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Dimension is compression

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Abstract

Effective fractal dimension was defined by Lutz (2003) in order to quantitatively analyze the structure of complexity classes. Interesting connections of effective dimension with information theory were also found, in fact the cases of polynomial-space and constructive dimension can be precisely characterized in terms of Kolmogorov complexity, while analogous results for polynomial-time dimension haven't been found.

In this paper we remedy the situation by using the natural concept of reversible time-bounded compression for finite strings. We completely characterize polynomial-time dimension in terms of polynomial-time compressors.

1 Introduction

Effective fractal dimension was defined in [10] in order to quantitatively analyze the structure of complexity classes. See [9, 14] for a summary of the main results.

In parallel, the connections of this effective dimension with algorithmic information started being patent. The cases of constructive, recursive and polynomial-space dimension were characterized precisely as the best case asymptotic compression rate when using plain, recursive, and polynomial-space-bounded Kolmogorov complexity, respectively [13, 11, 6].

But the case of polynomial-time bounds remained elusive [8]. This is not strange since computing even approximately the value of time-bounded Kolmogorov complexity seems to require an exponential search. The main difference with space-bounded Kolmogorov complexity is reversibility, in this later case the encoding phase can be performed within similar space-bounds.

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In this paper we look at the usual notion of compression algorithm for finite strings. A polynomial-time compression scheme is just a pair of encoder and decoder algorithms, both working in polynomial-time. We consider encoders that do not completely start from scratch when working on an extension of a previous input. This last condition is formalized in section 3 with a conditional-entropy like inequality.

We exactly characterize p-dimension as the best case asymptotical (that is, i.o.) compression ratio attained by these polynomial-time compression schemes.

Several results on the polynomial-time dimension of complexity classes can be now interpreted as compressibility results. For example, the (characteristic sequences of) languages in a class of p-dimension 1 cannot be i.o. compressed by more that a sublinear amount. Here we obtain results on the compressibility of complete and autoreducible languages.

Buhrman and Longprè have given a characterization of p-measure in terms of compressibility in [4], but in that case the compressors are restricted to extenders and the encoder is required to give several alternatives, one of them being the correct output. In the light of our present results we can view effective dimension as an information content measure for infinite strings, whereas resource-bounded measure can only distinguish the extreme case of measure 1 classes.

2 Preliminaries

The Cantor space $\mathbf{C}$ is the set of all infinite binary sequences. If $w \in \{0,1\}^*$ and $x \in \{0,1\}^* \cup \mathbf{C}$, $w \sqsubseteq x$ means that w is a prefix of x. For $0 \leq i \leq j$, we write $x[i \dots j]$ for the string consisting of the i-th through the j-th bits of x.

Let p be the set of polynomial-time computable functions. Let E= DTIME($2^{O(n)}$).

Definition. Let $s \in [0, \infty)$.

1. An *s-gale* is a function $d : \{0,1\}^* \to [0, \infty)$ satisfying

$$d(w) = 2^{-s}[d(w0) + d(w1)]$$

for all $w \in \{0,1\}^*$.

2. A *martingale* is a 0-gale, that is, a function $d : \{0,1\}^* \to [0, \infty)$ satisfying

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

for all $w \in \{0,1\}^*$.

$\square$

We will often use the following basic result.

Theorem 2.1 *[10] Let $s, s' \in [0, \infty)$. If d is an s-gale them $d'(w) = 2^{(s'-s)|w|}d(w)$ is an s'-gale.*

Definition. Let $s \in [0, +\infty)$ and d be an s-gale.

1. We say that d *succeeds* on a sequence $S \in \mathbf{C}$ if

$$\limsup_{n\to\infty} d(S[0\ldots n]) = \infty$$

   The success set of d is $S^{\infty}[d] = \{S \in \mathbf{C} \mid d \,\text{succeeds on}\, S\}$

2. We say that d *succeeds strongly* on a sequence $S \in \mathbf{C}$ if

$$\liminf_{n\to\infty} d(S[0\ldots n]) = \infty$$

   The strong success set of d is $S^{\infty}_{\mathrm{str}}[d] = \{S \in \mathbf{C} \mid d \,\text{succeeds strongly on}\, S\}$

$\square$

Definition. Let $X \subseteq \mathbf{C}$,

1. The *p-dimension* of X is

   $\dim_{\mathrm{p}}(X) = \inf\{s \in [0, +\infty) \mid \exists\, d \text{ p-computable } s\text{-gale s.t. } X \subseteq S^{\infty}[d]\}$

2. The *strong* p-*dimension* of X is

   $\mathrm{Dim}_{\mathrm{p}}(X) = \inf\{s \in [0, +\infty) \mid \exists\, d \text{ p-computable } s\text{-gale s.t. } X \subseteq S^{\infty}_{\mathrm{str}}[d]\}$

$\square$

For a complete introduction and motivation of effective dimension and effective strong dimension see [9].

3 Compressors that do not start from scratch

In this section we develop the notion of compressors that "do not start from scratch" in the sense that when encoding successively longer extensions of an input, the outputs are restricted in the way we make precise below. The extreme case of this behavior is when the compressor is a mere extender, that is, $C(w)$ is always a prefix of $C(wu)$. We consider here a much weaker restriction than extension.

Definition. A pair of functions (C, D) (C the encoder, D the decoder) $C, D : \{0,1\}^* \to \{0,1\}^*$ is a *polynomial-time compressor* if:

(i) C and D can be computed in polynomial-time on their corresponding input length.

(ii) For all $w \in \{0,1\}^*$, $D(C(w), |w|) = w$.

□

In this paper, we could make all codes prefix-free, that is, $C(\{0,1\}^n)$ is a prefix set for each n. For the asymptotic compression rates the difference is not significant.

Notice that in the previous definition there is no restriction whatsoever on the behavior of C, the encoder, when working on two inputs that are one an extension of the other. For instance, we can have $|C(wu)| \ll |C(w)|$ and $C(wu)$ can have no common prefix with $C(w)$.

We introduce a restriction on the compressor that has an effect on the variety of $C(wu)$ for different u, that will be controlled by $|C(w)|$. Compressors under this condition are still far more general that extenders or length increasing compressors.

Definition. A polynomial-time compressor (C, D) *does not start from scratch* if for all but finitely many $k \in \mathbb{N}$ and $w \in \{0,1\}^*$

$$\sum_{|u|=k} 2^{-|C(wu)|} \leq 2^{k/\log k} 2^{-|C(w)|}. \tag{1}$$

□

We will consider only compressors that do not start from scratch.

Notice that when $\sum_{|u|=k} 2^{-|C(wu)|} \leq 2^{-|C(w)|}$, condition (1) is trivial, while in general $\sum_{|u|=k} 2^{-|C(wu)|}$ can be as large as 1.

Remark 3.1 *Polynomial-time compressors (C, D) that satisfy the following two conditions don't start from scratch.*

i) For all $w, u \in \{0,1\}^*$, $|C(wu)| \geq |C(w)|$

ii) For all but finitely many $k \in \mathbb{N}$, $w \in \{0,1\}^*$, and $\forall i \geq 0$

$$N_i = N_i(w,k) = \#\left\{u \in \{0,1\}^k \;\middle|\; |C(wu)| = |C(w)| + i\right\} \leq 2^{i+\frac{k}{\log k}-\log k}$$

Example 3.2 *For the following polynomial-time compressors condition (1) holds*

- *C is an extender, that is, $\forall\, w, w' \in \{0,1\}^*$*

$$w \sqsubseteq w' \Rightarrow C(w) \sqsubseteq C(w').$$

- *Compressors with common prefixes: $\forall\, w, u \in \{0,1\}^*$ $C(wu)$ and $C(w)$ have a common prefix of length at least*

$$|C(w)| - \frac{|u|}{\log(|u|)} + \log(|u|).$$

In fact, we have a weaker restriction on compressors that still implies our main result of equivalence with p-dimension:

For all $\epsilon > 0$, for all but finitely many $w \in \{0,1\}^*$, there exist $k = k(w) = O(\log(|w|))$ that verifies

$$\sum_{|u|=k} 2^{-|C(wu)|} \leq 2^{\epsilon k} 2^{-|C(w)|}.$$

This weaker condition also restricts the behavior of C in the sense of not starting from scratch, but in this case in a much less local way. For instance we admit a compressor C in which for all w, u with $|u| = \log(|w|)$, $C(wu)$ and $C(w)$ have a common prefix of length $|C(w)| - \epsilon \log(|w|) + \log\log(|w|)$.

In this paper we have preferred to stick to simpler condition (1).

4 Main theorem

We first define the notion of a.e.(almost everywhere) and i.o.(infinitely often) compressibility for sets of infinite sequences as the asymptotic best (respectively worse) compression ratio.

Definition. For $\alpha \in [0,1]$ and $X \subseteq \mathbf{C}$,

1. X is *α-i.o. polynomial-time compressible* if there is a polynomial-time compressor (C, D) that does not start from scratch and such that for every $A \in X$
$$\liminf_n \frac{|C(A[0 \ldots n-1])|}{n} \leq \alpha$$

2. X is *α-a.e. polynomial-time compressible* if there is a polynomial-time compressor (C, D) that does not start from scratch and such that for every $A \in X$
$$\limsup_n \frac{|C(A[0 \ldots n-1])|}{n} \leq \alpha$$

□

Definition. Let $X \subseteq \mathbf{C}$,

1. X is *i.o. polynomial-time incompressible* if for every (C, D) polynomial-time compressor that does not start from scratch, there exist $A \in X$ such that
$$\liminf_n \frac{|C(A[0 \ldots n-1])|}{n} = 1$$

2. X is *a.e. polynomial-time incompressible* if for every (C, D) polynomial-time compressor that does not start from scratch, there exist $A \in X$ such that
$$\limsup_n \frac{|C(A[0 \ldots n-1])|}{n} = 1$$

□

Our main theorem characterizes p-dimension in terms of polynomial-time compression.

Theorem 4.1 *Let $X \subseteq \mathbf{C}$,*

$$\begin{aligned} \dim_{\mathrm{p}}(X) &= \inf\{\, \alpha \,|\, X \textit{ is } \alpha\textit{-i.o. polynomial-time compressible}\} \\ \mathrm{Dim}_{\mathrm{p}}(X) &= \inf\{\, \alpha \,|\, X \textit{ is } \alpha\textit{-a.e. polynomial-time compressible}\} \end{aligned}$$

We include a detailed proof of this result in section 6.

Hitchcock showd in [7] that p-dimension can be characterized in terms of on-line prediction algorithms, using the well-studied log-loss prediction ratio. Our result can thus be interpreted as a joining bridge between the performance of polynomial-time prediction and compression algorithms, both in the best and the worse case.

5 Applications of the main result

In this section we obtain interesting consequences of our characterization for the polynomial-time compressibility of complete and autoreducible sets from previously known p-dimension results.

Notice that in this section we identify each language A with its characteristic sequence χ_A, therefore compressibility of a class always means compressibility of the corresponding characteristic sequences.

We start by showing that no polynomial-time compressor works on all many-one complete sets.

Theorem 5.1 *The class of polynomial-time many-one complete sets for E is i.o. polynomial-time incompressible.*

Proof. Ambos-Spies et al. prove in [1] that the class has p-dimension 1. □

Next we consider $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$, the class of sets that are equivalent to A by $\leq_{\mathrm{m}}^{\mathrm{P}}$-reductions. The compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ and $\deg_{\mathrm{m}}^{\mathrm{P}}(B)$, for $A \leq_{\mathrm{m}}^{\mathrm{P}} B$, is related by the following theorem.

Theorem 5.2 *Let A, B be sets in E such that $A \leq_{\mathrm{m}}^{\mathrm{P}} B$, then*

1. *the i.o. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ is at most the i.o. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(B)$.*

2. *The a.e. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ is at most the a.e. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(B)$.*

Proof. Ambos-Spies et al. prove 1. in [1] for p-dimension. Athreya et al. prove in [2] the strong dimension result for 2. □

We next consider the property of autoreducibility. A set A is autoreducible if A can be decided by using A as an oracle but without asking query x on input x. We obtain incompressibility results both in the case of polynomial-time many-one autoreducibility and for the *complement* of i.o. p-Turing autoreducible sets. Therefore for each polynomial-time bound there are i.o. incompressible sets that are $\leq_{\mathrm{m}}^{\mathrm{P}}$-autoreducible and other that are not even i.o. $\leq_{\mathrm{T}}^{\mathrm{P}}$-autoreducible.

Theorem 5.3 *The class of polynomial-time many-one autoreducible sets are i.o. polynomial-time incompressible.*

Proof. Ambos-Spies et al. prove in [1] that the class has p-dimension 1. □

Theorem 5.4 *The class of sets that are NOT i.o. polynomial-time Turing autoreducible are i.o. polynomial-time incompressible.*

Proof. Beigel et al. prove in [3] that the class has p-dimension 1. □

We next show that there exist polynomial-time many-one degrees with every possible value for both a.e. and i.o. compressibility.

Theorem 5.5 *Let x, y be computable reals such that $0 \leq x \leq y \leq 1$. Then there is a set A in E such that the i.o. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ is x and the a.e. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ is y.*

Proof. Athreya et al. prove in [2] the result for p-dimension and strong p-dimension. □

This last theorem includes the extreme case for which the i.o. compression ratio is 1 whereas the a.e. ratio is 0.

Finally, the hypothesis "NP has positive p-dimension" can be interpreted in terms of incompressibility. This hypothesis has interesting consequences on the approximation algorithms for MAX3SAT.

Theorem 5.6 *If for some $\alpha > 0$ NP is not α-i.o-compressible in polynomial-time then any approximation algorithm $\mathcal{A}$ for MAX3SAT must satisfy at least one of the following*

1. *For some $\delta > 0$, $\mathcal{A}$ uses time at least 2^{n^δ}*
2. *For all $\epsilon > 0$, $\mathcal{A}$ has performance ratio less than $7/8 + \epsilon$ on an exponentially dense set of satisfiable instances.*

Proof. Hitchcock proves in [5] that the consequence follows from NP having positive p-dimension. □

6 Proof of theorem 4.1

We first proof that a dimension upper bound gives a compression upper bound.

Theorem 6.1 *Let $X \subseteq \{0,1\}^*$,*

$$\dim_{\mathrm{p}}(X) < s \Rightarrow X \text{ is } s-\textit{i.o. polynomial-time compressible.}$$

$$\mathrm{Dim}_{\mathrm{p}}(X) < s \Rightarrow X \text{ is } s-\textit{a.e. polynomial-time compressible.}$$

To prove theorem 6.1 we need to make sure simple gales are sufficient in the definition of p-dimension.

Lemma 6.2 *[12] Let d_1 be a martingale. Let $c : \{0,1\}^* \to [0,+\infty)$ be a polynomial-time computable function such that for each $w \in \{0,1\}^*$, $|c(w) - d_1(w)| \leq 2^{-|w|}$. Let d_2 be recursively defined as follows*

$$\begin{aligned} d_2(\lambda) &= c(\lambda) + 2 \\ d_2(wb) &= d_2(w) + \frac{c(wb) - c(w\bar{b})}{2} \end{aligned}$$

Then d_2 is a martingale in p such that $|d_1(w) - d_2(w)| \leq 4$.

Lemma 6.3 *Let $X \subseteq \mathbf{C}$. If $\dim_{\mathrm{p}}(X) = \alpha$ then $\forall s > \alpha$ there exists an s-gale d with $X \subseteq S^{\infty}[d]$ such that for all $w \in \{0,1\}^*$, there exists $m_w, n_w \in \mathbb{N}$ with $n_w \leq |w| + 1$ and*

$$d(w)2^{-|w|s} = m_w 2^{-(n_w + |w|)}$$

Proof. If $\dim_{\mathrm{p}}(X) = \alpha$ then $\forall s > \alpha$ there exists an s-gale d', with $d'(\lambda) = 1$, that succeeds on X.
Let d_1 be the martingale $d_1(w) = 2^{(1-s)|w|}d'(w)$ and let $c : \{0,1\}^* \to [0,+\infty)$ be such that $c(w) = m'_w 2^{-n'_w}$ where

$$n'_w = \min\{n \in \mathbb{N} \mid \exists m \text{ s.t. } |m2^{-n} - d_1(w)| < 2^{-|w|}\}$$

$$m'_w = min\{m \in \mathbb{N} \mid |m2^{-n_w} - d_1(w)| < 2^{-|w|}\}$$

Notice that $n'_w \leq |w| + 1$ because within an interval of length $2^{-|w|}$ there exists at at least least one dyadic number $m2^{-n}$ with $n = |w| + 1$.
Let d_2 be as in lemma 6.2. There exists $m_w, n_w \in \mathbb{N}$ such that $d_2(w) = m_w 2^{-n_w}$ with $n_w \leq |w| + 1$. We prove this by induction, if $|w| = 0$ then $d_2(\lambda) = 3 = 3 \cdot 2^{-0}$.

$$\begin{aligned} d_2(wb) &= d_2(w) + \frac{c(wb) - c(w\bar{b})}{2} \\ &= m_w 2^{-n_w} + \frac{m'_{wb} 2^{-n'_{wb}} - m'_{w\bar{b}} 2^{-n'_{w\bar{b}}}}{2} \\ &= 2^{-n_w b} m_{wb} \end{aligned}$$

where $n_{wb} = \max\{n_w, n'_{w0} + 1, n'_{w1} + 1\} \leq |w| + 2$.

Let $d(w) = 2^{(s-1)|w|}d_2(w)$ be an s-gale, then

(i) $d(w)2^{-|w|s} = 2^{-|w|}d_2(w) = m_w 2^{-(n_w+|w|)}$ is a dyadic number and $n_w \leq |w| + 1$.

(ii) $|d'(w) - d(w)| = 2^{(s-1)|w|}|d_1(w) - d_2(w)| \leq 2^{(s-1)|w|}4$ so $S^\infty[d'] = S^\infty[d]$.

□

Lemma 6.4 *Let a, b be dyadic numbers. and let $I = [a, b)$ be an interval of length $r \in [0, 1)$, then there exists a string z of length $-\lfloor \log(r) \rfloor + 1$ such that $a < 0.z < b$ and z can be computed in time polynomial in $|z|$.*

Proof. If we make a partition of the interval $[0, 1)$ in intervals of length $2^{\lceil \log r \rceil - 1}$ then the strings z with length $|z| = -\lfloor \log(r) \rfloor + 1$ are just the endpoints of those intervals and it is clear that there exists one of these endpoints inside our interval. We can compute z bit by bit by using the condition $a < 0.z < b$ and choosing bit 0 if both alternatives are valid. □

Proof Theorem 6.1. We prove the first inequality; the proof for strong dimension is analogous. Let X be such that $\dim_{\mathrm{p}}(X) < s$, then there exists d' a p-computable s-gale such that $d'(\lambda) = 1$, $X \subseteq S^\infty[d']$, and for all $w \in \{0, 1\}^*$, there exists $m_w, n_w \in \mathbb{N}$ with $n_w \leq |w| + 1$ and

$$d'(w)2^{-|w|s} = m_w 2^{-(n_w+|w|)}.$$

Then, if $d(w) = 2^{(1-s)|w|}d'(w)$, d is a p-computable martingale such that,

i. For all $A \in X$, $\quad d(A[0 \ldots n-1]) > 2^{(1-s)n}$ *i.o.* n

ii. For all $w \in \{0, 1\}^*$, there exists $m_w, n_w \in \mathbb{N}$ with $n_w \leq |w| + 1$

$$d(w) = m_w 2^{-n_w}$$

Let $h : \{0, 1\}^* \to \mathbb{R}$ be defined as follows.

$$h(w) := \sum_{|y|=|w|, y<w} d(y)2^{-|w|}$$

where $y < w$ means that y precedes x in lexicographic order. Denote by $succ(w)$ the successor of w in lexicographic order. Notice that $h(w)$ is a dyadic number $m2^{-n}$ with $n \leq 2|w| + 1$, therefore there is a $z \in \{0, 1\}^*$ such that $|z| \leq 2|w| + 2$ and $h(w) < 0.z < h(succ(w))$. Let z_w be the first shortest string such that $h(w) < 0.z < h(succ(w))$. We define the encoder as $C(w) = z_w$.

To define our decoder D, let $z \in \{0,1\}^*$ and $n \in \mathbb{N}$, then to generate a string of length n from (z, n), simulate the martingale starting at λ on successively longer strings. Suppose we have generated the string w so far. If $h(w1) \leq 0.z$, then append 0 to w, if $h(w1) > 0.z$, then append 1 to x. Continue until $|w| = n$. At the end of this process, we have the string w of length n such that $h(w) \leq 0.z < h(succ(w))$.

We next show that the polynomial-time compressor (C, D) does not start from scratch.

Notice that for each w the interval $[h(w), h(succ(w)))$ has length exactly $d(w)2^{-|w|}$. Then by lemma 6.4, there is a string z of length $-\lfloor \log(2^{-|w|}d(w)) \rfloor + 1 \leq |w| - \lfloor \log(d(w)) \rfloor + 1$ such that $h(w) < 0.z < h(succ(w))$. So,

$$|z_w| \leq |w| - \lfloor \log(d(w)) \rfloor + 1.$$

To see that C verifies condition (1), we will prove that C verifies the two conditions of remark 3.1.

i) It is clear that for all $w, u \in \{0,1\}^*$, $|C(wu)| \geq |C(w)|$ because the interval $[h(wu), h(succ(wu)))$ is included in $[h(w), h(succ(w)))$.

ii) Fix $k \in \mathbb{N}$, $w \in \{0,1\}^*$ and $i \in \mathbb{N}$,

$$N_i = \#\left\{u \in \{0,1\}^* \;\middle|\; |u| = k \,\text{and}\, |z_{wu}| = |z_w| + i\right\}$$

We have that

$$(N_i - 1)2^{-(|z_w|+i)} < d(w)2^{-|w|}$$
$$N_i < 1 + d(w)2^{-|w|+|z_w|+i}$$

but since $|z_w| \leq |w| - \lfloor \log d(w) \rfloor + 1$,

$$N_i < 1 + 2^{\log(d(w)) - \lfloor \log d(w) \rfloor} 2^{i+1} \leq 1 + 2^{i+2} \leq 2^{i+k/\log k - \log k}$$

for all but finitely many k.

Finally, let us see that (C, D) compresses X. For all $A \in X$,

$$\begin{aligned} |C(A[0 \ldots n-1]| &= |z_{A[0 \ldots n-1]}| \\ &\leq n - \lfloor \log(d(A[0 \ldots n-1])) \rfloor + 1 \\ &\leq n - \log(2^{(1-s)n}) + 1 \\ &= sn + 1 \end{aligned}$$

□

Next we show that compressibility is an upper bound on dimension.

Theorem 6.5 *Let $X \subseteq \mathbf{C}$.*

$$X \text{ is } s\text{-i.o. polynomial-time compressible} \Rightarrow \dim_{\mathrm{p}}(X) \leq s.$$

$$X \text{ is } s\text{-a.e. polynomial-time compressible} \Rightarrow \mathrm{Dim}_{\mathrm{p}}(X) \leq s.$$

Proof. We prove the first inequality; the proof for strong dimension is analogous. Let $s' > s$ and let $k \in \mathbb{N}$ be such that $s' - s > 1/\log(k)$, we define

$$d(w) := \frac{2^{|w|-|C(w)|}}{2^{|w|/\log k}}, \qquad \text{if } |w| = mk \text{ for some } m \in \mathbb{N},$$
$$d(w) := \sum_{|u|=mk-|w|} \frac{d(wu)}{2^{mk-|w|}} \quad \text{if } (m-1)k < |w| < mk \text{ for some } m \in \mathbb{N}$$

Notice that d is a martingale because of condition (1). d is also computable in polynomial-time. Let $d'(w) = 2^{(s'-1)|w|}d(w)$ be an s'-gale in p. For all $A \in X$,

$$\liminf_n \frac{|C(A[0 \dots n-1])|}{n} \leq s$$

so there exists $(b_n)_{n\in\mathbb{N}}$ a sequence of natural numbers such that

$$\lim_n \frac{|C(A[0 \dots b_n - 1])|}{b_n} \leq s$$

Let $(a_n)_{n\in\mathbb{N}}$ a sequence of natural numbers such that $a_n k < b_n \leq (a_n+1)k$. Then we have that

$$\begin{aligned} d'(A[0 \dots a_n k - 1]) &= 2^{(s'-1)a_n k} \cdot \frac{2^{a_n k - |C(A[0 \dots a_n k - 1])|}}{2^{a_n k/\log k}} \\ &= \frac{2^{s' a_n k}}{2^{a_n k/\log k}} \cdot 2^{-|C(A[0 \dots a_n k-1])|} \end{aligned}$$

By condition (1), $2^{-|C(A[0 \dots b_n - 1])|} \leq 2^{k/\log k} 2^{-|C(A[0 \dots a_n k - 1])|}$. Let $s'' = s' - s - 1/\log k$.

$$d'(A[0 \dots a_n k - 1]) \geq 2^{a_n k s'' - k/\log k}$$

and d' succeeds on X. $\square$

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