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Crown reductions for the Minimum Weighted Vertex Cover problem

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Abstract. The paper studies crown reductions for the Minimum Weighted Vertex Cover problem introduced recently for the unweighted case by Fellows et al. ([15], [1]). We show a close relation of crown reductions to Nemhauser and Trotter reductions based on the linear programming relaxation of the problem. So called strong crown reductions, suitable for finding (or counting) all minimum vertex covers, or finding a minimum vertex cover under some additional constraints, are also introduced and studied. We show how crown decompositions and strong crown decompositions can be computed in polynomial time. For weighted König-Egerváry graphs (G, w) we show how the set of vertices belonging to all minimum vertex covers, and the set of vertices belonging to no minimum vertex covers, can be efficiently computed.
Further, for some specific classes of graphs, simple algorithms for the Min-VC problem with a constant approximation factor $r < 2$ are provided. On the other hand, we conclude that for the regular graphs, or for the Hamiltonian connected graphs, the problem is as hard to approximate as for general graphs.
It is demonstrated how the results about strong crown reductions can be used to achieve a linear size problem kernel for some related vertex cover problems.

1 Introduction

The Minimum (weighted) Vertex Cover problem (shortly, Min-w-VC) is one of the fundamental NP-hard problems in the combinatorial optimization. As it cannot be solved exactly in polynomial time, unless P = NP, approaches have concentrated on the design of efficient approximation algorithms. In spite of a great deal of efforts, the tight bound on its approximability by a polynomial time algorithm is left open. Recall that the problem has a simple 2-approximation algorithm and, for any constant $r < 2$, no r-approximation algorithm is known, even in the unweighted case. Currently the best lower bound on polynomial time approximability is $10\sqrt{5} - 21 \approx 1.36067$, due to Dinur and Safra [12]. More precisely, to achieve smaller approximation factor is NP-hard.

Recently, there have been increasing interest and progress in lowering the exponential running time of algorithms that solve NP-hard optimization problems, like Min-VC, exactly. The theory of parametrized computation and fixed parameter tractability is a newly developed approach dealing with exact algorithms for such intractable problems. Many hard problems can be associated with a parameter in such way that the problems are tractable when the parameter is fixed or varies within a small range. Such parametrized problems are now known as fixed parameter tractable (FPT) [13]. The parametrized version of the Vertex Cover problem is a well known FPT problem and has received considerable interest: for a given graph and a positive integer k, the problem is to find a vertex cover of weight at most k or to report that no such vertex cover exists.

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Very important methods employed in the development of algorithms (both, exact and approximation) are reductions to problem kernel. These are efficient procedures that in approximation preserving way reduce input instances to instances of smaller size and special structure. Such techniques are referred to as *kernelization.* For example, in parametrized version of the MIN-VC problem they reduce in conjunction or independently both, the graph size and the parameter size. After kernelization as a preprocessing step, the branch-and-search process based on bounded search trees can be applied to design an exact algorithm, that is efficient if the parameter is fixed. Special structure of the problem kernel usually allows to design a simple polynomial time approximation algorithm for the general problem.

Historically, in kernelization techniques for the MIN-w-VC problem the role of the linear programming (LP) relaxation of the problem has been crucial. As it was firstly observed by Nemhauser and Trotter ([22]), any optimal solution to the corresponding linear program provides such reduction in a simple way. Moreover, the rounding procedure provides a 2-approximation algorithm. An interesting underlying structure of the problem is that any extreme point of the feasible region is half-integral. This additional structure helps to improve the efficiency of the reduction, so called *Nemhauser-Trotter* (NT) reduction, as the problem of finding a minimum half-integral solution can be reduced to the bipartite MIN-w-VC problem. Such problems can be solved as the MAXIMUM FLOW problems much faster than general linear programs.

A new technique, called the *crown reduction*, has been recently introduced by Fellows et al. ([15], [1]) in kernelizing of the parametrized (unweighted) MIN-VC problem. Seemingly, it is a very different approach from NT-reductions that are based on the LP-relaxation of the problem. This paper provides the first systematic study of crown reductions. It deals with the more general case of vertex weighted graphs. We prove, quite surprisingly, that crown reductions are closely related to the LP-relaxation of the problem. In fact, the crown reduction technique is essentially equivalent to the one based on NT-reductions. In particular, the problem kernels (i.e., instances that are irreducible) are the same for both approaches. We also introduce and study more special *strong crown reductions* that are suitable for finding (or counting) *all* minimum vertex covers, or finding a minimum vertex cover *under some additional constraints.* The paper is essentially self-contained and in some cases it contains also new (but different) proofs of some previously known results about the minimum half-integral vertex covers. We include them because we believe that our approach provides a better insight.

Preliminaries. Let $G = (V, E)$ be a graph with vertex weights $w : V \to (0, \infty)$. For a set of vertices $U \subseteq V$, let $N(U) := N_G(U) = \{v \in V : \exists u \in U \text{ such that } \{u, v\} \in E\}$ stand for the set of its neighbors, and $G[U]$ denote the subgraph of G induced by U. The weight of a vertex subset $U \subseteq V$ is defined by $w(U) := \sum_{u \in U} w(u)$.

MINIMUM WEIGHTED VERTEX COVER (MIN-w-VC)

Instance: A simple graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$.

Feasible solution: A vertex cover C for G, i.e., a subset $C \subseteq V$ such that for each $e \in E$, $e \cap C \neq \emptyset$.

Objective function: The weight $w(C) := \sum_{u \in C} w(u)$ of the vertex cover C.

The unweighted version of the MINIMUM VERTEX COVER problem (shortly, MIN-VC) is the special case of MIN-w-VC with uniform weights $w(u) = 1$ for each $u \in V$. Let $VC(G, w)$ be the set of all minimum vertex covers for (G, w) and $vc(G, w)$ stand for the weight of the minimum vertex cover for (G, w). In unweighted case we use shortly $VC(G)$ and $vc(G)$.

MIN-w-VC problem can be expressed as an Integer Program (IP) as follows: the goal is to *minimize* the function $w(x) := \sum_{u \in V} w(u) \cdot x(u)$, where $x(u) \in \{0,1\}$ for each $u \in V$, and *a feasible solution* $x : V \to \{0,1\}$ has to satisfy *edge constraints* $x(u) + x(v) \geq 1$ for each edge $\{u,v\} \in E$. The Linear Programming (LP) relaxation of the IP-formulation allows $x(u) \in \langle 0,1 \rangle$ (or even $x(u) \geq 0$). It is well known ([20], [22]) that there always exists an optimal solution of the LP-relaxation with the variables $x(u) \in \{0, \frac{1}{2}, 1\}$. The Half-Integral (HI) relaxation has exactly the same formulation as the IP-formulation, but it allows variables $x(u)$ from the set $\{0, \frac{1}{2}, 1\}$ for each $u \in V$. Hence a *feasible solution* is a half-integral vertex cover for G, i.e., a function $x : V \to \{0, \frac{1}{2}, 1\}$ satisfying edge constraints $x(u) + x(v) \geq 1$ for each edge $\{u,v\} \in E$. Let $VC^*(G,w)$ be the set of all minimum half-integral vertex covers $x : V \to \{0, \frac{1}{2}, 1\}$, and $vc^*(G,w)$ stand for the weight of a minimum half-integral vertex cover for (G,w). For a minimum half-integral vertex cover x for (G,w), we denote $V_i^x := \{u \in V : x(u) = i\}$ for each $i \in \{0, \frac{1}{2}, 1\}$.

Clearly, $vc^*(G,w) \leq vc(G,w)$, as for any vertex cover C its indicator function x^C is a feasible solution for the HI-relaxed problem with $w(x^C) = w(C)$. Further, $vc^*(G,w) \leq \frac{1}{2} w(V)$, as the function $x \equiv \frac{1}{2}$ on V is always feasible solution for the HI-relaxation. Recall that a weighted graph (G,w) is called a *König-Egerváry graph* (KEG) if $vc(G,w) = vc^*(G,w)$.

MAXIMUM FRACTIONAL w-MATCHING (MAX-w-FM)
Instance: A simple graph $G = (V,E)$ with vertex weights $w : V \to (0, \infty)$.
Feasible solution: A fractional w-matching $\lambda : E \to \langle 0, \infty)$ such that $\sum_{u \in N(v)} \lambda(\{u,v\}) \leq w(v)$ for every $v \in V$.
Objective function: The sum $\lambda(E) := \sum_{\{u,v\} \in E} \lambda(\{u,v\})$ of the fractional w-matching.

Let $\nu^*(G,w)$ denote the value of a maximum fractional w-matching for (G,w). The MAXIMUM FRACTIONAL w-MATCHING problem is precisely the dual linear program of the linear relaxation of MIN-w-VC for (G,w), hence $\nu^*(G,w) = vc^*(G,w)$ $(\leq vc(G,w))$.

Overview. The following simple local sufficient condition of optimality for the MIN-w-VC problem was first mentioned in [22].

Commitment Reduction. *If I is a nonempty independent set for a graph $G = (V,E)$ with vertex weights $w : V \to (0,\infty)$ such that $N(I)$ is a minimum vertex cover for $(G[I \cup N(I)], w)$, then there is a minimum vertex cover C for (G,w) that contains all vertices in $N(I)$ and none of the vertices in I.*

Obviously, a nonempty independent set I described in commitment reduction always exists. In particular, any maximum independent set for (G,w) has this property. Unfortunately, the problem to find such I is NP-hard in this general situation. To be able to use these reductions in a computationally efficient way, we have to confine ourselves to more special configurations of $G[I \cup N(I)]$ that could be efficiently recognized in weighted graphs. The commitment reduction becomes the crown reduction if $N(I)$ is a minimum weight vertex cover in the bipartite graph $G[I, N(I)]$ (the graph obtained from $G[I \cup N(I)]$ removing all edges within $N(I)$). The ordered triple (I,H,K), where $H = N(I)$ and $K = V \setminus (I \cup N(I))$ is called a (nontrivial) *crown decomposition* of G. In the case of unweighted graph $G = (V,E)$ the condition for the crown reduction can be equivalently stated as that $N(I)$ is matched into I. This is the way how the crown reductions and decompositions have been introduced in [1], [15] in the unweighted case. However, the authors left open problem whether determining if a graph G admits a nontrivial crown decomposition is in P, or not. From our results in Section 4 it easily follows that not only this recognition problem is polynomial but also that an optimal crown decomposition (i.e., the one with $(G[K],w)$ already crown-free) can be computed in polynomial time. Furthermore, we show that the problem kernel K that can be obtained for an instance (G,w) in this way is unique, and it is the same as the one that we can obtain using Nemhauser-Trotter reductions. Namely, it is $K_{\min}(G,w) = \cap_{y\in VC^*(G,w)} V^y_{\frac{1}{2}}$, the only inclusionwise minimal set among all $V^y_{\frac{1}{2}}$, $y \in VC^*(G,w)$.

However, crown reductions, NT-reductions, and other kernelization techniques ([1], [23]) can be hardly used if the problem is to find (resp., to count) *all* minimum vertex covers, or to find one minimum vertex cover for (G,w) *under some additional constraints.* For such problems stronger reductions have to be introduced. A natural substitution for the commitment reduction is the following *strong commitment reduction*: Assume that for a nonempty independent set I in (G,w), $N(I)$ is the only minimum vertex cover for $(G[I \cup N(I)], w)$. Then minimum vertex covers for (G,w) are exactly the sets $C = N(I) \cup C'$, where C' is a minimum vertex cover for (G',w) and $G' := G[V \setminus (I \cup N(I))]$. However in the general situation it is NP-hard even to decide if (G,w) contains such independent set I. Hence we confine ourselves to more special configuration of $G[I \cup N(I)]$. The strong commitment reduction becomes the *strong crown reduction*, if $N(I)$ is supposed to be the only minimum weight vertex cover in the bipartite graph $G[I, N(I)]$ introduced earlier. Results given in Section 3 also imply that for an instance (G,w) there is the only optimal strong crown decomposition (I,H,K) (i.e., the one with $G[K]$ already strong crown-free). This decomposition can be computed in polynomial time and the corresponding problem kernel can be obtained also using Nemhauser-Trotter reductions. Namely, it is $K_{\max} = \cup_{y\in VC^*(G,w)} V^y_{\frac{1}{2}}$, the only inclusionwise maximal set among all $V^y_{\frac{1}{2}}$, $y \in V^*(G,w)$. In this case not only the problem kernel is unique for a given (G,w), but also $y \in V^*(G,w)$ for which $V^y_{\frac{1}{2}} = K_{\max}(G,w)$. These results about Nemhauser-Trotter reductions are closely related to the stronger variant of Nemhauser-Trotter Theorem given in [17], and provide its alternative proof.

In the unweighted case the problem of determining the set $V_1(G)$ of vertices belonging to all minimum vertex covers in G, and the set $V_0(G)$ of vertices belonging to no minimum vertex covers, are also well studied NP-hard problems [3]. Let us mention that our results improve also those of [3] on the lower bound of $|V_0(G)|$. Under the efficiently decidable assumption $def^*(G) > 0$ (with $def^*(G) := |V| - 2vc^*(G)$), we provide a lower bound $|V_0(G)| \geq |V_0^*(G)| > def^*(G)$ for any graph G without isolated vertices. Moreover, the subset $V_0^*(G)$ of $V_0(G)$ of cardinality $def^*(G) + |V_1^*(G)|$ can be efficiently computed, as well as the subset $V_1^*(G) \neq 0$ of $V_1(G)$.

It is known ([3]) that if $\mathcal{F}$ is a fixed hereditary family of graphs for which computing $vc(G)$ is polynomial, one can determine $V_0(G)$, $V_1(G)$ efficiently for G in $\mathcal{F}$. The question has been raised in [3] how to find $V_0(G)$ and/or $V_1(G)$ efficiently for some other classes of graphs. Our results contribute to this theory as well, proving that $V_0(G,w)$ and $V_1(G,w)$ can be computed effieciently for König-Egerváry graphs (G,w). Let us note, that the KEG graphs is *not* a hereditary (i.e., induced subgraph closed) family. Our Theorem 7 in combination with Theorem 4 provide a decomposition of a weighted graph (G,w) into “irreducible parts”. It describes how *all* minimum vertex covers for (G,w) are structured in the König-Egerváry part $(G[V \setminus K_{\min}], w)$ of (G,w).

In Section 5 we show how reductions studied in this paper provide for some specific classes of graphs efficient approximation algorithm for the Min-VC problem with a constant approximation factor $r < 2$. We show, for example, that for the matching number $\nu(G)$ and the fractional matching number $\nu^*(G)$ it holds $\nu(G) \geq \frac{2}{3}\nu^*(G)$, and for the class $G_\delta := \{G : \nu(G) \leq (1-\delta)\nu^*(G)\}$ of graphs (for any fixed constant $\delta \in \langle 0, \frac{1}{3} \rangle$) we describe a simple $2\frac{1-\delta}{1+\delta}$-approximation algorithm. On the other hand, we conclude that for the graphs with perfect matchings, or for the regular graphs, or for the Hamiltonian connected graphs, the MIN-VC problem is as hard to approximate as for general graphs.

In Section 6 we demonstrate how strong crown reductions (or, strong NT-reductions) can be used for fixed-parameter tractable problems related to the MIN-VC problem (unweighted, for simplicity). It can be used as efficient reduction to find (or to count) *all* minimum vertex covers in G, or to find one minimum vertex cover in G *under some additional constraints.*

2 Kernelization by minimum half-integral vertex covers

It is well known (see [22]) that the problem of finding a minimum half-integral vertex cover for a weighted graph (G, w) can be reduced to the problem of finding a minimum vertex cover in a related weighted bipartite graph (G^b, w^b), defined by the following construction.

Definition 1. *For a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ we define the weighted bipartite graph (G^b, w^b) with $G^b = (V^b, E^b)$, as follows: there are two copies u^L and u^R of each vertex $u \in V$ of the same weight $w^b(u^L) = w^b(u^R) = w(u)$ in (G^b, w^b), $V^L{:=}\{u^L : u \in V\}$, $V^R{:=}\{u^R : u \in V\}$, and $V^b{:=}V^L \cup V^R$. Each edge $\{u, v\} \in E$ of G creates two edges in G^b, namely $\{u^L, v^R\}$ and $\{v^L, u^R\}$. Hence $E^b{:=}\{\{u^L, v^R\}, \{v^L, u^R\} : \{u, v\} \in E\}$. For $U \subseteq V$ we use also U^L, U^R, and $U^b := U^L \cup U^R$ for the corresponding sets of vertices.*

For any set $C \subseteq V^{\mathrm{L}} \cup V^{\mathrm{R}}$ we associate a map $x_C : V \to \{0, \frac{1}{2}, 1\}$ in the following way: $x_C(u) = \frac{1}{2}|C \cap \{u^{\mathrm{L}}, u^{\mathrm{R}}\}|$ for any $u \in V$. Clearly, $w(x_C) = \frac{1}{2}w^b(C)$ for any $C \subseteq V^{\mathrm{L}} \cup V^{\mathrm{R}}$.

Lemma 1. ([22]) *The mapping $C \mapsto x_C$ maps the set of vertex covers in (G^b, w^b) onto the set of half-integral vertex covers in (G, w). Moreover, it maps $VC(G^b, w^b)$ onto $VC^*(G, w)$. Consequently, $vc^*(G, w) = \frac{1}{2} vc(G^b, w^b)$.*

Remark 1. In bipartite graphs the MINIMUM (WEIGHTED) VERTEX COVER problem can be solved in polynomial time. The optimal solution for MIN-w-VC can be identified from the solution of the corresponding MINIMUM CUT problem, that can be found by efficient algorithms for the MAXIMUM FLOW problem on bipartite graphs (see Lawler [19]). For instance, the problem is solvable in time $O(|E||V| \log \frac{|V|^2}{|E|})$ using Goldberg and Tarjan's algorithm [16]. When the problem is unweighted, Dinic's algorithm for the MAXIMUM FLOW problem runs in $O(|E|\sqrt{|V|})$ time. Another approach in unweighted case is based on the bipartite graph matching theory. A maximum matching of a bipartite graph can be constructed in time $O(|E|\sqrt{|V|})$ by the algorithm of Hopcroft and Karp ([18]) (or even for general graphs by the algorithm of Micali and Varizani), and a minimum vertex cover for a bipartite graph can be constructed from a maximum matching in time $O(|E|)$.

As it follows from Lemma 1 and efficient solvability of MIN-w-VC in bipartite graphs, $vc^*(G, w)$ for a graph (G, w) can be computed efficiently.

Definition 2. *Let $G = (V, E)$ be a graph with vertex weights $w : V \to (0, \infty)$. Denote by $V_0(G, w)$ the set of vertices avoided by each minimum vertex cover for (G, w) and $V_1(G, w)$ the set of vertices contained in each minimum vertex cover for (G, w). Similarly, for $i \in \{0, 1\}$, denote by $V_i^*(G, w)$ the set of vertices with value i in each minimum half-integral vertex cover for (G, w).*

Remark 2. For a fixed weighted graph (G, w) let $\Phi : V^b \to V^b$ denote the automorphism of (G^b, w^b) defined by $\Phi(u^{\mathrm{L}}) = u^{\mathrm{R}}$, $\Phi(u^{\mathrm{R}}) = u^{\mathrm{L}}$ for each $u \in V$. For a fixed $u \in V$ we obtain

$u^{\mathrm{L}} \in V_0(G^b, w^b)$ iff $u^{\mathrm{R}} \in V_0(G^b, w^b)$ iff (using Lemma 1) $u \in V_0^*(G, w)$; $u^{\mathrm{L}} \in V_1(G^b, w^b)$ iff $u^R \in V_1(G^b, w^b)$ iff $u \in V_1^*(G, w)$. In other words, for each $i \in \{0, 1\}$, $V_i(G^b, w^b)$ is the union of sets $\{u^L, u^R\}$ over all $u \in V_i^*(G, w)$.

Remark 3. If $G = (V, E)$ is a bipartite graph with bipartition $V = A \cup B$ and with vertex weights $w : V \to (0, \infty)$, then (G^b, w^b) consists of two disjoint copies of (G, w), namely $(G^b[A^{\mathrm{L}} \cup B^{\mathrm{R}}], w^b)$ and $(G^b[A^{\mathrm{R}} \cup B^{\mathrm{L}}], w^b)$. Therefore $vc(G^b, w^b) = 2vc(G, w)$, and $vc^*(G, w) = vc(G, w)$ by Lemma 1. Moreover, $u \in V_0(G, w)$ iff $u^{\mathrm{L}}, u^{\mathrm{R}} \in V_0(G^b, w^b)$ iff $u \in V_0^*(G, w)$, hence $V_0(G, w) = V_0^*(G, w)$. In the same way we get $V_1(G, w) = V_1^*(G, w)$.

Definition 3. *A minimum half-integral vertex cover x with the property that no $y \in VC^*(G, w)$ satisfies both, $V_0^y \subsetneq V_0^x$ and $V_1^y \subsetneq V_1^x$, is called a* pivot.

The existence of a pivot is clear. For example, $x \in VC^*(G, w)$ such that V_0^x is inclusion-wise minimal among sets $\{V_0^y : y \in VC^*(G, w)\}$, is a pivot. We will prove later that there is only one pivot in $VC^*(G, w)$ and it can be found efficiently. The following lemma describes basic properties of decompositions generated by minimum half-integral vertex covers and their relation to crown decompositions studied later.

Lemma 2. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ and a partition $V = V_0^x \cup V_1^x \cup V_{\frac{1}{2}}^x$ according to a fixed minimum half-integral vertex cover x for (G, w). Then the following statements hold:*

(i) V_0^x is an independent set, every isolated vertex of G (if any) belongs to V_0^x, and $N(V_0^x) = V_1^x$.
(ii) $vc^*(G[V_{\frac{1}{2}}^x], w) = \frac{1}{2} w(V_{\frac{1}{2}}^x)$
(iii) For each $U \subseteq V_1^x$, $w(N(U) \cap V_0^x) \geq w(U)$. If x is a pivot, then $\emptyset \neq U \subseteq V_1^x$ implies $w(N(U) \cap V_0^x) > w(U)$.

Proof. (i) As x satisfies all edge constraints, there are no edges within V_0^x or between V_0^x and $V_{\frac{1}{2}}^x$. Hence V_0^x is an independent set, every isolated vertex of G (if any) belongs to V_0^x, and $N(V_0^x) \subseteq V_1^x$. If there is $u \in V_1^x \setminus N(V_0^x)$, then changing x on u to $\frac{1}{2}$ results in a half-integral vertex cover $\widetilde{x}$ with $w(\widetilde{x}) < w(x)$, a contradiction with minimality of x.

(ii) If there is $y \in VC^*(G[V_{\frac{1}{2}}^x], w)$ with $w(y) < \frac{1}{2} w(V_{\frac{1}{2}}^x)$, then changing x on $V_{\frac{1}{2}}^x$ to y implies a half integral vertex cover $\widetilde{x}$ for (G, w) with $w(\widetilde{x}) < w(x)$, a contradiction.

(iii) Let us consider a fixed nonempty set $U \subseteq V_1^x$. We define a new half-integral vertex cover y changing x on $U \cup (N(U) \cap V_0^x)$ to $\frac{1}{2}$. Clearly, $w(y) = w(x) + \frac{1}{2}(w(N(U) \cap V_0^x) - w(U)) \geq w(x)$, as $x \in VC^*(G, w)$. Hence $w(N(U) \cap V_0^x) \geq w(U)$ and the equality occurs iff $y \in VC^*(G, w)$ as well. Obviously, $V_0^y \subsetneq V_0^x$ and $V_1^y \subsetneq V_1^x$. Therefore such y doesn't exist if x is a pivot and necessarily $w(N(U) \cap V_0^x) > w(U)$ in that case. □

Given $x \in VC^*(G, w)$, from Lemma 2(iii) it easily follows that V_1^x is a minimum vertex cover for $(G[V_0^x \cup V_1^x], w)$, and that it is the only minimum vertex cover in $(G[V_0^x \cup V_1^x], w)$, if x is a pivot. (Such results, also for minimum half-integral vertex covers for $(G[V_0^x \cup V_1^x], w)$ are proved in Lemma 3, 4 in a more general situation of crown reductions.) From the first part, Nemhauser-Trotter Theorem [22] follows. Namely, a minimum vertex cover for (G, w) exists that is of the form $C = V_1^x \cup C'$, where C' is a minimum vertex cover for $(G[V_{\frac{1}{2}}^x], w)$. If x is a pivot, we can conclude that even all minimum vertex covers for (G, w) are of that form.

3 Kernelization by crown reductions

A new kernelization technique, called crown reduction, has been introduced in [1], [15] for the unweighted MIN-VC problem. In this section we study crown reductions and their properties in the more general case of vertex weighted graphs. We describe the relation of crown reductions to the LP-relaxation of the MIN-w-VC problem and to the NT-reductions.

Definition 4. *For an independent set I in G let $G[I, N(I)]$ denote the bipartite graph obtained from $G[I \cup N(I)]$ removing all edges within $N(I)$ (if any).*

We define a special version of commitment reduction (resp., strong commitment reduction) when the assumption on $N(I)$ to be a minimum vertex cover (resp., the only minimum vertex cover) for $(G[I \cup N(I)], w)$ is strengthened to be true even for its bipartite subgraph $(G[I, N(I)], w)$.

Definition 5. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. A* crown *(resp., a* strong crown*) in (G, w) is a nonempty independent set I of G such that $w(N(U) \cap I) \geq w(U)$ (resp., $w(N(U) \cap I) > w(U)$) holds for every nonempty set $U \subseteq N(I)$.*

If I is a (strong) crown in (G, w) then the ordered triple (I, H, K), where $H = N(I)$ (the head *of the crown I) and $K = V \setminus (I \cup H)$ (the* rest*), is called a* (strong) crown decomposition *of (G, w). All such (strong) crown decompositions are called nontrivial, and the triple $(\emptyset, \emptyset, V)$ is called a trivial (strong) crown decomposition. If a graph (G, w) has only trivial (strong) crown decomposition, then G is called crown-free (resp. strong crown-free).*

In the unweighted case the condition for a crown is equivalent, due to Hall's Theorem, that $N(I)$ is matched into I. This is exactly the context and the way how the crown reduction has been introduced in [1], [15].

Lemma 2 yields that for any $x \in VC^*(G, w)$ the triple $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$ is a crown decomposition of (G, w). Hence every Nemhauser-Trotter reduction is also a crown reduction. Furthermore, if x is a pivot then $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$ is a strong crown decomposition. A crown decomposition is trivial if and only if $x \equiv \frac{1}{2}$.

Lemma 3. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. For every nonempty independent set I in G the following conditions (i)–(iv) are equivalent and any of them implies that $N(I) \in VC(G[I \cup N(I)], w)$.*

(i) *I is a crown in (G, w),*
(ii) *$N(I) \in VC(G[I, N(I)], w)$,*
(iii) *$x \in VC^*(G[I, N(I)], w)$ for x defined by $x|_I \equiv 0$, $x|_{N(I)} \equiv 1$,*
(iv) *$x \in VC^*(G[I \cup N(I)], w)$ for x defined by $x|_I \equiv 0$, $x|_{N(I)} \equiv 1$.*

Proof. (i)⇒(ii): $N(I)$ is a vertex cover in $G[I, N(I)]$. To prove its optimality, take another vertex cover C in $G[I, N(I)]$ and show that $w(N(I)) \leq w(C)$ as follows. Let $U := N(I) \setminus C$, then clearly $N(U) \cap I \subseteq C \cap I$ and using (i) we get $w(C \cap I) \geq w(N(U) \cap I) \geq w(U)$.

As $N(I) \cup (C \cap I) = C \cup U$ and both unions are disjoint, $w(N(I)) + w(C \cap I) = w(C) + w(U)$. That combined with the previous gives $w(N(I)) \leq w(C)$.

(ii)⇒(iii): x defined in (iii) is a half-integral vertex cover for $G[I, N(I)]$ and $w(x) = w(N(I))$. Hence it suffices to prove that $vc^*(G[I, N(I)], w) = w(N(I))$. As $G[I, N(I)]$ is bipartite, $vc^*(G[I, N(I)], w) = vc(G[I, N(I)], w)$ (Remark 2), and this is equal to $w(N(I))$, by (ii).

(iii)⇒(iv): As x is a half-integral vertex cover for $(G[I \cup N(I)], w)$ that is optimal even in its subgraph $G[I, N(I)]$, it has to be in $VC^*(G[I \cup N(I)], w)$.

(iv)⇒(i): It follows from Lemma 2(iii) applied to the graph $(G[I \cup N(I)], w)$.

Hence (i)–(iv) are equivalent. Moreover, $N(I) \in VC(G[I \cup N(I)], w)$ is trivially implied by any of them.□

Lemma 4. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. For every nonempty independent set I in G the following conditions (i)–(iv) are equivalent and any of them implies that $N(I)$ is the only element of $VC(G[I \cup N(I)], w)$.*

(i) I is a strong crown in (G, w),
(ii) the only element of $VC(G[I, N(I)], w)$ is $N(I)$,
(iii) the only element of $VC^*(G[I, N(I)], w)$ is x defined by $x|_I \equiv 0$, $x|_{N(I)} \equiv 1$,
(iv) the only element of $VC^*(G[I \cup N(I)], w)$ is x defined by $x|_I \equiv 0$, $x|_{N(I)} \equiv 1$.

Proof. We will focus on the uniqueness part of (i)–(iv), the rest follows from the previous Lemma 3.

(i)⇒(ii): $N(I) \in VC(G[I, N(I)], w)$ is clear by previous Lemma 3. To prove the uniqueness, take any vertex cover C in $G[I, N(I)]$ and prove that either $C = N(I)$, or $w(C) > w(N(I))$ as follows. The case $N(I) \subseteq C$ being clear, so we can assume that $U := N(I) \setminus C \neq \emptyset$. Now we can argue as in the proof in Lemma 3, but we have now the strict inequality $w(C \cap I) \geq w(N(U) \cap I) > w(U)$ assuming (i). It results in the strict inequality $w(N(I)) < w(C)$.

(ii)⇒(iii): Again, $x \in VC^*(G[I, N(I)], w)$ is clear. To prove the uniqueness, take any $y \in VC^*(G[I, N(I)], w)$ and prove that $y = x$ as follows. By Lemma 1, y is determined by some $C \in VC(G[I, N(I)]^b, w^b)$ for which $y(u) = \frac{1}{2}|\{u^L, u^R\} \cap C|$ for each $u \in I \cup N(I)$. As $G[I, N(I)]$ is bipartite $(G[I, N(I)]^b, w^b)$ consists of two disjoint copies of $(G[I, N(I)], w)$ (Remark 3). Assuming (ii), C being the optimal has to choose $N(I)^L$ from one copy and $N(I)^R$ from the another one. Hence $C = N(I)^L \cup N(I)^R$ and $y = x$ follows.

(iii)⇒(iv): Again, $x \in VC^*(G[I \cup N(I)], w)$. Then $w(y) = w(x)$, and y is a half-integral vertex cover also in $G[I, N(I)]$, which is a subgraph of $G[I \cup N(I)]$. Thus y is a feasible solution with the same weight as (assuming (iii)) the only element $x \in VC^*(G[I, N(I)], w)$. Consequently, $y = x$.

(iv)⇒(i): Consider $U \subseteq N(I)$ and define a half-integral vertex cover y for $(G[I \cup N(I)], w)$ changing x on $U \cup (N(U) \cap I)$ to $\frac{1}{2}$. Then $w(N(U) \cap I) - w(U) = 2(w(y) - w(x))$ that has to be positive as x was the only minimum, assuming (iv).

Hence (i)–(iv) are equivalent, and any of them implies that the only element of $VC(G[I \cup N(I)], w)$ is $N(I)$.□

Lemma 5. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$, and a crown decomposition (I, H, K) of (G, w).*

(i) Every (minimum) vertex cover for $(G[K], w)$ together with H forms a (minimum) vertex cover for (G, w). Every (minimum) half-integral vertex cover for $(G[K], w)$ extended by 1 on H and by 0 on I forms a (minimum) half-integral vertex cover for (G, w).
(ii) For every minimum vertex cover C for (G, w), $C \cap K$ and $C \cap (I \cup H)$ are minimum vertex covers for $(G[K], w)$ and $(G[I \cup H], w)$, respectively. For every minimum half-integral vertex cover y for (G, w), $y|_K$ and $y|_{(I \cup H)}$ are minimum half-integral vertex covers for $(G[K], w)$ and $(G[I \cup H], w)$, respectively.

(iii) *If I is a strong crown then minimum vertex covers for (G, w) are exactly the sets $C = H \cup C'$, where C' is a minimum vertex cover for $(G[K], w)$. Minimum half-integral vertex covers for (G, w) are exactly the mappings $x : V \to \{0, 1, \frac{1}{2}\}$ such that $x|_I \equiv 0$, $x|_H \equiv 1$, and $x|_K = x'$, where x' is a minimum half-integral vertex cover for $(G[K], w)$. In particular, $I \subseteq V_0^*(G, w) := \cap_{y \in VC^*(G,w)} V_0^y$ and $H \subseteq V_1^*(G, w) := \cap_{y \in VC^*(G,w)} V_1^y$.*

Proof. It quite easily follows from Lemma 3 and Lemma 4. (i) being trivial, let us give a sketch of proof of (ii), (iii) for vertex covers. For half-integral vertex covers we can argue in the same way.

(ii) Consider $C \in VC(G, w)$. Then $C \cap (I \cup H)$ is a vertex cover for $(G[I \cup H], w)$, and it is a minimum (i.e., of weight $w(H)$) because otherwise $C_1 := H \cup (C \cap K)$ is a vertex cover in G with $w(C_1) < w(C)$, a contradiction. Consequently, $w(C_1) = w(C)$. Now a vertex cover $C \cap K = C_1 \cap K$ for $(G[K], w)$ is a minimum, because otherwise $C_2 := H \cup C'$ for any $C' \in VC(G[K], w)$ is a vertex cover in G with $w(C_2) < w(C)$, a contradiction.

(iii) If I is a strong crown and $C \in VC(G, w)$, then (ii) implies that $C \cap (I \cup H) \in VC(G[I \cup H], w)$, hence $C \cap (I \cup H) = H$ due to properties of a strong crown derived in Lemma 4.□

The following theorem summarizes the most important properties of crown decompositions

Theorem 1. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ and (I, H, K) be a crown decomposition of (G, w). Then*

(i) $vc(G[I \cup H], w) = vc^*(G[I \cup H], w) = w(H)$,
(ii) $vc(G, w) = vc(G[I \cup H], w) + vc(G[K], w)$,
(iii) $vc^*(G, w) = vc^*(G[I \cup H], w) + vc^*(G[K], w)$,
(iv) $VC^*(G[K], w)$ *consists of restrictions to the set K of minimum half-integral vertex covers from* $VC^*(G, w)$,
(v) $VC(G[K], w)$ *consists of intersections of K with minimum vertex covers from* $VC(G, w)$.

Proof. (i) follows from Lemma 3, (ii), (iii), (iv), (v) from Lemma 5. □

In [15] Fellows left open the problem whether determining if a graph admits a nontrivial crown decomposition belongs to P, or if it is NP-complete. In the next subsection we solve this problem for strong crown decompositions, and in Section 4 for crown decompositions. We show that even the search version of this problem is in P. Given an instance (G, w), one can find in polynomial time a (strong) crown decomposition (I, H, K) with $(G[K], w)$ already (strong) crown-free.

Notation and terminology. Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. Let us denote $K_{\max} := K_{\max}(G, w) = \cup_{y \in VC^*(G,w)} V_{\frac{1}{2}}^y$, and $K_{\min} := K_{\min}(G, w) = \cap_{y \in VC^*(G,w)} V_{\frac{1}{2}}^y$.

3.1 The properties of strong crown reductions

To characterize efficiently graphs (G, w) that admit a nontrivial strong crown decomposition we introduce a notion of *fractional deficiency*. The standard (i.e., integral and unweighted) definition of the deficiency is $def(G) = |V(G)| - 2\nu(G)$, where $\nu(G)$ is the size of a maximum matching in an unweighted graph G. We generalize it to the fractional theory of (vertex) weighted graphs.

Definition 6. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. The fractional deficiency of (G, w), denoted by $def^*(G, w)$, is defined as $def^*(G, w) = w(V) - 2vc^*(G, w)$.*

Clearly, $def^*(G, w) \geq 0$. For each $x \in VC^*(G, w)$, $vc^*(G, w) = w(x) = w(V_1^x) + \frac{1}{2}w(V_{\frac{1}{2}}^x)$ and hence $def^*(G, w) = w(V_0^x) - w(V_1^x)$ follows. In particular, we have $def^*(G, w) = w(V_0^*(G, w)) - w(V_1^*(G, w))$. The following result can be viewed as the fractional version of the Tutte-Berge Formula in our setting.

Theorem 2. *For a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ the following holds: $def^*(G, w) = \max\{w(I) - w(N(I)) : I$ is an independent set of $G\} = \max\{w(Z) - w(N(Z)) : Z \subseteq V\}$. Moreover, an independent set I satisfies $w(I) - w(N(I)) = def^*(G, w)$ if and only if $I \in \{V_0^y : y \in VC^*(G, w)\}$.*

Proof. Put $d = \max\{w(I) - w(N(I)) : I$ is an independent set of $G\}$, $D = \max\{w(Z) - w(N(Z)) : Z \subseteq V\}$. As it was observed above, $w(V_0^x) - w(V_1^x) = def^*(G, w)$ holds for each $x \in VC^*(G, w)$. Hence $d \geq def^*(G, w)$.

Now assume that I is an independent set such that $w(I) - w(N(I)) = d$ and prove that necessarily $d = def^*(G, w)$ and $I \in \{V_0^y : y \in VC^*(G, w)\}$. Define $x : V \to \{0, 1, \frac{1}{2}\}$ by $x|_I \equiv 0$, $x|_{N(I)} \equiv 1$, and $x|_{V\setminus(I\cup N(I))} \equiv \frac{1}{2}$. Clearly, x is a half-integral vertex cover and $2w(x) = w(V) - d$. Hence $d = w(V) - 2w(x) \leq w(V) - 2vc^*(G, w) = def^*(G, w)$ $(\leq d)$ and both inequalities are, in fact, equalities. Thus $def^*(G, w) = d$ and $w(x) = vc^*(G, w)$. That means $x \in VC^*(G, w)$ and recall that $I = V_0^x$.

Now we will prove that $d = D$. Obviously, $d \leq D$. To prove $d \geq D$ consider any $Z \subseteq V$ such that $w(Z) - w(N(Z)) = D$. Let I be the set of isolated vertices of $G[Z]$. Clearly, $Z \setminus I \subseteq N(Z)$, $N(I) \subseteq N(Z)$, $(Z \setminus I) \cap N(I) = \emptyset$, and $w(I) - w(N(I)) \leq d$, hence $w(Z) - D = w(N(Z)) \geq w(Z \setminus I) + w(N(I)) \geq w(Z \setminus I) + w(I) - d = w(Z) - d$, and $d \geq D$ follows.□

As follows from the definition, $def^*(G, w)$ is efficiently computable. Hence, strong crown-free graphs can be efficiently recognized due to the following lemma

Lemma 6. *For a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ the following statements are equivalent:*

(i) (G, w) *does not admit a nontrivial strong crown decomposition,*
(ii) $def^*(G, w) = 0$,
(iii) $w(N(I)) \geq w(I)$ *holds for each independent set* I *of* G,
(iv) $w(N(Z)) \geq w(Z)$ *holds for each* $Z \subseteq V$.

Proof. (ii)–(iv) are equivalent due to Theorem 2. ¬(i)⇒¬(iii) is trivial, as any strong crown I in (G, w) by its definition satisfies $w(I) > w(N(I))$.

¬(ii)⇒¬(i): Suppose $def^*(G, w) > 0$. Then G admits the nontrivial strong crown decomposition with a crown V_0^*, as $w(V_0^*) - w(V_1^*) = def^*(G, w) > 0$.□

Lemma 7. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ and a crown decomposition (I, H, K) of (G, w). Then*

(i) $def^*(G, w) = def^*(G[I \cup H], w) + def^*(G[K], w) = w(I) - w(H) + def^*(G[K], w)$,
(ii) $def^*(G[K], w) = 0$ *if and only if* $(I, H, K) = (V_0^x, V_1^x, V_{\frac{1}{2}}^x)$ *for some* $x \in VC^*(G, w)$.

Proof. (i) It follows from Theorem 1(i) and (iii). (ii) By (i) we get that $def^*(G[K], w) = 0$ iff $w(I) - w(H) = def^*(G, w)$. This is, due to Theorem 2 equivalent to $I = V_0^x$ for some $x \in VC^*(G, w)$.□

Theorem 3. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. Then the following hold:*

(i) Among minimum half-integral vertex covers for (G, w) there is exactly one pivot x. The corresponding partition $V = V_0^x \cup V_1^x \cup V_{\frac{1}{2}}^x$ according to x has the following properties: $V_0^x = V_0^* = \cap_{y \in VC^*(G,w)} V_0^y$, $V_1^x = V_1^* = \cap_{y \in VC^*(G,w)} V_1^y = N(V_0^x)$, and $V_{\frac{1}{2}}^x = K_{\max} = \cup_{y \in VC^*(G,w)} V_{\frac{1}{2}}^y$.

(ii) $(V_0^*(G, w), V_1^*(G, w), K_{\max}(G, w))$ is the only strong crown decomposition (I, H, K) of (G, w) such that $(G[K], w)$ is strong crown-free.

Proof. (i) If $vc^*(G, w) = \frac{1}{2}w(V)$ then $x \equiv \frac{1}{2}$ belongs to $VC^*(G, w)$ and it is clearly the only pivot in $VC^*(G, w)$.

Assume that $vc^*(G, w) < \frac{1}{2}w(V)$ and $x \in VC^*(G, w)$ be a pivot. By Lemma 2(iii), V_0^x is a strong crown in (G, w). Using Lemma 5(iii), $V_0^x \subseteq \cap_{y \in VC^*(G,w)} V_0^y$ $(\subseteq V_0^x)$ and $V_1^x \subseteq \cap_{y \in VC^*(G,w)} V_1^y$ $(\subseteq V_1^x)$. Hence, $V_0^x = V_0^*(G, w)$ and $V_1^x = V_1^*(G, w)$ hold for a pivot, thus its uniqueness is obvious.

(ii) Recall that by Lemma 5(iii), $I \subseteq V_0^*$ and $H \subseteq V_1^*$. If $(G[K], w)$ is strong crown-free, i.e., $def^*(G[K], w) = 0$, then $w(I) - w(N(I)) = def^*(G, w)$ by Lemma 7 which implies $I \in \{V_0^y : y \in VC^*(G, w)\}$. However, $V_0^y \subsetneq V_0^*$ is impossible, hence $I = V_0^*$.□

In the following lemma we prove that a strong crown decomposition $(V_0^*, V_1^*, K_{\max})$ can be constructed efficiently for a graph (G, w).

Lemma 8. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. The strong crown decomposition $(V_0^*(G, w), V_1^*(G, w), K_{\max}(G, w))$ can be constructed in polynomial time; in unweighted case in time $O(|E|\sqrt{|V|})$.*

Proof. Let $G = (V, E)$ be a graph with vertex weights $w : V \to (0, \infty)$. Trivially, it suffices to construct V_0^*, as then $V_1^* = N(V_0^*)$ and $K_{\max} = V \setminus (V_0^* \cup V_1^*)$ can be simply found.

First, it is easy to see, that for any $v \in V$

$$v \notin V_0(G, w) \iff \exists C \in VC(G, w) \text{ s.t. } v \in C \implies vc(G \setminus v, w) + w(v) = vc(G, w).$$

Therefore $V_0(G, w) = \{v \in V : vc(G \setminus v, w) + w(v) > vc(G, w)\}$.

(a) Assume now that G is bipartite. Then one can compute in polynomial time $vc(G, w)$ and $vc(G \setminus v, w)$ for every $v \in V$ using maximum flow techniques mentioned in Remark 1. Therefore computing $V_0(G, w)$, that is for bipartite graphs (G, w) the same as $V_0^*(G, w)$ by Remark 3, can be done in polynomial time.

For later applications, let us sketch an estimate on time-complexity in the unweighted case. Using the bipartite version of Gallai-Edmonds Structure Theorem we can see that the task is now to compute the set D of all vertices in G which are avoided (i.e., unmatched) by at least one maximum matching in G. The set D can be easily computed via Edmonds matching algorithm, as follows:

Construct any maximum matching M of G (it can be done in time $O(|E|\sqrt{|V|})$ by algorithm from [18] or [21]). Let D_M be the set of vertices avoided by M. An alternating path with respect to M is a simple path $\{v_1, v_2, \ldots, v_r\}$ in G such that $v_1 \in D_M$ and the edges $\{v_{2k}, v_{2k+1}\}$ are in M for all integers k such that $1 \le k \le \frac{r-1}{2}$. The length of such a path is $r-1$ (possibly 0). One can prove that the set D is exactly the set of vertices in G that are reachable from at least one vertex in D_M via an alternating path of even length. The total complexity to construct D $(= V_0^*)$ is majorized by that of constructing the maximum matching M, hence $O(|E|\sqrt{|V|})$.

(b) If G is not bipartite, then computing $K_{\max}(G, w)$ reduces to computing $V_0^*(G^b, w^b)$ (Remark 2). Here G^b is bipartite with $|V^b| = 2|V|$ and $|E^b| = 2|E|$, and the proof used in (a) can be applied to determine $V_0^*(G^b, w^b)$.□

We can summarize our previous results as follows:

Theorem 4. *There exists a polynomial time algorithm that partitions the vertex set V of a given graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ into three subsets V_0, V_1, $K_{\max}$ with no edges between V_0 and $K_{\max}$ or within V_0, such that*

(i) $vc(G[K_{\max}], w) \ge vc^*(G[K_{\max}], w) = \frac{1}{2}w(G[K_{\max}])$,
(ii) every *minimum vertex cover C for (G, w) satisfies $V_1 \subseteq C \subseteq V_1 \cup K_{\max}$ and $C \cap K_{\max}$ is a minimum vertex cover for $(G[K_{\max}], w)$,*
(iii) every (minimum) vertex cover for $(G[K_{\max}], w)$ together with V_1 forms a (minimum) vertex cover for (G, w),
(iv) $V_0 = \cap_{y \in VC^*(G,w)} V_0^y$, *and* $V_1 = \cap_{y \in VC^*(G,w)} V_1^y = N(V_0)$,
(v) $\emptyset \ne U \subseteq V_1$ *implies* $w(N(U) \cap V_0) > w(U)$,
(vi) $V_0 = \emptyset$ *if and only if* $def^*(G, w) = 0$,
(vii) $w(V_0) - w(V_1) = def^*(G, w)$.

The parts (i)–(iii) yield to the strengthened version of Nemhauser-Trotter theorem proved already in [17].

Remark 4. In unweighted case the question of whether in a given graph G there are vertices belonging to all maximum independent sets of G is well studied. The problem is known to be NP-hard, so one of the questions studied is to provide lower bounds on $|V_0(G)|$ for specific classes of graphs (see [3], and references therein). Essentially the best result of this kind in [3] is that if G is a graph without isolated vertices and with $\alpha(G) > \nu(G)$, then $|V_0(G)| > \alpha(G) - \nu(G)$ (here α is independence number, ν is matching number).

Unfortunately, this bound depends on $\alpha(G)$ that is NP-hard to compute. Using Theorem 4 one can provide results at least as strong, but with efficiently computable bounds. As $V_0^*(G) \subseteq V_0(G)$, one can provide a lower bound $|V_0(G)| \ge |V_0^*(G)| = def^*(G) + |V_1^*(G)|$.

By a bound given by Lorentzen (see Theorem 64.12 in [25]) $def^*(G) \ge \alpha(G) - \nu(G)$. Hence under a weaker and efficiently decidable assumption $def^*(G) > 0$ we obtain at least as good lower bound $|V_0(G)| \ge |V_0^*(G)| > def^*(G)$ for a graph G without isolated vertices. Moreover, the subset $V_0^*(G)$ of $V_0(G)$ of cardinality $def^*(G) + |V_1^*(G)|$ can be efficiently computed, at the same time with the subset $V_1^*(G) \ne \emptyset$ of $V_1(G)$.

General crown reduction strategy. Suppose we have an algorithm that for a given input (G, w) provides a nontrivial strong crown decomposition. Hence for (G, w) as an input, we write $G_1 := G$ and if (G_1, w) has a strong crown, we obtain a strong crown decomposition (I_1, H_1, K_1). If $K_1 \ne \emptyset$ and $G_2 := G[K_1]$ is such that (G_2, w) has a strong crown, we get a strong crown decomposition (I_2, H_2, K_2). We continue repeatedly finding a new strong crown decomposition (I_k, H_k, K_k) in (G_k, w). We stop when $K_k = \emptyset$, or $(G[K_k], w)$ is strong crown-free. If we write $I = \cup_{i=1}^k I_i$, $H = \cup_{i=1}^k H_i$, then (I, H, K_k) is a strong crown decomposition of

(G, w). As clearly $def^*(G[K_k], w) = 0$ (otherwise we can find new strong crown), Lemma 7(ii) shows that $(I, H, K_k) = (V_0^x, V_1^x, V_{\frac{1}{2}}^x)$ for some $x \in VC^*(G)$. This implies that using strong crown reduction technique we cannot obtain smaller kernels than those obtainable using only Nemhauser-Trotter reductions for (G, w). There are possibly many strong crowns in (G, w), but all are dominated by the unique strong crown $V_0^*(G, w)$ and this can be found efficiently.

3.2 Fractional w-matchings

In this subsection we alternatively prove that there exists only one pivot which can be found efficiently.

Definition 7. *Let $G = (V, E)$ be a graph with vertex weights $w : V \to (0, \infty)$ and $\lambda : E \to \langle 0, \infty)$ be a fractional w-matching. The vertex $v \in V$ is said to be* saturated *by λ, when $\sum_{u \in N(v)} \lambda(\{u, v\}) = w(v)$. A fractional w-matching λ is called* perfect*, when all vertices are saturated by λ. Further, an edge $\{u, v\} \in E$ is* active *in λ if $\lambda(\{u, v\}) > 0$, otherwise it is called* passive.

The Maximum w-Matching problem is the integral version of the Max-w-FM problem, i.e., all weights w are integral together which the additional constraints that each $\lambda(\{u, v\})$ is integral. Let $\nu(G, w)$ denote the optimal value of the Maximum w-Matching problem. (We drop the acronym w in case $w \equiv 1$).

Clearly, $\nu(G, w) \leq \nu^*(G, w)$. It is known that $\nu(G, w) = \nu^*(G, w)$ whenever w is integral and $vc(G, w) = vc^*(G, w)$. Hence, for integral weights w a weighted graph (G, w) is KEG if and only if $\nu(G, w) = vc(G, w)$.

Fractional Tutte-Berge and Gallai-Edmonds sets. Let a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ be given and consider a subset S of V. The *trivial S-components* are isolated vertices of the graph $G \setminus S$; the set I_S denotes their union, which is clearly an independent set of G. The *nontrivial S-components* are the other connected components of $G \setminus S$. The quantity $def^*(S) := w(I_S) - w(S)$ denotes the *fractional deficiency of S in* (G, w). Observe that $N(I_S) \subseteq S$ and if $N(I_S) \neq S$ then $S' := N(I_S)$ satisfies $I_{S'} = I_S$, $N(I_{S'}) = S'$, and $def^*(S') > def^*(S)$.

Now Theorem 2 implies that $\max_{S \subseteq V} def^*(S) = def^*(G, w)$. Moreover, $S \subseteq V$ satisfies $def^*(S) = def^*(G, w)$ if and only if $S \in \{V_1^y : y \in VC^*(G, w)\}$.

Given $S \subseteq V$, it is important to notice that

(1) every fractional w-matching λ in (G, w) satisfies

$$\begin{aligned} w(V) - 2\lambda(E) &= \sum_{v \in V} \left(w(v) - \sum_{u \in N(v)} \lambda(\{u, v\}) \right) \\ &\geq \sum_{v \in I_s} \left(w(v) - \sum_{u \in N(v)} \lambda(\{u, v\}) \right) = w(I_S) - \sum_{v \in I_s} \sum_{u \in N(v)} \lambda(\{u, v\}) \geq w(I_S) - w(S) = def^*(S), \end{aligned}$$

(2) if a fractional w-matching λ in (G, w) satisfies $w(V) - 2\lambda(E) = def^*(S)$, then $def^*(S) = def^*(G, w)$, λ is a maximum fractional matching for (G, w), it contains a perfect fractional w-matching of each nontrivial S-component, and it saturates all vertices of S. In particular, all edges within S, or between S and $V \setminus (S \cup I_S)$, are passive in λ.

We generalize some notions that are standard in the literature on matching theory to the fractional w-matchings theory for vertex weighted graphs (G, w). The books [20] and [25]) provide comprehensive treatments of such theory, including various methods suitable for weighted problems as well.

A set $S \subseteq V$ is called *a fractional Tutte-Berge set for* (G, w) if a (maximum) fractional w-matching λ for (G, w) exists, that satisfies (2). These are exactly the sets $\{V_1^y : y \in VC^*(G, w)\}$. Further, $S \subseteq V$ is called *a fractional Gallai-Edmonds set for* (G, w) if

(a) nontrivial S-components, if any, have a perfect fractional w-matching,
(b) if $S \neq \emptyset$, then S satisfies generalized Hall's condition with positive surplus in $G[I_S, S]$, i.e.,

$$w(N(U) \cap I_S) > w(U) \text{ for every nonempty set } U \subseteq S.$$

It is easy to see that for a fractional Gallai-Edmonds set S for (G, w) the underlying vertex set I_S of trivial S-components is exactly the set of vertices left unsaturated by at least one maximum fractional w-matching and $S = N(I_S)$. Hence if such set S exists, it is unique.

By linear programming duality, $def^*(G, w) = 0$ iff G has a perfect fractional w-matching. This together with Lemma 2 implies that V_1^x is a fractional Gallai-Edmonds set for (G, w) if x is a pivot. This is another proof of uniqueness of pivot for (G, w). Hence, by Theorem 3, $V_1^*(G, w)$ is the (unique) fractional Gallai-Edmonds set for (G, w).

4 Decomposition into Irreducible Subgraphs

After applying strong crown decompositions to (G, w) one can obtain efficiently partition V_0, V_1, and $K_{\max}$ of V such that every minimum vertex cover C for (G, w) contains all vertices of V_1 and none of V_0 (Theorem 4). Hence, the MIN-w-VC problem for (G, w) can be reduced to $(G[K_{\max}], w)$. This partition V_0, V_1, and $K_{\max}$ corresponds to the case in which the problem kernel $G[K_{\max}]$ is the largest among all $V_{\frac{1}{2}}^y$, $y \in VC^*(G, w)$, in fact $K_{\max} = \cup_{y \in VC^*(G,w)} V_{\frac{1}{2}}^y$. The graph $G[K_{\max}]$ is strong crown-free, but it can have a variety of crown decompositions. However, one can again applying crown decompositions to $(G[K_{\max}], w)$ to find efficiently partition V_0', V_1', and $K_{\min}$ such that $K_{\min}$ is crown-free. We will see later that in this case is the problem kernel $K_{\min}$ smallest among all $V_{\frac{1}{2}}^y$, $y \in VC^*(G, w)$, namely $K_{\min} = \cap_{y \in VC^*(G,w)} V_{\frac{1}{2}}^y$.

Now we prove that similarly as for a strong crown the decision whether a graph (G, w) has a crown can be done in polynomial time.

Theorem 5. *Given a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$. Then the following statements are equivalent:*

(i) (G, w) *does not admit a nontrivial crown decomposition,*
(ii) $x \equiv \frac{1}{2}$ *is the only element of* $VC^*(G, w)$*,*
(iii) $w(N(I)) > w(I)$ *holds for each nonempty independent set* I *of* G*,*
(iv) $w(N(Z)) > w(Z)$ *holds for each* $Z \subseteq V$ *such that* $N(Z) \neq Z$*.*

Proof. (ii)⇔(iii) has been proved in [22], ¬(i)⇒¬(iii) is trivial, as any crown I in (G, w) satisfies $w(I) \geq w(N(I))$ by its Definition 5.

$\neg$(ii)$\Rightarrow \neg$(i) If $x \in VC^*(G,w)$ and $x \not\equiv \frac{1}{2}$, then by Lemma 2 $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$ is a nontrivial crown decomposition of (G,w).

(iv)$\Rightarrow$(iii) trivial, (iii) $\Rightarrow$ (iv) If $Z \subseteq V$ is such that $N(Z) \neq Z$, the same is true for the restriction of Z to at least one of connectivity components of G. Hence to prove (iv) from (iii) we can assume that G is connected. Let $Z \subseteq V$ with $N(Z) \neq Z$ be fixed and I be the set of isolated vertices of $G[Z]$. Clearly $Z \setminus I$ and $N(I)$ are disjoint subsets of $N(Z)$, hence $w(N(Z)) \geq w(Z \setminus I) + w(N(I))$. If $I \neq \emptyset$, then $w(N(I)) > w(I)$ by (iii), and $w(N(Z)) > w(Z \setminus I) + w(I) = w(Z)$ follows. If $I = \emptyset$, then clearly $Z \subset N(Z)$ and $w(N(Z)) > w(Z)$ follows as well. □

The condition (ii) in Theorem 5 can be tested efficiently (see [22]), hence the problem to decide of whether (G,w) has a crown belongs to P. In what follows we explain, in particular, how a crown decomposition (I,H,K) can be found efficiently with the property, that $(G[K],w)$ is already crown-free. We apply first the strong crown decomposition described in previous section, and search for a crown in the graph $G[K_{\max}]$. In this way we can confine ourselves to graphs (G,w) that are strong crown-free. That means, to graphs with $def^*(G,w) = 0$ or, equivalently said, to graphs satisfying $w(N(I)) \geq w(I)$ for each independent set I of G.

Definition 8. *Let $G = (V,E)$ be a graph with vertex weights $w : V \to (0,\infty)$ such that $def^*(G,w) = 0$. A crown I in G is called* irreducible*, if the bipartite graph $(G[I,N(I)],w)$ has exactly two minimum vertex covers, namely I and $N(I)$. Define the set $\mathcal{I} = \{I : I$ is a nonempty independent set such that $w(N(I)) = w(I)\}$.*

Lemma 9. *Let $G = (V,E)$ be a graph with vertex weights $w : V \to (0,\infty)$ such that $def^*(G,w) = 0$. Then*

(i) I is a crown in (G,w) if and only if $I \in \mathcal{I}$,
(ii) a crown I in (G,w) is irreducible if and only if I is an inclusionwise minimal set in $\mathcal{I}$.

Proof. (i) If I is a crown in (G,w) then I is a nonempty independent set of G satisfying $w(N(I)) \leq w(I)$ (Definition 5). The opposite inequality also holds, due to our assumption $def^*(G,w) = 0$ (Lemma 6), and $I \in \mathcal{I}$ follows.

Assume that $I \in \mathcal{I}$. We want to show that $w(N(U) \cap I) \geq w(U)$ for every $U \subseteq N(I)$. Keep $U \subseteq N(I)$ fixed, and put $S = I \setminus N(U)$. As S is independent, recall that from Lemma 6 $w(N(S)) \geq w(S)$. Clearly $N(S) \subseteq N(I) \setminus U$, hence $w(N(I) \setminus U) \geq w(N(S)) \geq w(S)$.

As $w(I) = w(N(I))$, $w(N(U) \cap I) = w(I) - w(S) = w(N(I)) - w(S) \geq w(N(I)) - w(N(I) \setminus U) = w(U)$.

(ii) Assume that $I \in \mathcal{I}$ is not inclusionwise minimal, i.e., there exists nonempty independent set $T \subsetneq I$, such that $w(N(T)) = w(T)$. Then $(I \setminus T) \cup N(T)$ is a minimum vertex cover for $(G[I,N(I)],w)$ distinct from I and H, and I is not irreducible.

If I is not irreducible, there is a minimum vertex cover C for $(G[I,N(I)],w)$ such that $\emptyset \neq I \setminus C \subsetneq I$. Clearly, $N(I \setminus C) \subseteq C \cap N(I)$, hence we obtain $w(N(I \setminus C)) \leq w(C \cap N(I))$ and $w(N(I \setminus C)) \geq w(I \setminus C)$ (as $def^*(G,w) = 0$). As $w(C) = w(I)$, we get $w(I \setminus C) = w(C \cap N(I))$ and $w(I \setminus C) = w(C \cap N(I)) \geq w(N(I \setminus C)) \geq w(I \setminus C)$, and $I \setminus C \in \mathcal{I}$ follows, hence I is not inclusionwise minimal.□

If I is an irreducible crown, then the Min-w-VC problem for $(G[I \cup N(I)],w)$ has at most two solutions: $N(I)$ is always a solution, and if there are no edges within $N(I)$ then I is a solution as well. One can recognize an irreducible crown I (more precisely its image in (G^b, w^b)) using so called *allowed* and *forbidden* edges of this bipartite graph. This terminology comes from decomposition theorems related to maximal matchings in unweighted graphs [20].

Definition 9. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = L \cup R$ and vertex weights $w : V \to (0, \infty)$ such that $vc(G, w) = \frac{1}{2}w(V)$. An edge $\{u, v\} \in E$ is called* allowed *for (G, w), if every minimum vertex cover C in (G, w) contains only one of the vertices u and v, otherwise it is called* forbidden*. Removing forbidden edges from G we obtain a graph whose connected components are called elementary blocks. Further, (G, w) is called elementary, if it has exactly two minimum vertex covers, namely L and R.*

Clearly, if a bipartite graph (G, w) is elementary, then G is connected and every edge is allowed. If $w \equiv 1$, then we deal exactly with bipartite graphs with perfect matching. In this context an edge of G is called allowed if it is contained in some perfect matching of G, and G is called elementary if it is connected and every edge of G is allowed. (See [20, Thm. 4.1.1] for the proof that these notions, and those given in Definition 9, are equivalent in the case of unweighted bipartite graphs with perfect matching.)

Now we need to prove the generalization of the classical bipartite Dulmage-Mendelsohn Decomposition Theorem for weighted bipartite graphs with focus on minimum vertex covers instead of maximum matchings.

Theorem 6. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = L \cup R$ and vertex weight $w : V \to (0, \infty)$ such that $vc(G, w) = \frac{1}{2}w(V)$. The subgraph of G which contains all allowed edges for (G, w) consists of elementary blocks, $B_i = G[L_i \cup R_i]$ for $i = 1, 2, \dots, r$ (here $\cup_{i=1}^r L_i = L$ and $\cup_{i=1}^r R_i = R$ are partitions).The ordering B_1, B_2, $\dots$, B_r can be chosen with the following property: every edge in G between two blocks B_i and B_j with $i < j$ must have its R-vertex in B_i and L-vertex in B_j. The decomposition into blocks and their admissible ordering can be constructed in polynomial time; in unweighted case in time $O(|E|\sqrt{|V|})$.*

Proof. (a) Let us start with the structural part of the theorem.

If $\emptyset \neq A \subsetneq L$ implies $w(N(A)) > w(A)$, then (G, w) is elementary and the decomposition trivially consists of one block. Otherwise we can find L_1, $\emptyset \neq L_1 \subsetneq L$, such that $w(N(L_1)) = w(L_1)$. We can take such L_1 which is minimal (on inclusion) with respect this property, i.e., $\emptyset \neq A \subsetneq L_1$ implies $w(N(A)) > w(A)$. Put $R_1 := N(L_1)$. Clearly, $(G[L_1 \cup R_1], w)$ is elementary. It is not difficult to see that for every $C \in VC(G, w)$ $C \cap (R_1 \cup L_1)$ is a minimum vertex cover for $(G[L_1 \cup R_1], w)$ and $C \setminus (R_1 \cup L_1)$ is a minimum vertex cover for $(G[V \setminus (L_1 \cup R_1)], w)$. Hence, $C \cap (L_1 \cup R_1)$ is either L_1 or R_1. It implies that each edge between L_1 and R_1 is allowed for (G, w).

Now we can do the same with the graph $G_1 := G[V \setminus (L_1 \cup R_1)]$, as (G_1, w) satisfies the same structural assumptions as (G, w). Continuing by induction we will end in finitely many steps with $\emptyset \neq L_i \subseteq L$, $\emptyset \neq R_i \subseteq R$ such that $V = \cup_{i=1}^r (L_i \cup R_i)$ is a partition in G with the following properties:

(i) for $i < j$ there is no edge between L_i and R_j,
(ii) for every $i \in \{1, 2, \dots, r\}$ and for every $C \in VC(G, w)$ $C \cap (L_i \cup R_i)$ is either L_i or R_i,
(iii) for every $i \in \{1, 2, \dots, r\}$ $(G[L_i \cup R_i], w)$ is an elementary graph.

The property (ii) implies that all edges inside a fixed block $B_i = G[L_i \cup R_i]$ are allowed for (G, w). The property (i) implies that all edges between two blocks B_j and B_i $(j > i)$

are forbidden; namely (for a fixed $i = 1, \dots, r-1$) $C_i := \cup_{k=1}^{i} R_k \bigcup \cup_{k=i+1}^{r} L_k$ is an element of $VC(G, w)$ containing both vertices from each edge between blocks $B_{j'}$ and $B_{i'}$, where $i' \in \{1, \dots, i\}$, $j' \in \{i+1, \dots, r\}$.

It easily follows that each B_i is a component of the subgraph of G obtained by restricting G to the edges allowed for (G, w). This completes the proof of the structural part of the theorem.

(b) Let us focus on the complexity of the problem to construct the above decomposition into blocks. Identifying blocks basically reduces to identifying edges which are allowed for (G, w). But $\{u, v\} \in E$ is allowed for (G, w) iff $vc(G \setminus (\{u\}, \{v\}), w) + w(u) + w(v) > vc(G, w)$, which can be tested efficiently as G is bipartite. It easily follows that the above decomposition can be constructed in polynomial time.

Let us briefly discuss time complexity of the following algorithm that provides decomposition into blocks and their admissible ordering in unweighted case:

Step 1: Find a perfect matching M of G.
Step 2: Build the directed graph G_M from G by replacing each edge $\{u, v\}$ in M by two arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$, and by orienting all other edges from R to L.
Step 3: Compute the strongly connected components of G_M, each of them corresponds to a block of G (independently of M chosen).
Step 4: Find an admissible ordering of blocks, build the reduced digraph from G_M by contracting each block of G_M in a vertex. The resulting digraph is acyclic, that induces a partial order between blocks. Hence a compatible total order of blocks can be obtained by any topological sorting of that acyclic graph.

Steps 2-4 can be computed in time $O(|V|+|E|)$, e.g., by depth first search. Hence running time of the whole algorithm is dominated by the time complexity $O(|E|\sqrt{|V|})$ of the search for a maximum matching. $\square$

Remark 5. As follows from the proof of Theorem 6 L and R are always in $VC(G, w)$, but if (G, w) is not elementary there are also "intermediate" minimum vertex covers. Namely, each set $C_i := \bigcup_{k=1}^{i} R_k \cup \bigcup_{k=i+1}^{r} L_k$, $\quad i = 0, 1, \dots, r$ is a minimum vertex cover for (G, w). Moreover, for each $C \in VC(G, w)$ and each $i \in \{1, 2, \dots, r\}$, $C \cap (L_i \cup R_i)$ is either L_i or R_i.

As follows from Theorem 5 if a graph (G, w) has a nontrivial crown decomposition, then there exists also a crown decomposition of the form $(V_0^x, V_1^x, V_{\frac{1}{2}}^x)$ for some $x \in VC^*(G, w)$. Now we are interested in those minimum half-integral vertex covers x for which a partition V_0^x, V_1^x, $V_{\frac{1}{2}}^x$ of G is such that $(G[V_{\frac{1}{2}}^x], w)$ does not have a crown, or, equivalently, $z \equiv \frac{1}{2}$ on $V_{\frac{1}{2}}^x$ is the *unique* element of $VC^*(G[V_{\frac{1}{2}}^x], w)$. There are possibly many such $x \in VC^*(G, w)$, but we will see that $V_{\frac{1}{2}}^x$ is uniquely determined, namely $V_{\frac{1}{2}}^x = K_{\min}(G, w)$. In [22] Nemhauser and Trotter described how to find such x efficiently, and the uniqueness of $V_{\frac{1}{2}}^x$ was proved by Picard and Queyranne in [24]. More comprehensive treatment on this topic in unweighted case was given by Bourjolly and Pulleyblank in [4].

In the following theorem we provide an efficient decomposition of (G, w) into irreducible parts, that nicely describes the structure of all minimum vertex covers in the subgraph $(G[K_{\max} \setminus K_{\min}], w)$. In some cases this theorem allows to find several vertices of $V_i(G, w) \setminus V_i^*(G, w)$, $i = 0, 1$. Therefore it can be useful also for the problem to find all minimum vertex covers for (G, w) to reduce further the kernel $(G[K_{\max}], w)$ obtained in Theorem 4.

Theorem 7. *Let $G = (V, E)$ be a graph with vertex weights $w : V \to (0, \infty)$ and suppose $vc^*(G, w) = \frac{1}{2}w(V)$. Then there exists a polynomial time algorithm (running in time $O(|E|\sqrt{|V|})$ in unweighted case) that constructs a partition*

$$V = K_{\min} \cup \bigcup_{i=1}^{s} I_i \cup \bigcup_{i=1}^{s} H_i$$

with the following properties:

(i) For each $i \in \{1, 2, \dots, s\}$ the following holds true:
  (a) $H_i = N(I_i) \setminus \cup_{j=1}^{i-1} H_j$,
  (b) $w(I_i) = w(H_i) = vc(G[I_i \cup H_i], w)$,
  (c) $\emptyset \neq T \subsetneq I_i$ implies $w(N(T) \cap H_i) > w(T)$,
  (d) for every $C \in VC(G, w)$ $C \cap (I_i \cup H_i)$ is either I_i or H_i, and if H_i is not an independent set then $C \cap (I_i \cup H_i) = H_i$.

(ii) There is a minimum half-integral vertex cover x for (G, w) such that $V_0^x = \cup_{i=1}^{s} I_i$, $V_1^x = \cup_{i=1}^{s} H_i$, and $V_{\frac{1}{2}}^x = K_{\min}$.

(iii) $K_{\min} = \cap_{y \in VC^*(G,w)} V_{\frac{1}{2}}^y$. Moreover, if $K_{\min} \neq \emptyset$, $z \equiv \frac{1}{2}$ on $K_{\min}$ is the unique element of $VC^*(G[K_{\min}], w)$ and $vc(G[K_{\min}], w) > vc^*(G[K_{\min}], w) = \frac{1}{2}w(K_{\min})$.

Proof. First, we apply the Theorem 6 to the bipartite graph (G^b, w^b) of (G, w). For any $i = 1, \dots, r$ a block $B_i = G^b[L_i \cup R_i]$ of (G^b, w^b) has $L_i = I_i^L$, $R_i = H_i^R$ for some I_i, $H_i \subseteq V$ with $w(I_i) = w(H_i)$. The structure of allowed edges and blocks of (G^b, w^b) now additionally reflects the presence of the automorphism Φ of (G^b, w^b), described in Remark 2. For any $\{u, v\} \in E$, $\{u^L, v^R\}$ is allowed edge of (G^b, w^b) iff $\{v^L, u^R\}$ is allowed. Hence vertices $I_i^L \cup H_i^R$ induce a block of (G^b, w^b) iff $H_i^L \cup I_i^R$ do. It follows, in particular, that for any such block either $H_i \cap I_i = \emptyset$ (a block is called *simple*), or $H_i = I_i$ (a block is called *hard*). Simple blocks of (G^b, w^b) appear in pairs, $G^b[I_i^L \cup H_i^R]$ together with $G^b[H_i^L \cup I_i^R]$. Hard blocks are of the form $G^b[H_i^{\mathrm{L}} \cup H_i^{\mathrm{R}}]$. Let there be exactly s pairs of simple blocks and $(r - 2s)$ hard blocks. Now we can change the order of blocks guaranteed by Theorem 6 to fulfill in addition the following symmetry related to the automorphism Φ: simple blocks form initial and end segments of B_1, B_2, …, B_r, with (B_i, B_{r+1-i}) being a pair of twin simple blocks for each $i = 1, 2, \dots, s$, i.e., if $B_i = G^b[I_i^{\mathrm{L}} \cup H_i^{\mathrm{R}}]$ then $B_{r+1-i} = G^b[H_i^{\mathrm{L}} \cup I_i^{\mathrm{R}}] =: \widetilde{B}_i$.

Let us explain briefly how any admissible ordering B_1, B_2, …, B_r of blocks can be converted to desired symmetric one. If $s = 0$ there is nothing to prove, hence suppose $s \geq 1$. Assume B_1 is hard, $B_1 = G^b[H_1^L \cup H_1^R]$, it follows that there are no edges in G^b between H_1^L and $V^b \setminus H_1^R$ (hence no edges in G between H_1 and $V \setminus H_1$). Consequently, B_1 is a component of G^b and we can move it freely and still preserve admissibility of the ordering. The same applies to any block from initial segment (if any) of hard blocks, hence we can completely ignore them in our symmetric conversion (and we can place them later to the middle group of hard blocks).

Now take the first simple block, say B_i. Place it at the beginning, and its twin $\widetilde{B}_i$ place at the end, and relabel. It is easy to check that admissibility of the ordering is preserved. Now we can forget also this pair of blocks and apply the same rule for the rest. This procedure leads simply to desired symmetric admissible ordering of blocks of (G^b, w^b), B_1, B_2, …, B_r, with $B_i = G^b[I_i^{\mathrm{L}} \cup H_i^{\mathrm{R}}]$, B_1, B_2, …, B_s, being simple $B_{r+1-i} = \widetilde{B}_i$ for $i = 1, 2, \dots, s$ as well and $B_i = G^b[H_i^{\mathrm{L}} \cup H_i^{\mathrm{R}}]$ for $s + 1 \leq i \leq r - s$ being hard.

Now (i) follows directly from Theorem 6 and we prove the properties (ii) and (iii).

Let $K := \bigcup_{i=s+1}^{r-s} H_i$. In what follows we show that $K = K_{\min}(G, w)$ and that there is $x \in VC^*(G, w)$ for which $V^x_{\frac{1}{2}} = K$.

As any $C \in VC(G^b, w^b)$ contains from each hard block B_i exactly one of sets H_i^{L}, H_i^{R}, it easily follows that for each $y \in VC^*(G, w)$ $y \equiv \frac{1}{2}$ on K. Equivalently, $K \subseteq K_{\min}$.

Now we will show that, in fact, the equality holds here, finding $x \in VC^*(G, w)$ for which $V^x_{\frac{1}{2}} = K$ holds.

As was mentioned in Remark 5, each $C_i := \cup_{k=1}^{i} H_k^R \bigcup \cup_{k=i+1}^{r} I_k^{\mathrm{L}} \in VC(G^b, w^b)$ for $i = 0$, $1, \ldots, r$, is a minimum vertex cover for (G, w). If moreover $0 \le i \le s$, then an element $x_i \in VC^*(G, w)$ satisfies $V_0^{x_i} = \cup_{k=1}^{i} I_k$, $V_1^{x_i} = \cup_{k=1}^{i} H_k$, $V^{x_i}_{\frac{1}{2}} = K \bigcup \cup_{k=i+1}^{s} (I_k \cup H_k)$. In particular, for $x := x_s \in VC^*(G, w)$ we have $V_0^x = \cup_{k=1}^{s} I_k$, $V_1^x = \cup_{k=1}^{s} H_k$, and $V^x_{\frac{1}{2}} = K$ $(\subseteq \cap_{y \in VC^*(G,w)} V^y_{\frac{1}{2}} \subseteq V^x_{\frac{1}{2}})$. Hence $V^x_{\frac{1}{2}} = K_{\min} = \cap_{y \in VC^*(G,w)} V^y_{\frac{1}{2}}$.$\square$

Remark 6. Under the assumptions of Theorem 7, we have $V_0^*(G, w) = V_1^*(G, w) = \emptyset$ and we cannot say much, in general, about $V_0(G, w)$ and $V_1(G, w)$. But in many cases the theorem gives us nontrivial information about $V_0(G, w)$ and $V_1(G, w)$. Let the corresponding partition $V = K_{\min} \bigcup \cup_{i=1}^{s} I_i \bigcup \cup_{i=1}^{s} H_i$ be fixed. We will say that $i \in \{1, 2, \ldots, s\}$ is *determined* if for every $C \in VC(G, w)$, $C \cap (I_i \cup H_i) = H_i$ (i.e., $H_i \subseteq C$ and $I_i \cap C = \emptyset$). Clearly, if H_i is not an independent set, then i is *determined.* Further, if i is *determined* and $j < i$ is such that there exists an edge between I_i and H_j, then j is *determined.* Also, if i is such that for some $k > i$ there exists an edge between H_i and H_k as well as an edge between H_i and I_k, then i is *determined.* These and similar observations allow in some cases to further reduce the kernel $(G[V_{\frac{1}{2}}], w)$ obtained in Theorem 4, as we can tell à priori for some i that $I_i \subseteq V_0(G, w)$ and $H_i \subseteq V_1(G, w)$.

Let us mention some of the consequences of Theorem 7. For a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ we firstly apply Theorem 4 to obtain $(G[V_{\frac{1}{2}}], w)$ satisfying assumptions of Theorem 7. Applying Theorem 7 reduces the MIN-w-VC problem for (G, w) to the one for $(G[K_{\min}], w)$, for which $x \equiv \frac{1}{2}$ on K is the unique element of $VC^*(G[K_{\min}], w)$. Moreover, the difference $vc(G, w) - vc^*(G, w)$ is preserved during this reduction. It is the same as $vc(G[K_{\min}], w) - vc^*(G[K_{\min}], w)$, which is zero iff $K_{\min} = \emptyset$. Hence we have obtained as a byproduct a new polynomial time algorithm that recognizes w-KEG graphs and solves MIN-w-VC problem on them.

Moreover, for a weighted KEG graph (G, w) we can find $V_0(G, w)$ and $V_1(G, w)$ in polynomial time. This problem reduces, by Theorem 4, to the one for $(G[K_{\max}], w)$. Hence we can assume that $G = G[K_{\max}]$. Applying Theorem 7 we see that $K_{\min} = \emptyset$ and to find $V_0(G, w)$ and $V_1(G, w)$ means to find the set J of all $i \in \{1, 2, \ldots, s\}$ that are *determined* (with the meaning introduced in Remark 6). Clearly, $V_0(G, w) = \cup_{i \in J} I_i$ and $V_1(G, w) = \cup_{i \in J} H_i$. We will proceed by induction. We start with $i = 1$ and try to find if i is *determined*, or it is not. If 1 is not *determined*, then there is $C \in VC(G, w)$ such that $I_1 \subseteq C$ and $H_1 \cap C = \emptyset$. Consider one such C fixed, we can derive some properties about the sets $J_I := \{i : I_i \subseteq C\}$, $J_H := \{i : H_i \subseteq C\}$. We either get a contradiction $J_I \cap J_H \neq \emptyset$, or find such C.

We will build up J_I, J_H starting from $J_I := \{1\}$, $J_H := \emptyset$, as follows:

Step 1. If $i \in J_I$ and $j \notin J_I$ are such that there exists an edge between H_i and I_j, add j to J_I. Repeat till J_I can be enlarged this way. Then go to Step 2.

Step 2. If $i \in J_I$ and $j \in \{1, 2, \ldots, s\}$ are such that there exists an edge between H_i and H_j (i.e., within H_i, if $j = i$), add j to J_H if $j \notin J_I$, else go to Step 5. Repeat till J_H can be enlarged this way. Then go to Step 3.

Step 3. If $i \in J_H$ and $j < i$ are such that there exists an edge between I_i and H_j, add j to J_H if $j \notin J_I$, else go to Step 5. Repeat till J_H can be enlarged this way. Then go to Step 4.

Step 4. 1 is not *determined*. There is $C \in VC(G, w)$ such that $I_1 \subseteq C$. Any such C contains each I_i, $i \in J_I$, and each H_i, $i \in J_H$. If there are still some $i \in \{1, 2, \ldots, s\} \setminus (J_I \cup J_H)$, one can define one such C, taking $C \cap (I_i \cap H_i) = H_i$ for any such i.

Step 5. 1 is *determined*.

Once the question of whether 1 is *determined* was answered, we remember the answer, remove $I_1 \cup H_1$ from G and continue with the rest of the graph and with $i = 2$, and so on. The minimum vertex covers in the graph after each removal are exactly restrictions of $VC(G, w)$ to that smaller vertex set. Hence the set J of all $i \in \{1, 2, \ldots, s\}$ that are *determined*, can be computed inductively.

Corollary 1. *There is a polynomial time algorithm (of time complexity $O(|E|\sqrt{|V|})$ in unweighted case) that for a graph $G = (V, E)$ with vertex weights $w : V \to (0, \infty)$ decides whether $vc(G, w) = vc^*(G, w)$, and if the equality holds, finds one minimum vertex cover for (G, w). Moreover, for graphs (G, w) for which the equality holds, i.e., for König-Egerváry graphs, $V_0(G, w)$ and $V_1(G, w)$ can be computed in polynomial time.*

Remark 7. Since a (maximum) independent set for (G, w) is a complement of a (minimum) vertex cover for (G, w), all results above can be translated in obvious way to the ones for the Maximum Weighted Independent Set problem.

5 Applications

The best known application of Nemhauser-Trotter reduction is that it provides a simple 2-approximation algorithm for the Min-w-VC problem. Consider the following algorithm for a given instance (G, w): Find $x \in VC^*(G, w)$ such that $V^x_{\frac{1}{2}} = K_{\min}(G, w)$, pick *any* vertex cover C' of $G[K_{\min}]$, and return a vertex cover $C := V^x_1 \cup C'$ of G.

As $vc(G, w) = w(V^x_1) + vc(G[V^x_{\frac{1}{2}}], w) \geq w(V^x_1) + \frac{1}{2}w(V^x_{\frac{1}{2}})$, we obtain that the approximation factor of this algorithm is at most $\frac{w(V^x_1)+w(V^x_{\frac{1}{2}})}{w(V^x_1)+\frac{1}{2}w(V^x_{\frac{1}{2}})} \leq 2$. This factor is even strictly less than 2 on instances for which V^x_1 is at least a fixed fraction of $V^x_{\frac{1}{2}}$. More precisely, if $\delta|V^x_{\frac{1}{2}}| \leq 2(1-\delta)|V^x_1|$, then its approximation factor is at most $2 - \delta$.

Up to now only the property that $vc(G[V^x_{\frac{1}{2}}], w) \geq \frac{1}{2}w(V^x_{\frac{1}{2}})$ was used. But the fact that $vc^*(G[V^x_{\frac{1}{2}}], w) = \frac{1}{2}w(V^x_{\frac{1}{2}})$ (or, equivalently, that $G[V^x_{\frac{1}{2}}]$ has a perfect fractional w-matching) can be used more efficiently. The stronger condition, that $y \equiv \frac{1}{2}$ on $V^x_{\frac{1}{2}}$ is the only element of $VC^*(G[V^x_{\frac{1}{2}}], w)$ if x was such that $V^x_{\frac{1}{2}} = K_{\min}(G, w)$, can be used as well.

The unweighted case

In what follows we will discuss the unweighted case, i.e., the case when $w \equiv 1$. For a fixed $x \in VC^*(G)$ we obtain from Lemma 2 that

$$\nu(G) = \nu(G[V_0^x \cup V_1^x]) + \nu(G[V_{\frac{1}{2}}^x]),$$
$$\nu^*(G) = \nu^*(G[V_0^x \cup V_1^x]) + \nu^*(G[V_{\frac{1}{2}}^x]),$$
$$\nu(G[V_0^x \cup V_1^x]) = \nu^*(G[V_0^x \cup V_1^x]) = |V_1^x|, \text{ and } \nu^*(G[V_{\frac{1}{2}}^x]) = \frac{1}{2}|V_{\frac{1}{2}}^x|.$$

(Recall that by the LP duality $vc^*(G) = \nu^*(G) \geq \nu(G)$.)

Moreover, it is clear that for any maximum matching M in G all vertices of V_1^x are matched, and no edge within V_1^x belongs to M. It is well known that in the unweighted case the extremal maximum fractional matchings are half-integral. Hence $G[V_{\frac{1}{2}}^x]$ has a perfect half-integral matching, and therefore the vertex set of $G[V_{\frac{1}{2}}^x]$ can be covered by a set of vertex disjoint edges and odd cycles. This in turn shows that $\nu(G_{\frac{1}{2}}^x) \geq \frac{1}{3}|V_{\frac{1}{2}}^x|$, with the equality iff all components of $G[V_{\frac{1}{2}}^x]$ are triangles. In conjunction with inequalities above it implies that $\nu(G) \geq \frac{2}{3}\nu^*(G)$, with the equality iff all nontrivial components of G are triangles.

Consider the following simple approximation algorithm for Min-VC for a given input instance $G = (V, E)$: find $x \in VC^*(G)$ with $V_{\frac{1}{2}}^x = K_{\min}(G)$ and pick any inclusionwise maximal matching M in $G[V_{\frac{1}{2}}^x]$. Let V_M be the set of vertices matched by M. Return a vertex cover $C := V_1^x \cup V_M$ of G. Clearly, $|C| \leq |V_1^x| + 2\nu(G[V_{\frac{1}{2}}^x])$. Using Lorentzen bound $vc(G) \geq 2\nu^*(G) - \nu(G)$ (applied to the graph $G[V_{\frac{1}{2}}^x]$) we get

$$vc(G) = |V_1^x| + vc(G[V_{\frac{1}{2}}^x]) \geq |V_1^x| + |V_{\frac{1}{2}}^x| - \nu(G[V_{\frac{1}{2}}^x]).$$

For $\delta \in \langle 0, \frac{1}{3} \rangle$ let $\mathcal{G}_\delta = \{G : \nu(G) \leq (1-\delta)\nu^*(G)\}$. Hence $\mathcal{G}_0$ is the set of all graphs, and $\mathcal{G}_{\frac{1}{3}} = \{G :$ each nontrivial component of G is a triangle$\}$. We have proved in [9] that the approximation threshold for the Min-VC problem restricted to graphs with a perfect matching is the same as for the problem in general graphs. This suggests that instances $G = (V, E)$ with $\nu^*(G) = \nu(G) = \frac{|V|}{2}$ are hardest to approximate. We will show that the algorithm described above performs on graphs from $\mathcal{G}_\delta$ with an approximation factor at most $2\frac{1-\delta}{1+\delta}$. Notice that this varies from 2 to 1 when δ varies from 0 to $\frac{1}{3}$.

Indeed, $G \in \mathcal{G}_\delta$ means $|V_1^x| + \nu(G[V_{\frac{1}{2}}^x]) \leq (1-\delta)(|V_1^x| + \frac{1}{2}|V_{\frac{1}{2}}^x|)$, or equivalently $\nu(G[V_{\frac{1}{2}}^x]) \leq \frac{1-\delta}{2}|V_{\frac{1}{2}}^x| - \delta|V_1^x|$. Using that in the above inequalities for $|C|$ and $vc(G)$ we get $|C| \leq (1-2\delta)|V_1^x| + (1-\delta)|V_{\frac{1}{2}}^x|$, and $vc(G) \geq (1+\delta)|V_1^x| + \frac{(1+\delta)}{2}|V_{\frac{1}{2}}^x|$. Thus the approximation factor of this algorithm is at most

$$\frac{(1-2\delta)|V_1^x| + (1-\delta)|V_{\frac{1}{2}}^x|}{(1+\delta)|V_1^x| + \frac{(1+\delta)}{2}|V_{\frac{1}{2}}^x|} \leq 2\frac{1-\delta}{1+\delta}.$$

The left hand side provides even better estimate, if $|V_1^x|$ is a significant fraction of $|V_{\frac{1}{2}}^x|$.

Definition 10. *A graph $G = (V, E)$ is called* regularizable *if it is possible to replace each edge $e \in E$ with $n(e) \geq 1$ multiple edges so that the resulting multigraph is regular. A graph G is called* Hamiltonian-connected *if every two distinct vertices are connected in G by a Hamiltonian path.*

Recall that the problem kernel $G[K_{\min}]$ has the property that $y \equiv \frac{1}{2}$ on $K_{\min}$ is the only element of $VC^*(G[K_{\min}])$. This in turn implies that instances $G = (V, E)$ of Min-VC for which $y \equiv \frac{1}{2}$ on V is the only minimum half-integral vertex cover in G, are as hard to approximate as the general ones. Additionally to Theorem 5, the following characterization of these graphs was given by Berge [2] in unweighted case. The solution $y \equiv \frac{1}{2}$ on V is the only minimum half-integral vertex cover in G if and only if G is regularizable and each component of G is non-bipartite. It is easy to make such regularization of a regularizable graph $G = (V, E)$ in polynomial time, and using standard multiplication techniques convert this regular multigraph to a regular graph in such way that the minimum vertex cover increases by a multiplicative factor that is easy to compute. This shows that regular instances of Min-VC are as hard to approximate as the general ones.

Similar result can be obtained for Hamiltonian-connected graphs, as well. Given a graph $G = (V, E)$ for which $y \equiv \frac{1}{2}$ on V is the only element of $VC^*(G)$. Then for some $k \leq |V|$ the graph $G[k]$ is Hamiltonian-connected, where $G[k]$ is obtained from G by replacing each vertex with an independent set of size k (see [6]). But Min-VC for G and $G[k]$ are equally hard to approximate. This shows that Hamiltonian-connected instances of Min-VC are as hard to approximate as the general ones.

Corollary 2. *The threshold on polynomial time approximability of the* Min-VC *problem is the same as the one for this problem restricted to regular graphs or, alternatively, to Hamiltonian-connected graphs.*

6 Parametrized Complexity and Vertex Covers

To describe some applications of our strong version of crown reductions, we will confine ourselves to the unweighted Min-VC problem in this section. But the results can be generalized in a straightforward way to the case of real weights $w \geq 1$.

The Minimum Vertex Cover problem and its variants play a very special role among fixed-parameter tractable problems. Let us recall the basic parametrized version of the problem:

Instance: A graph $G = (V, E)$ and a nonnegative integer k

Decision version: Is there a vertex cover for G with at most k vertices?

Search version: Either find a vertex cover for G with at most k vertices or report that no such vertex cover exists.

Recall for the parametrized decision version of the vertex cover problem the reduction to a problem kernel means to apply an efficient preprocessing on the instance (G, k) to construct another instance (G_1, k_1), where G_1 is a subgraph of G, $k_1 \leq k$, and G_1 has a vertex cover with at most k_1 vertices iff G has a vertex cover with at most k vertices. As observed in [8], the Nemhauser-Trotter Theorem allows to find efficiently a linear size problem kernel for Min-VC. Namely, there is an algorithm of running time $O(k|V| + k^3)$ that, given an instance $(G = (V, E), k)$, constructs another instance $(G' = (V', E'), k')$ with the following properties: G' is an induced subgraph of G, $|V'| \leq 2k'$, $k' \leq k$, and G admits a vertex cover of size k iff G' admits a vertex cover of size k'. Clearly, using the same technique one can solve the parametrized search version of the vertex cover problem, or the problem: to find a *minimum vertex cover* of G if $vc(G) \leq k$, or report that $vc(G) > k$.

Parametrized All-Min-VC problem

Unlike the Nemhauser-Trotter Theorem, Theorem 4 can be used as efficient reduction to linear size problem kernel for the following problem: to find *all minimum vertex covers* if $vc(G) \leq k$ or report that $vc(G) > k$.

Instance: $(G = (V, E), k)$ and a nonnegative integer k

Search version: Either find all minimum vertex covers for G if $vc(G) \leq k$, or report that $vc(G) > k$.

Theorem 8. *There is an algorithm of running time $O(k|V| + k^3)$ that for a given instance $(G = (V, E), k)$ either reports that $vc(G) > k$, or finds a partition $V = N \cup Y \cup V'$ such that $G' := G[V']$, $k' := k - |Y|$, $vc(G') \geq \frac{1}{2}|V'|$, and $|V'| \leq 2k'$. Moreover, $vc(G) \leq k$ iff $vc(G') \leq k'$, and assuming $vc(G) \leq k$:*
(i) for every minimum vertex cover C' for G': $C' \cup Y \in VC(G)$, and
(ii) for every minimum vertex cover C for G: $Y \subseteq C \subseteq Y \cup V'$ and $C \cap V' \in VC(G')$.

Proof. Let an instance $(G = (V, E), k)$ be given. Clearly, every vertex $v \in V$ of degree at least $k+1$ has to belong to every vertex cover of size at most k, provided $vc(G) \leq k$. Denote Y'', the set of vertices of G of degree at least $(k + 1)$, N'', the set of isolated vertices of $G \setminus Y''$, $V'' := V \setminus (Y'' \cup N'')$, and $k'' = k - |Y''|$. Firstly, in running time $O(k|V|)$ we can construct a graph $G'' = (V'', E'') := G[V'']$ (see, e.g., Buss [5] for such simple algorithm). Clearly, $vc(G) \leq k$ iff $vc(G'') \leq k''$, and assuming $vc(G) \leq k$: (i) for every $C'' \in VC(G'')$, $C'' \cup Y'' \in VC(G)$, and (ii) for every $C \in VC(G)$, $Y'' \subseteq C \subseteq Y'' \cup V''$ and $C \cap V'' \in VC(G'')$.

Each vertex of G'' has degree at most k. Hence $vc(G'') \leq k''$ is only possible if $|E''| \leq kk''$. If $|E''| > kk''$, we can report that $vc(G) > k$ and the algorithm terminates. Otherwise, we have $|E''| \leq k \cdot k''$ $(\leq k^2)$, and since G'' does not contain isolated vertices, it follows that $|V''| \leq 2|E''| \leq 2k^2$. Now we apply Theorem 4 to the graph G'' (with $w \equiv 1$). Namely, we partition the vertex set V'' into three subsets V_0, V_1, $V_{\frac{1}{2}}$ in time $O(|E''|\sqrt{|V''|}) = O(k^3)$. Further, we put $Y := Y'' \cup V_1$, $N := N'' \cup V_0$, $V' := V_{\frac{1}{2}}$, $G' := G[V']$, and $k' := k'' - |V_1| = k - |Y|$. Obviously, $vc(G) \leq k$ iff $vc(G') \leq k'$, and from Theorem 4 also $vc(G') \geq \frac{1}{2}|V'|$. It means if $|V'| > 2k'$, we can report that $vc(G') > k'$, hence $vc(G) > k$, and the algorithm terminates. Otherwise $|V'| \leq 2k'$ holds, as was required. All other properties follow directly from Theorem 4.$\square$

Theorem 8 can be used to many other parametrized problems related to Min-VC as reduction to linear size problem kernel. The typical example is the problem, whose task is to find one minimum vertex cover for G under some additional constraints.

Parametrized Constrained-Min-VC problem

Instance: $(G = (V, E), k)$, k a nonnegative integer, and finitely many linear constraints P_1, P_2, $\ldots$, P_r of the form P_i: $\sum_{v \in V} a_i(v)x(v) \leq b_i$, $i = 1, 2, \ldots, r$, where $a_i(v)$, $b_i \in \mathbb{R}$.

Task: If $vc(G) \leq k$ find C from $VC(G)$, whose indicator function $x = x^C$ satisfies all constraints P_1, P_2, $\ldots$, P_r, otherwise report that no such minimum vertex cover exists.

The most natural case is when each $a_i(v)$ is either 0 or 1, and b_i are nonnegative integers. Then constraint P_i says, that $|C \cap A_i| \leq b_i$ for a set $A_i := \{v \in V : a_i(v) = 1\}$ and for a vertex cover $C \in VC(G)$ to be found. The problem has received considerable attention even in its very simplified version, when $G = (V, E)$ is a bipartite graph with bipartition (L, R), and two nonnegative integers k_{L} and k_{R} (with $k = k_{\mathrm{L}} + k_{\mathrm{R}}$) are given as an input. The k_{L} and k_{R} represent constraints $|C \cap L| \leq k_{\mathrm{L}}$, $|C \cap R| \leq k_{\mathrm{R}}$ on $C \in VC(G)$ to be found. This problem arises from the extensively studied fault coverage problem for reconfigurable memory arrays in VLSI design, see [8] and references therein.

Theorem 8 clearly allows efficient reduction to the linear size problem kernel for PARAMETRIZED CONSTRAINED MIN-VC. Namely, $(G = (V, E), k)$ with constraints $P_i : \sum_{v \in V} a_i(v)x(v) \leq b_i$, $i = 1, 2, \ldots, r$ is reduced using Theorem 8 to $(G' = (V', E'), k')$ with $|V'| \leq 2k'$ $(\leq 2k)$, and with constraints $P'_i : \sum_{v \in V'} a_i(v)x(v) \leq b'_i$ $(:= b_i - \sum_{v \in Y} a_i(v))$, $i = 1, 2, \ldots, r$.

Further research

Let us mention that the crown reduction technique can be applied effectively to other parametrized problems (see [10]). Moreover, our new decomposition theorems for vertex covers have connections with "parametrized enumeration" in the sense of listing all minimal solutions, as also discussed in [14]. We believe that the technique of this paper may be a powerful tool to designing algorithms for other fixed parameter tractable problems that are related to the MIN-VC problem.

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