{N}\}$ for any oracle A, and each $N_{(k),i}$ is clocked by $|i| \cdot n^k + |i|$, where n is the length of an input. In the following proof, we fix such a list.

Proof of Proposition 8.8. Let k be any number in $\mathbb{N}^+$. Letting G be any Cohen-Feferman generic set, we show the existence of an infinite set in P^G that has no infinite $\mathrm{NP}^G_{(k)}$-subset. Consider the set $L^X_k = \{x \mid x0^{|x|^{k+2}} \in X\}$ for each oracle X. Clearly, L^G_k belongs to P^G. For each fixed number $n \in \mathbb{N}$, let $\mathcal{D}^n_1$ denote the set of all forcing conditions σ that force "$\exists x \in \Sigma^* \, [x \in \Sigma^{<n} \vee x0^{|x|^{k+2}} \in X]$," where X is a variable running over all subsets of Σ^*. Since $\mathcal{D}^n_1$ is obviously dense for every n, L^G_k is indeed an infinite set.

Now, let S be any $\mathrm{NP}^G_{(k)}$-subset of L^G_k. We need to show that S is finite. Since $S \in \mathrm{NP}^G_{(k)}$, there exists an index $i \in \mathbb{N}$ such that $S = L(N_{(k),i}, G)$. Take any sufficiently large natural number c such that $|i| \cdot |x|^k + |i| < |x|^{k+2}$ for every $x \in \Sigma^*$ with $|x| \geq \max\{|i|, c\}$. Hence, the machine $N_{(k),i}$ does not query any string of length $\geq |x|^{k+2}$ on input x with any oracle A. Consider the following two statements:

- $\varphi_0(X) :\equiv \exists n \in \omega \, \forall x \in \Sigma^* \, [x \notin L(N_{(k),i}, X) \vee |x| \leq n]$, and
- $\varphi_1(X) :\equiv \exists x \in \Sigma^* \, [x \in L(N_{(k),i}, X) \wedge x0^{|x|^{k+2}} \notin X]$.

Let $\mathcal{D}_2$ be the set of all forcing conditions σ that force $(\varphi_0(X) \vee \varphi_1(X))$. We want to claim that $(\varphi_0(G) \vee \varphi_1(G))$ is true. To prove this claim, it suffices to show that $\mathcal{D}_2$ is dense.

Now, we want to prove that $\mathcal{D}_2$ is dense. In case where there is an extension τ of σ that forces $\varphi_0(X)$, this τ is in $\mathcal{D}_2$ and also forces $(\varphi_0(X) \vee \varphi_1(X))$. Next, assume that no extension of σ forces $\varphi_0(X)$. By the property of weak forcing, this implies that σ forces $\neg\varphi_0(X)$, which means "$\forall n \in \omega \, \exists x \in \Sigma^* \, [N^X_{(k),i}(x) = 1 \wedge |x| > n]$." Choose any sufficiently large number n_0 such that σ forces "$\exists x \in \Sigma^* \, [x \in L(N_{(k),i}, X) \wedge |x| > n_0]$." Furthermore, choose a string x and an extension ρ of σ such that ρ forces "$N^X_{(k),i}(x) = 1 \wedge |x| > n_0$." Since N^ρ on input x cannot query any string of length $\geq |x|^{k+2}$, we can assume that $\mathrm{dom}(\rho)$ does not include $x0^{|x|^{k+2}}$. Therefore, we can define the extension τ of ρ such that $\mathrm{dom}(\tau) = \mathrm{dom}(\rho) \cup \{x0^{|x|^{k+2}}\}$ and $\tau(x0^{|x|^{k+2}}) = 0$. Clearly, τ forces $\varphi_1(X)$ and thus, it forces $(\varphi_0(X) \vee \varphi_1(X))$. Therefore, $\mathcal{D}_2$ is dense.

Since G is generic, either $L(N_{(k),i}, G)$ is finite or $L(N_{(k),i}, G) - L^G_k$ is nonempty. Recall that $S = L(N_{(k),i}, G)$. Since S is a subset of L^G_k, S is therefore finite, as required. □

9 Epilogue

The class NP and the polynomial-time hierarchy built over NP have been a major subject in the theory of computational complexity since their introduction to the theory. Since an early work of Flajolet and Steyaert [16], the notions of resource-bounded immunity and simplicity have made significant contributions to the development of the theory. In this paper, we have further explored these notions for a better understanding of the classes lying in the polynomial-time hierarchy. Although our research is foundational in nature, we hope that our research will draw general-audience's attention to the importance of resource-bounded immunity and simplicity. There are, of course, many open problems waiting to be solved. For the future research, we list six important directions on the study of resource-bounded immunity and simplicity.

1. Separating weak notions of completeness. The completeness notions in general have played an essential role in computational complexity theory for the past three decades. Recall that Post originally introduced immunity and simplicity in an attempt to fill the gap between the class of recursive sets and the class of complete sets. One of the open problems is the differentiation of weak completeness within the polynomial-time hierarchy. We hope that weak completeness notions can be separated by different types of simplicity notions.
2. Finding new connections to other notions. Although we have mentioned a close connection of immunity to several notions, such as complexity cores and instance complexity, we do not know many connections to the existing complexity-theoretical notions. We still need to discover new connections to any other important notions; for instance, resource-bounded measure and pseudorandom generators.
3. Studying the nature of classes of immune sets. We have shown several closure properties of the classes of certain types of immune sets under weak reductions. For a better understanding of immunity, it is also important to study *classes* of immune sets rather than each individual immune set alone. There are few systematic studies along this line. We hope to discover useful closure properties that are unique to these classes of immune sets. Such properties may reveal the characteristics of immune sets.
4. Exploring limited immunity and simplicity notions. We have reopened the study of k-immunity and further introduced the notion of feasibly k-immunity to analyze the structure of NP. An open question is to determine whether there exists a feasible k-simple set. Along a similar line of research, we can naturally introduce strongly k-immunity, almost k-immunity, and k-hyperimmunity. We hope to explore these notions and study their roles within NP.
5. Exploring new relativized worlds. An oracle separation by immune or simple sets is sometimes called a *strong separation*. Such a separation usually requires intricate tools and proof techniques. We need to develop new tools and techniques to exhibit tight separations of complexity classes. An important open problem, for instance, is to find a relativized world where there exists a $\Sigma_k^{\mathrm{P}} \cap \Pi_k^{\mathrm{P}}$-immune Σ_k^{P}-simple set for every $k \geq 2$.
6. Discovering good working hypotheses. We have proposed the k-immune hypothesis, which implies the existence of an NP-simple set. We hope that finding a good working hypothesis will boost the research on immunity and simplicity.

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