

# Bounded Color Multiplicity Graph Isomorphism is in the #L Hierarchy\*

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## Abstract

In this paper we study the complexity of *Bounded Color Multiplicity Graph Isomorphism*  $\text{BCGI}_b$ : the input is a pair of vertex-colored graphs such that the number of vertices of a given color in an input graph is bounded by  $b$ . We show that  $\text{BCGI}_b$  is in the #L hierarchy (more precisely, the  $\text{Mod}_k\text{L}$  hierarchy for some constant  $k$  depending on  $b$ ). Combined with the fact that Bounded Color Multiplicity Graph Isomorphism is logspace many-one hard for every set in the  $\text{Mod}_k\text{L}$  hierarchy for any constant  $k$ , we get a tight classification of the problem using logspace-bounded counting classes.

## 1 Introduction

In the last decade or so, logspace bounded classes, particularly logspace counting classes, have played an important role in classifying natural problems in  $\text{NC}^2$ . A well-known example is the problem of computing the determinant. We know from Toda and Vinay's work [Tod91, Vin91] that the complexity class  $\text{GapL}$  *exactly* captures the complexity of computing integer determinants. Likewise, computing determinants over finite fields of characteristic  $p$  is captured by the class  $\text{Mod}_p\text{L}$  [BDHM92]. More recently, the results of [ABO99] classify important linear algebraic problems like finding the rank and checking feasibility of linear equations. Also, the complexity of perfect matching is now quite well characterized in [ARZ99] using logspace counting classes and the isolation lemma. For other results in this area, including connections to circuit complexity questions, we refer the reader to the recent survey article by Allender [Al04].

### 1.1 Graph Isomorphism and Logspace Counting Classes

In this paper we study the following restricted version of Graph Isomorphism problem: The input is a pair of vertex-colored graphs  $(G_1, G_2)$  with the additional property that the number of vertices of a given color in the graphs is less than a prescribed constant bound  $b$ . And, the restricted Graph Isomorphism problem is to check if there is a color-preserving isomorphism between  $G_1$  and  $G_2$ .

This problem is known as the *Bounded Color Multiplicity Graph Isomorphism* problem (which we denote by  $\text{BCGI}_b$ ). It is a well-studied restriction of Graph Isomorphism: This was the first

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version of Graph Isomorphism studied using group-theoretic methods when Babai gave a randomized polynomial time algorithm for  $\text{BCGI}_b$  for each constant  $b$  [Bab79]. This result was improved to a deterministic polynomial time algorithm in [FHL80]. Subsequently, Luks in [Lu86] gave a remarkable NC algorithm for the  $\text{BCGI}_b$  problem. Building on earlier work by McKenzie and Cook [MC87]) and Luks and McKenzie [LM88], in [Lu86] Luks introduced group-theoretic techniques to handle the nonabelian parts of permutation groups in the class NC. The earlier papers [MC87, LM88] developed techniques to show that several permutation group problems for abelian and solvable groups are in the class NC. These methods are linear-algebraic and do not extend to general permutation groups.

Luks' NC algorithm in [Lu86] is actually for a more general problem (of which  $\text{BCGI}_b$  is a special case). In [Lu86] the actual parallel time taken by the algorithm is not specified. But we can notice that the NC algorithm runs in  $O(\log n)$  stages and in each stage it uses Luby's  $\text{NC}^2$  algorithm for the maximal independent set problem. Thus the running time of the parallel algorithm appears to be at least  $\log^3 n$ .

In a different line of research, Torán in [Tor04] developed a nice graph gadget that enables the simulation of addition modulo  $k$  using automorphisms of the graph gadget. As a consequence, in [Tor04] several hardness results for Graph Isomorphism are shown. In particular, it is shown that  $\text{BCGI}_b$  is  $\text{AC}^0$ -many one hard for the logspace counting class  $\text{Mod}_k\text{L}$  for each constant  $k$ . The construction in [Tor04] requires  $b$  to be  $k^2$ .

In a related paper [JKMT04], the  $\text{BCGI}$  problem is examined for color classes of sizes 2 and 3, and show that in these cases it is in  $\text{SL}$  (which puts it in  $\text{L}$  as  $\text{SL} = \text{L}$  [Re04]).

## 1.2 The results of this paper

The question that naturally arises is whether the gap between the NC upper bound result of [Lu86] and the lower bound result of [Tor04] for  $\text{BCGI}_b$  can be closed. In this paper, we examine Luks' NC algorithm carefully using logspace complexity classes. As our main result we show that  $\text{BCGI}_b$  is in  $\text{Mod}_k\text{L}$  hierarchy, where the constant  $k$  and the level of the hierarchy in which  $\text{BCGI}$  sits depends on  $b$ .

It turns out that the we can decompose the problem into a constant number of stages. Each stage corresponds to either handling an abelian simple group or a nonabelian simple group. We are able to show that the stages involving abelian groups can be solved in the complexity class  $\text{Mod}_k\text{L}$  for a suitable constant  $k$ . Surprisingly for handling nonabelian groups, although the group theory is more involved, the actual computation can be done in deterministic logspace  $\text{L}$ . This is because in the nonabelian stages we can do away with finding maximal independent sets and show that the computations involved are actually reducible to *undirected graph connectivity* (which is recently shown to be in  $\text{L}$  [Re04]).

Since the  $\text{Mod}_k\text{L}$  hierarchy is contained in the  $\#\text{L}$  hierarchy which, in turn, is contained in  $\text{NC}^2$  (even in  $\text{TC}^1$ ) [ABO99], our result puts  $\text{BCGI}_b$  in  $\text{NC}^2$  (indeed in  $\text{TC}^1$ ).

Below we depict a chain known inclusions of the relevant complexity classes inside  $\text{NC}^2$ .

$$\text{BCGI}_b \in \text{Mod}_k\text{L} \overset{\text{Mod}_k\text{L}}{\subseteq} \overset{\dots}{\subseteq} \#\text{L} \overset{\#\text{L}}{\subseteq} \overset{\dots}{\subseteq} \text{TC}^1 \subseteq \text{NC}^2.$$

While the ideas and the underlying group-theoretic machinery in our result is based on Luks [Lu86] and [LM88], to prove the  $\text{Mod}_k\text{L}$  hierarchy upper bound we had to make several crucial changes in the NC algorithm of [Lu86].

**Remark.** *How well does our  $\text{Mod}_k\text{L}$  hierarchy upper bound classify the problem  $\text{BCGI}_b$ ?*

*We can apply Torán’s result [Tor04, Lemma 4.7] to show that for every  $k$  if  $A$  is a set in the  $j$ th level of the  $\text{Mod}_k\text{L}$  hierarchy, then  $A$  is logspace many-one reducible to  $\text{BCGI}_b$ , where  $b$  depends only on  $k$  and  $j$  (a proof sketch is in the Appendix). Thus,  $\text{BCGI}_b$  is intimately related to the  $\text{Mod}_k\text{L}$  hierarchy. The question whether the  $\text{Mod}_k\text{L}$  hierarchy collapses for composite  $k$  is open (for prime  $k$  it is known to collapse to  $\text{Mod}_k\text{L}$ ).*

## 2 Preliminaries and Notation

We assume some familiarity with basic complexity theory and basic notions in group theory and linear algebra. For details on the complexity classes discussed here we refer the reader to [ABO99] and for group-theoretic definitions we refer the reader to [Ha, Wie64].

### 2.1 Complexity Classes

We write  $\text{L}$  and  $\text{FL}$  for the class of languages (respectively, functions) computable by deterministic logspace-bounded Turing machines. The class of languages accepted by nondeterministic logspace machines is denoted by  $\text{NL}$ , and the class of languages accepted by symmetric logspace machines is denoted by  $\text{SL}$  (equal to  $\text{L}$  in [Re04]). The class  $\text{NL}$  is known to be closed under complement.

We now recall the definitions for the complexity classes  $\text{GapL}$  and  $\text{Mod}_k\text{L}$  with  $k \geq 2$ . For a nondeterministic Turing machine  $M$  we denote by  $\text{acc}_M(x)$  and  $\text{rej}_M(x)$  the number of accepting and rejecting computation paths of  $M$  on input  $x$ , respectively. We define  $\text{gap}_M(x)$  as the difference  $\text{acc}_M(x) - \text{rej}_M(x)$ .

**Definition 2.1.** *The class  $\text{GapL}$  is defined as the class of functions  $f : \Sigma^* \rightarrow \mathbb{Z}$  such that for some nondeterministic logspace-bounded Turing machine  $M$ ,  $f(x) = \text{gap}_M(x)$  for every  $x \in \Sigma^*$ .*

$\text{GapL}$  can also be defined equivalently as the closure of  $\#\text{L}$  under subtraction, where  $\#\text{L}$  denote the set of all functions  $\text{acc}_M(x)$  for nondeterministic logspace-bounded Turing machines  $M$ .

**Definition 2.2.** *Let  $k \geq 2$  be an integer.  $\text{Mod}_k\text{L}$  is the class of sets  $L$  such that there is an  $f \in \#\text{L}$  with  $x \in L \Leftrightarrow f(x) \not\equiv 0 \pmod{k}$ .*

We now define the reductions that are discussed in this paper: for languages  $L$  and  $L'$ , we say that  $L$  is *logspace many-one reducible* to  $L'$  if there is an  $f \in \text{FL}$  such that  $x \in L$  if and only if  $f(x) \in L'$ . We say that  $L$  is *logspace Turing reducible* to  $L'$  if there is a deterministic logspace oracle Turing machine  $M$  such that  $M$  with oracle  $L'$  accepts  $L$ .

Nondeterministic logspace oracle Turing machines are defined using the “Ruzzo-Simon-Tompa” oracle access [RST84] (also see [AO96, ABO99]) which restricts the machine to write the oracle queries deterministically. We define the relativized classes  $\#\text{L}^A$  and  $\text{Mod}_k\text{L}^A$  in this model. We can now inductively define the  $\text{Mod}_k\text{L}$  hierarchy: A language  $L$  is in the  $j$ th level of the  $\text{Mod}_k\text{L}$  hierarchy if there is an oracle set  $A$  in the  $(j - 1)$ th level such that  $L \in \text{Mod}_k\text{L}^A$ .

For languages  $L$  and  $L'$ , we say that  $L$  is  $\text{NC}^1$ -Turing reducible to  $L'$  if there is a logspace uniform circuit family of polynomial size and logarithmic depth with bounded fanin gates that computes  $L$  with oracle gates for  $L'$ .  $\text{AC}^0$ -Turing reductions are similarly defined using logspace uniform  $\text{AC}^0$  circuit families. It is known that the  $\#\text{L}$  hierarchy equals  $\text{AC}^0(\#\text{L})$  [AO96].

## 2.2 Permutation Groups

We recall useful definitions and basic facts about permutation groups. For details we refer the reader to the book by Wielandt [Wie64].

A permutation group  $G$  is a subgroup of  $\text{Sym}(\Omega)$ , where  $\text{Sym}(\Omega)$  is the group of all permutations on  $\Omega$ , and  $\Omega$  is an  $n$ -element set. We write  $H \leq G$  when  $H$  is a subgroup of  $G$ . The subgroup  $H$  of  $G$  is *normal*, denoted  $H \triangleleft G$ , if  $gHg^{-1} = H$  for all  $g \in G$ .

The image of a point  $\alpha \in \Omega$  under  $g \in G$  is denoted  $\alpha^g$ , and we apply permutations from left to right so that  $\alpha^{gh} = (\alpha^g)^h$ . The set  $\alpha^G = \{\alpha^g \mid g \in G\}$  is the  $G$ -orbit of  $\alpha$ , and  $G$  is *transitive* on  $\Omega$  if  $\alpha^G = \Omega$ . The *group generated* by a set of permutations  $S \subseteq \text{Sym}(\Omega)$  is the smallest subgroup of  $\text{Sym}(\Omega)$  containing  $S$ , denoted  $\langle S \rangle$ . For a group  $G$  and  $S \subseteq G$ , the *normal closure* of  $S$  in  $G$  is the smallest normal subgroup of  $G$  containing  $S$  and is denoted by  $\text{NCL}_G(S)$ .

A group  $G$  is *cyclic* if  $G = \langle g \rangle$  for some  $g$ ,  $G$  is *abelian* if  $gh = hg$  for all  $g, h \in G$ .

Let  $G \leq \text{Sym}(\Omega)$  be transitive on  $\Omega$ . A subset  $\Delta \subseteq \Omega$  is called a  $G$ -block if for every  $g \in G$  either  $\Delta^g = \Delta$  or  $\Delta^g \cap \Delta = \emptyset$ . For any transitive group  $G$ , the set  $\Omega$  and the singleton sets  $\{\omega\}$ ,  $\omega \in \Omega$  are called *trivial* blocks. A transitive group  $G$  is called *primitive* if it does not have any nontrivial blocks otherwise it is called *imprimitive*.

Consider a permutation group  $G$  that is transitive on  $\Omega$ . If  $\Delta$  is a  $G$ -block, then for every  $g \in G$   $\Delta^g$  is also a  $G$ -block. The collection of blocks  $\{\Delta^g : g \in G\}$  forms a partition of  $\Omega$  and is called the  $\Delta$  block system. Notice that the permutation group  $G$  acts transitively on the  $\Delta$  block system (since every element of  $G$  in its natural action maps blocks to blocks). We call  $\Delta \subset \Omega$  a *maximal block* if there is no other block  $\Sigma$  such that  $\Delta \subset \Sigma \subset \Omega$ . In this case, we call  $\{\Delta^g : g \in G\}$  a *maximal block system*.

If  $G$ 's action is imprimitive on a maximal block system, then we can again find a maximal block system here and continue further, resulting in a hierarchy of block systems. We formalize this process below.

Let  $\Delta$  and  $\Sigma$  be two  $G$ -blocks in  $\Omega$ , such that  $\Delta \subseteq \Sigma$ . Notice that the collection of blocks  $\{\Delta^g : g \in G \text{ and } \Delta^g \subseteq \Sigma\}$  forms a partition of  $\Sigma$  which we call the  $\Delta$  block system of  $\Sigma$ .

We now define a structure tree of the transitive group  $G$ . Structure trees play an important role in algorithms for permutation groups [Luk93]. The *structure tree* of  $G$  is a rooted tree nodes of which are labeled by  $G$ -blocks satisfying the following conditions:

1. The root is labeled  $\Omega$ .
2. The leaves are labeled with singleton sets  $\{\alpha\}$ ,  $\alpha \in \Omega$ .
3. For each internal node labeled by  $\Sigma$ , the labels of its children constitute a  $\Delta$  block system of  $\Sigma$ , where  $\Delta$  is a maximal block properly contained in  $\Sigma$ .

Note that there is a natural action of  $G$  on nodes at each level:  $g \in G$  maps node  $u$  to  $v$  iff there is a  $G$ -block  $\Delta$  such that the labels of  $u$  and  $v$  are  $\Delta$  and  $\Delta^g$  respectively. Notice that  $G$  acts primitively on the children of the root.

Let  $G$  be a permutation group on  $\Omega$  with orbits  $\Omega_1, \dots, \Omega_r$ . The *structure forest* is a collection of structure trees  $T_1, \dots, T_r$ , where  $T_i$  is the structure tree of the transitive action of  $G$  restricted to  $\Omega_i$ .

### 3 Overview of the algorithm

Let  $\Omega = \{1, 2, \dots, n\}$  and  $G \leq \text{Sym}(\Delta)$  be a group. The group  $G_{\{\Delta\}}$  is the *pointwise stabilizer* of  $\Delta$ , that is the subgroup of elements in  $G$  that fix each point in  $\Delta$ .

We now define BCGI and a closely related problem POINTSET<sub>b</sub>.

**Definition 3.1.** • *The problem BCGI<sub>b</sub> has input instances as pairs of graphs  $(G_1, G_2)$  where the two graphs  $G_1$  and  $G_2$  are vertex colored such that each color class is of size at most  $b$ . The problem is to test if there is a color-preserving isomorphism between  $G_1$  and  $G_2$ . We sometimes write BCGI to stand for the problem when the constant  $b$  is not specified.*

- *The problem POINTSET<sub>b</sub> has input instances consisting of a subset  $\Delta \subseteq \Omega$  along with a permutation group  $G \leq \text{Sym}(\Omega)$  given by its generator set, such that  $G$  has orbits of size bounded by a constant  $b$ . The problem is to find the pointwise stabilizer subgroup  $G_{\{\Delta\}}$  of  $G$ .*

It is well known that for each constant  $b$  there is a constant  $c$  such that BCGI<sub>b</sub> reducible to POINTSET<sub>c</sub> [LM88, Lu86]. If we examine the reduction it is easy to see that it is indeed a logspace many-one reduction. Thus we have the following proposition.

**Proposition 3.2.** *BCGI<sub>b</sub> is logspace many-one reducible to POINTSET<sub>c</sub>, where the constant  $c$  depends on the size of the color classes  $b$ .*

Thus, in order to prove the upper bound result for BCGI, it suffice to show that the problem POINTSET<sub>b</sub> is in the Mod<sub>k</sub>L hierarchy for some constant  $k$ . The rest the paper is devoted to this. We first give an overview of the algorithm, breaking it up into different tasks. In the subsequent sections we describe algorithms for these tasks in detail.

Let  $\Omega = \cup_{i=1}^m \Omega_i$  be the partition of  $\Omega$  into  $G$ -orbits, where  $|\Omega_i| \leq b$  for each  $i$ . Let  $G_i$  be the permutation group obtained by projecting  $G$  onto  $\Omega_i$  for each  $i$ . Since  $|G_i| \leq b!$  for each  $i$ , the following proposition is immediate.

**Proposition 3.3.** *Given an instance  $G \leq \text{Sym}(\Omega)$  of POINTSET<sub>b</sub>, the orbits  $\Omega_i$  and the groups  $G_i$  obtained by projecting  $G$  on  $\Omega_i$  can be computed in deterministic logspace.*

Furthermore, for each  $G_i$  (acting transitively on  $\Omega_i$ ) we can compute a structure tree of  $G_i$  in deterministic logspace: we can simply search for a minimal block  $B_1$  and find the corresponding block system by letting  $G_i$  act on it. Thus we obtain  $\Omega_i = B_1 \cup \dots \cup B_l$ , where  $\{B_i\}$  is the block system. Either  $G_i$  acts primitively on  $\{B_1, \dots, B_l\}$  or we can again find a minimal block and a corresponding block system. Continuing thus, we finally get a list of blocks on which  $G_i$  acts primitively. We get a rooted tree corresponding to the above algorithm: the leaves are elements of  $\Omega_i$ , the next level has  $l$  nodes corresponding to each of  $B_1, \dots, B_l$  and, the children of  $B_j$  are the elements of  $\Omega_i$  it contains and so on. At the  $k^{\text{th}}$  level of the tree are the nodes corresponding to a minimal block system for  $G_i$ 's action on the block system at level  $k - 1$ .

**Proposition 3.4.** *For each orbit of  $G$ , an instance of POINTSET<sub>b</sub> a structure tree can be computed in deterministic logspace.*

Since the orbit sizes are constant, the above proposition is immediate.

Let  $T_1, \dots, T_m$  be the computed structure trees for the respective orbits  $\Omega_1, \dots, \Omega_m$  of  $G$ . Every  $T_i$  has constant height (bounded by  $\log b$ ). Notice that  $G_{\{\Delta\}}$ , the subgroup we need to compute,

fixes every ancestor of points in  $\Delta$  of a structure tree  $T_i$ , ( $1 \leq i \leq m$ ). This property will allow us to reduce the problem further as follows:

Consider  $G$ 's action on the set  $\Omega' = \cup_{i=1}^m \Omega'_i$ , where  $\Omega'_i$  is the set of children of the root of  $T_i$ , ( $1 \leq i \leq m$ ). By definition  $G$  acts primitively on each  $\Omega'_i$ .

Let  $\Delta' \subseteq \Omega'$  be the ancestors of  $\Delta$  in all the structure trees  $T_i$ . Then  $(G, \Delta', \Omega')$  is an instance of  $\text{POINTSET}_b$  with the further restriction that  $G$  is primitive on its orbits. We refer to this restricted problem as primitive- $\text{POINTSET}_b$ . Suppose we have an algorithm for primitive- $\text{POINTSET}_b$  in the  $\text{Mod}_k\text{L}$  hierarchy for some constant  $k$ . This algorithm will compute  $G_{\{\Delta'\}} = H$ . We will assume that the algorithm will actually compute a generating set  $\{h_1, \dots, h_s\}$  for  $H$ , where it keeps track of  $h_i$ 's action on the entire set  $\Omega$ .

Notice that the group  $G_{\{\Delta\}}$  is a subgroup of  $H$ . Now consider the structure forest of  $H$ . For  $\delta \in \Delta$  let  $T_\delta$  be the tree with  $\delta$  as a leaf in the structure forest for  $H$ . It is easy to see that the height of  $T_\delta$  is 1 less than the height of the corresponding tree in the structure forest for  $G$ .

Thus, in computing  $H$  we have made progress. We can replace  $G$  by  $H$ , and recompute the structure forest for  $H$  and repeat the above. If we invoke the  $\text{Mod}_k\text{L}$  hierarchy upper bound for primitive- $\text{POINTSET}_b$  a constant number of nested oracle levels, where the constant is bounded by  $\log b$ , then we would have computed  $G_{\{\Delta\}}$ .

Putting it together, we have shown the following reduction.

**Theorem 3.5.** *If primitive- $\text{POINTSET}_b$  is in the  $\text{Mod}_k\text{L}$  hierarchy then  $\text{POINTSET}_b$  is also in the  $\text{Mod}_k\text{L}$  hierarchy.*

**Remark.** *Both the level of the hierarchy and  $k$  are constants that depend on the orbit bound  $b$ . We do not make it explicit here.*

Thus the problem that we need to upper bound is primitive- $\text{POINTSET}_b$ . In the remaining sections we will show that this problem is in the  $\text{Mod}_k\text{L}$  hierarchy for a constant  $k$ .

## 4 Basic group theoretic lemmas

For proving our upper bound on primitive- $\text{POINTSET}_b$  we need to recall some group-theoretic machinery from [Lu86]. We shall also need to assemble specific properties of permutation groups with bounded size orbits.

Let  $G \leq \text{Sym}(\Omega)$  and let  $\Omega_i$ ,  $1 \leq i \leq m$  be its orbits such that  $|\Omega_i| \leq b$  for all  $i$ . Let  $G_i$  be the projection of  $G$  onto  $\Omega_i$ . Clearly,  $G \leq G_1 \times \dots \times G_m$ . For any  $h \in G_1 \times \dots \times G_m$  let  $pr_i(h)$  mean the action of  $h$  restricted to  $\Omega_i$ . For a subgroup  $H$  of  $G_1 \times \dots \times G_m$ , we will use  $pr_i(H)$  to denote projection of the subgroup  $H$  on  $\Omega_i$ .

We have  $G_i \leq \text{Sym}(\Omega_i)$  and  $|\Omega_i| \leq b$  for each  $i$ , the set of possible simple subgroups that can occur as quotients of consecutive groups in any composition series of  $G_1 \times \dots \times G_m$  is a constant sized collection  $\mathcal{T} = \{T_1, \dots, T_c\}$  where  $c \leq 2^{b!}$ . A deterministic logspace machine can explicitly compute and keep  $\mathcal{T}$  as a table.

**Definition 4.1 (T-semisimple groups).** *A finite group  $G$  is  $T$ -semisimple, for a simple group  $T$ , if  $G = G_1 \times \dots \times G_r$  where each  $G_i$  is isomorphic to  $T$ .*

**Lemma 4.2.** [Lu86] *For any simple group  $T$  and any finite group  $G$ , there is a unique minimal normal subgroup  $N$  such that  $G/N$  is  $T$ -semisimple.*

We call  $N$  as the  $T$ -residue of  $G$  and denote it by  $\text{Res}_T(G)$ . By Lemma 4.2, notice that  $\text{Res}_T(G)$  is invariant under any automorphism of  $G$ .

**Definition 4.3.** *A subgroup  $H$  of a finite group  $G$  is a characteristic subgroup if  $H$  is invariant under all automorphisms of  $G$ .*

We recall two useful properties.

**Proposition 4.4.** *1. If  $H$  is a characteristic subgroup of  $G$  then  $H$  is a normal subgroup of  $G$ .*

*2. If  $K$  is a characteristic subgroup of  $H$  and  $H$  is a characteristic subgroup of  $G$  then  $K$  is a characteristic subgroup of  $G$ .*

For any simple group  $T$ , notice that  $\text{Res}_T(G)$  is a characteristic subgroup of  $G$ .

**Definition 4.5.** *A residual tower of  $G$  is a normal series  $G = R_0 \triangleright R_1 \dots \triangleright R_l = 1$  where for all  $1 \leq i \leq l$ ,  $R_i = \text{Res}_{T_i}(R_{i-1})$  for some simple group  $T$ .*

Using Proposition 4.4 each  $R_i$  is a characteristic subgroup of  $G$  and hence is normal.

**Definition 4.6.** *The socle  $\text{Soc}(G)$  of a finite group  $G$  is the subgroup generated by the minimal normal subgroups of  $G$ .*

**Proposition 4.7.** *The socle  $\text{Soc}(G)$  is a characteristic subgroup of  $G$ .*

The following lemma is a crucial property of residual towers for primitive groups.

**Lemma 4.8.** [Lu86] *If  $H$  is a primitive permutation group acting on  $\Omega$  then the smallest nontrivial group in any residual tower is always  $\text{Soc}(H)$ .*

Now we consider an instance  $(G, \Delta, \Omega)$  of primitive-POINTSET $_b$ . As the orbits  $\Omega_i$  are of constant size, we can in deterministic logspace compute a residual tower  $\{R_{i,j}\}_{j=1}^l$  for each  $G_i$ . We assume, without loss of generality, that the length  $l$  of residue sequences of  $G_i$ 's are all same. Notice that  $l$  is a constant bounded by  $\log b$ .

We describe a series of groups for  $G$  obtained from these residual towers.<sup>1</sup>

First consider the groups  $S_j = R_{1,j} \times \dots \times R_{m,j}$ . Notice that  $S_0 = G_1 \times \dots \times G_m$ . Furthermore, since  $R_{i,j} \triangleleft G_i$  for  $1 \leq i \leq m$  and  $1 \leq j \leq l$ , it follows easily that we have a normal series

$$1 = S_l \triangleleft S_{l-1} \triangleleft \dots \triangleleft S_1 \triangleleft S_0 = G_1 \times \dots \times G_m.$$

We refine this normal series further as follows. Consider the list of simple groups  $\{T_1, \dots, T_c\}$  as an ordered list, we insert a series of length at most  $c$  between  $S_j$  and  $S_{j+1}$  for each  $j$ .

$$S_j = Q_0 \triangleright Q_1 \triangleright Q_2 \dots \triangleright Q_c = S_{j+1}.$$

where, for  $1 \leq k \leq c$ , the  $Q_k = R_{1,j}^{(k)} \times \dots \times R_{m,j}^{(k)}$  is given by

$$R_{i,j}^{(k)} = \begin{cases} R_{i,j+1} & \text{if } R_{i,j}/R_{i,j+1} \cong T_k \\ R_{i,j}^{(k-1)} & \text{otherwise} \end{cases}$$

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<sup>1</sup>This is different from the series used in [Lu86], and is more convenient for proving the results of this paper.

Notice that each  $Q_k$  is a normal subgroup of  $S_0$ . Furthermore  $Q_k/Q_{k+1}$  is either trivial or is  $T_k$ -semisimple. As a result we have a normal series of length at most  $cl$  (a constant) of the following form

$$G_1 \times \dots \times G_m = R_0(G) \triangleright R_1(G) \triangleright \dots \triangleright R_k(G) = 1, \quad (1)$$

where  $R_i(G)/R_{i+1}(G)$  is  $T$ -semisimple for some  $T \in \{T_1, \dots, T_c\}$ .

Taking  $N_i = G \cap R_i(G)$  we have the following *residue series*

$$G = N_0 \triangleright \dots \triangleright N_k = 1, \quad (2)$$

where  $N_i/N_{i+1}$  can be embedded  $R_i(G)/R_{i+1}(G)$ . It follows that  $N_i/N_{i+1}$  is also  $T$ -semisimple for some  $T \in \mathcal{T}$ . Furthermore,  $N_i \triangleleft G$  for all  $i$ .

We need the following properties of this series (as in [Lu86]).

**Lemma 4.9.** [Lu86] *Let  $N$  be a group in the normal series of Equation 2. Then for any subgroup  $H \leq G$  the projection  $pr_i(H) = G_i$  if and only if  $pr_i(H \cap N) = pr_i(N)$ .*

The following characterization of the socles of primitive permutation groups is also needed.

**Theorem 4.10 (O’Nan-Scott theorem).** *Let  $G \leq \text{Sym}(\Omega)$  be a primitive permutation group with socle  $\text{Soc}(G) = K$ . Let  $\omega \in \Omega$ . Then  $K$  is  $T$ -semisimple for some simple group  $G$  and one of the following holds:*

- (i)  $K$  is abelian in which case  $K$  is elementary abelian, regular on  $\Omega$ , and is the unique minimal normal subgroup of  $G$ . Furthermore  $K_{\{\omega\}} = 1$ .
- (ii)  $K$  is nonabelian, transitive on  $\Omega$ , and is the unique minimal normal subgroup of  $G$ .
- (iii)  $K$  is nonabelian and is a product  $K = K_1 \times K_2$ , where  $K_1$  and  $K_2$  are isomorphic.  $K_1$  and  $K_2$  are the only minimal normal subgroup of  $G$ , and each  $K_i$  acts regularly on  $\Omega$ . For each  $\omega \in \Omega$ ,  $K_{\{\omega\}} = \text{Diag}(K_1 \times K_2)$ .

Another crucial result is Scott’s lemma as stated in [Lu86]. We state a version useful for us. Let  $G$  and  $H$  be finite isomorphic groups. Then for an isomorphism  $\varphi$  from  $G$  to  $H$ , the diagonal subgroup  $\text{Diag}(G \times H)$  of  $G \times H$  defined as follows

$$\text{Diag}(G \times H) = \{(x, \varphi(x)) \mid x \in G\}.$$

Clearly,  $\text{Diag}(G \times H)$  is isomorphic to each of  $G$  and  $H$ . If  $\{H_i \mid 1 \leq i \leq r\}$  is a set of mutually isomorphic finite groups, we can similarly define  $\text{Diag}(\times_{i=1}^r H_i)$  as a subgroup of  $\prod_{i=1}^r H_i$ , given isomorphisms from  $H_1$  to each of the  $H_i$ ,  $i \neq 1$ .

**Lemma 4.11 (Scott’s Lemma).** *Let  $T$  be a finite simple group and let  $G$  be any subgroup of  $\prod_{i=1}^r T_i$  such that each  $T_i$  is isomorphic to  $T$ . Then  $G$  is a direct product of diagonal subgroups. More precisely, there is a partition  $\cup_{j=1}^s I_j$  of  $\{1, \dots, r\}$  such that*

$$G = \prod_{j=1}^s \text{Diag}(\times_{i \in I_j} T_i).$$



We now give a brief overview of the algorithm for primitive-POINTSET<sub>b</sub>. Let  $(G, \Delta)$  be an instance of primitive-POINTSET<sub>b</sub> and let  $G = N_0 \triangleright \dots \triangleright N_l$  be the residue series for  $G$  as defined in Equation 2. The algorithm claimed in the following theorem is the key subroutine for solving primitive-POINTSET<sub>b</sub>.

**Theorem 4.12.** *Given a instance  $(G, \Delta)$  of primitive-POINTSET<sub>b</sub> and a residue series  $G = N_0 \triangleright \dots \triangleright N_l = \{1\}$  for the group  $G$ , there is a sequence of subgroups  $G = H_0, H_1, \dots, H_l$  of  $G$  such that each  $H_s$  in the sequence contains  $G_{\{\Delta\}}$ , and  $|pr_i(H_s)| \leq |G_i|/2$  for all indices  $i$  such that  $pr_i(N_s) = \{1\}$  and  $\Omega_i$  contains a point of  $\Delta$ . Furthermore, for each  $s \geq 0$ , given the subgroup  $H_s$  of this sequence by a generating set the subgroup  $H_{s+1}$  can be computed by an  $L^{\text{Mod}_t L}$  algorithm.*

For an instance  $(G, \Delta)$  of primitive-POINTSET<sub>b</sub>, call an orbit  $\Omega_i$  a *target orbit* if  $\Omega_i$  contains a point of  $\Delta$ .

We compute the sequence of subgroups  $H_0, \dots, H_l$  as follows: To begin with  $H_0 = G$ . Then,  $H_{i+1}$  is computed from  $H_i$  by applying the  $L^{\text{Mod}_t L}$  algorithm of Theorem 4.12.

As a consequence of this theorem, the subgroup  $H_l$  is such that for all target orbits  $\Omega_i$ ,  $|pr_i(H_l)| \leq |G_i|/2$ . Thus, if we consider the group  $H = H_l$ , then notice that  $G_{\{\Delta\}} \leq H$ , and  $H$  has been cut down on all target orbits. We now recompute the residue series with  $G$  replaced by  $H = H_l$  and repeat the above process. Since in every such iteration the group is cut down in size on each target orbit, in a *constant* number of rounds the algorithm will have converged to  $G_{\{\Delta\}}$ . This requires only a constant number of iterations as each  $G_i$  is a constant-sized group. We will prove Theorem 4.12 in Section 6.

For putting primitive-POINTSET<sub>b</sub> in the  $\text{Mod}_k L$  hierarchy we first need to compute the residue series of  $G$  for an instance  $(G, \Delta)$  of primitive-POINTSET<sub>b</sub>. In the next section we explain how to compute it in the  $\text{Mod}_k L$  hierarchy using a general notion of strong generating sets as in [LM88]. Combined with the algorithm sketched above this will prove the upper bound for primitive-POINTSET<sub>b</sub>.

## 5 The Strong Generating Set problem

Let  $G \leq \text{Sym}(\Omega)$ , where  $G$  has orbits  $\Omega_i$ ,  $1 \leq i \leq m$  of size bounded by a constant  $b$ . Consider the *residue series* for  $G$  defined by Equation 2. In this section we show that computing a generating set for each subgroup in the residue series is in the  $\text{Mod}_k L$  hierarchy. The method we use is via strong generating sets and sifts, as invented by Luks and McKenzie [LM88, Lu86]. We begin with a general definition.

**Definition 5.1 (Strong generator set).** *Let  $G$  be a permutation group and let  $H$  be a subgroup of  $G$ . For a tower of subgroup  $G = G_0 \geq \dots \geq G_r = H$  let  $B_i$ ,  $0 \leq i < r$ , be the set of all left coset representatives of  $G_{i+1}$  in  $G_i$ . Every element  $g \in G$  can be uniquely expressed as a product  $g = g_0 g_1 \dots g_{r-1} h$  where  $g_i \in B_i$  and  $h \in H$ . The collection of sets  $B_i$ ,  $0 \leq i < r$  is called a strong generating set of  $G$  relative to  $H$  (hereafter abbreviated as SGS for  $G$  rel  $H$ ). If  $H = 1$  then  $\cup B_i$  is a generating set for  $G$  which is called a strong generating set for  $G$ .*

Given an element  $g \in G$  and an SGS (say,  $\cup B_i$ ) of  $G$  rel  $H$ , there is a unique  $h \in H$  such that  $g$  can be written as a product  $g = g_0 g_1 \dots g_{r-1} h$  where  $g_i \in B_i$ . By the *sift* of  $g$ , denoted by  $\text{Sift}(g)$ , we mean this element  $h$ . Note that the sift of an element depended on the choice of coset representatives  $B_i$  in the SGS of  $G$ .

We are interested in the special case when  $H \triangleleft G$ . For an SGS of  $G$  rel  $H$  arising from a *normal series* between  $G$  and  $H$ , we have the following lemma.

**Lemma 5.2.** *Let  $G$  be a permutation group and  $H$  be any normal subgroup of  $G$ . Let  $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_k = H$  be a normal series between  $G$  and  $H$  and let  $B_i$ 's be the left (right) coset representatives of  $G_{i+1}$  in  $G_i$ . Let  $A = \{g_1, g_2, \dots, g_l\}$  be a set of generators for  $G$ . Let  $S$  be the set consisting of the following elements*

1.  $\text{Sift}(g)$  for all  $g \in A$ .
2.  $\text{Sift}(x^{-1}yx)$  for all  $x \in B_i$  and  $y \in B_j$ ,  $i \leq j$ .
3.  $\text{Sift}(xy)$  for all  $x, y \in B_i$  for all  $i$ .

Then  $\text{NCL}_G(S) = H$ .

*Proof.* Since  $H$  is a normal subgroup of  $G$  and since all elements of  $S$  are contained in  $H$  it is clear that  $\text{NCL}_G(S) \leq H$ . To prove the converse consider any element  $h \in H$ . There exists elements  $y_1, y_2, \dots, y_m$  in  $A$  such that  $h = y_1 y_2 \dots y_l$ . Now since  $S$  contains the sifts of all  $y$ 's we have

$$h = \prod_{i=1}^m x_{i,1} x_{i,2} \dots x_{i,r} s_i.$$

where  $x_{i,j} \in B_j$  and  $s_i \in \text{NCL}_G(S)$ . Using the fact that  $\text{NCL}_G(S)$  is a normal subgroup and using properties 2 and 3 repeatedly we can rewrite the above product as  $h = x_1 x_2 \dots x_r s$  where  $x_i \in B_i$  and  $s \in \text{NCL}_G(S)$ . Now since  $h \in H$  we have  $x_1 x_2 \dots x_r = 1$  and hence  $h = s$ .  $\square$

## 5.1 Computing the Residue series

Let  $G \leq \text{Sym}(\Omega)$  with bounded orbits, given by a set of generators  $A$ . Let  $\Omega_1, \dots, \Omega_m$  be its orbits of size at most  $b$ . Consider the residue series

$$G = N_0 \triangleright N_1 \triangleright \dots \triangleright N_k = \{1\}.$$

Recall that  $k$  is a constant depending on  $b$ , and each quotient  $N_i/N_{i+1}$  is  $T$ -semisimple for some simple group  $T \in \{T_1, \dots, T_c\}$ , where  $c$  is another constant depending on  $b$ .

The algorithm proceeds in stages: at the  $i$ th stage, we assume that an SGS for  $G$  rel  $N_i$  is already computed, and the task at this stage is to extend it to an SGS for  $G$  rel  $N_{i+1}$ . To do this, following the approach of [LM88], we will compute an SGS for  $N_i$  rel  $N_{i+1}$  and append it to the SGS for  $G$  rel  $N_i$ . If we can show that this step can be carried out in  $L^{\text{Mod}_t L}$  for a suitable constant  $t$ , the claimed  $\text{Mod}_k L$  hierarchy upper bound will follow (where the level of the hierarchy is the number of stages).

In order to compute an SGS for  $N_i$  rel  $N_{i+1}$ , notice that it suffices to solve the problem defined in the theorem below (we can choose, in the theorem statement below,  $L = R_i(G)$  and  $M = R_{i+1}(G)$  as defined in Equation 1).

**Theorem 5.3.** *Let  $G \leq \text{Sym}(\Omega)$  with orbits  $\Omega_i$ ,  $1 \leq i \leq m$  given by generator set  $A$ . Let  $L = L_1 \times \dots \times L_r$  and  $M = M_1 \times \dots \times M_r$  be such that  $\text{Sym}(\Omega_i) \geq L_i \geq M_i$  and  $G$  normalizes both  $L$  and  $M$  and  $L_i/M_i$  is  $T$ -semisimple for some simple group  $T$ . Given a set  $S$  of elements*

of  $G$  such that  $\text{NCL}_G(S) = G \cap L$  there is an  $L^{\text{Mod}_t L}$  algorithm to compute an SGS for  $G \cap L$  rel  $G \cap M$ , for some constant  $t$ .

Furthermore, for any  $x \in G \cap L$  there is an  $L^{\text{Mod}_t L}$  algorithm to compute the  $\text{Sift}(x)$  w.r.t. the SGS for  $G \cap L$  rel  $G \cap M$  computed above.

*Proof.* The group  $L/M$  is  $T$ -semisimple as each  $L_i/M_i$  is  $T$ -semisimple. Consequently,  $L/M$  is of the form  $T_1 \times T_2 \times \dots \times T_s$  where  $T_i \cong T$  for all  $1 \leq i \leq s$ , for some  $s = O(m)$ , where the constant factor depends on  $b$ . Moreover,  $G \cap L/G \cap M$  can be faithfully embedded into  $\prod_{i=1}^m L_i/M_i$ .

We handle the proof of the theorem in two cases:

### **$T$ is nonabelian.**

By Scott's Lemma 4.11 we know that  $G \cap L/G \cap M$  is a product of diagonal groups, each isomorphic to  $T$ . Let  $\phi_i : L \mapsto T_i$  be the map obtained by composing the quotient map from  $L$  to  $L/M$  and the projection map to  $T_i$ . We say that  $T_i$  and  $T_j$  are *linked* if  $G \cap L/G \cap M$  restricted to  $T_i \times T_j$  is the diagonal group. Consider an undirected graph  $\mathcal{G}$  with vertex set  $V = \{T_i : 1 \leq i \leq t\}$  and edge set  $\{(T_i, T_j) : T_i \text{ and } T_j \text{ are linked}\}$ . The groups  $T_i$  and  $T_j$  are of bounded size hence we can check whether  $T_i$  and  $T_j$  are linked in deterministic logspace as follows:

Find  $G \cap L/G \cap M$  restricted to  $T_i \times T_j$ . Since this is a constant size group one can easily determine whether the group is  $T_i \times T_j$  or  $\text{diag}(T_i \times T_j)$  (by checking the order for example). To find  $G \cap L/G \cap M$  we first compute the set  $\phi_{ij}(S) = \{\langle \phi_i(s), \phi_j(s) \rangle : s \in S\}$ . We keep on adding new elements by taking the  $G$  conjugates of elements of  $S$  and restricting it to the  $T_i$  and  $T_j$  until no more elements can be added. This will require only a constant number of such additions, and hence can be done in logspace.

Now in the graph  $\mathcal{G}$  we find the connected components (in  $L^{\text{SL}} = L$ ). Let the connected components be  $C_k$ ,  $1 \leq k \leq l$ . For each connected component  $C_k$  of  $\mathcal{G}$  we pick a vertex, say  $T_{i_k}$ .

The main algorithmic step is the computation of elements  $g_k \in G \cap L$ ,  $1 \leq k \leq l$  such that  $\phi_{i_j}(g_k) \in T_{i_j}$  is identity for all  $j$  except  $j = k$ . Assuming that we have these elements, consider the normal subgroup  $\text{NCL}_G(g_k)$  of  $G \cap L$ . Since  $\phi_{i_k}$  is an onto homomorphism and  $T_{i_k}$  is simple, it follows that  $\phi_{i_k}(\text{NCL}_G(g_k))$ , being a nontrivial normal subgroup of  $T_{i_k}$ , must in fact be  $T_{i_k}$ . We compute a set  $B_k$  of distinct inverse images of  $\phi_{i_k}(\text{NCL}_G(g_k))$ ,  $1 \leq k \leq l$ . It is easy to see that  $\cup_{k=1}^l B_k$  is an SGS of  $G \cap L$  rel  $G \cap M$ .

In the sequel, we explain how to compute the  $g_k$ 's. It suffices to explain how to compute  $g_1$ .

To begin with we define *iterated commutators*. Let  $[h_1, \dots, h_k]$  be defined as

$$\begin{aligned} [h_1, h_2] &= h_1^{-1} h_2^{-1} h_1 h_2, \\ [h_1, \dots, h_i, h_{i+1}] &= [[h_1, \dots, h_i], h_{i+1}]. \end{aligned}$$

We will compute a sequence of elements  $h_1, h_2, \dots, h_l$  satisfying the following property.

1.  $\phi_{i_1}(h_1) \neq 1$ ,
2.  $\phi_{i_1}([h_1, h_2, \dots, h_k]) \neq 1$  for all  $k$  and
3.  $\phi_{i_k}(h_k) = 1$ .

It is easy to see that given  $h_1, h_2, \dots, h_l$  as above,  $g_1 = [h_1, h_2, \dots, h_l]$  has the required properties:  $\phi_{i_1}(g_1) \neq 1$ , and  $\phi_{i_k}(g_1) = 1$  for  $2 \leq k \leq l$ .

```

 $x := h_1$  such that  $\phi_{i_1}(h_1) \neq 1$  ;
 $\alpha := \phi_{i_1}(h_1)$ ;
for  $k = 2$  to  $l$  do
  compute  $h_k$  such that  $\phi_{i_k}(h_k)$  does not commute with  $\alpha$  and  $\phi_{i_k}(h_k) = 1$ ;
   $\beta := \phi_{i_1}(h_1)$  ;
   $\alpha := [\alpha, \beta]$  ;
  output  $h_k$ ;
end

```

We now give a logspace algorithm to actually find such a sequence  $h_1, h_2, \dots, h_l$ .

Having got the sequence  $h_1, \dots, h_l$  we now need to compute the iterated commutator  $[h_1, \dots, h_l]$  in logspace. Then, combining the two logspace computations we get a logspace algorithm for computing  $g_1$ .

**Claim.** *Let  $G$  be a permutation group with bounded-size orbits. Given  $h_1, \dots, h_n \in G$ , the iterated commutator  $[h_1, \dots, h_n]$  can be computed in deterministic logspace.*

It suffices to compute  $[h_1, \dots, h_n]$  restricted to each  $G$ -orbit  $\Omega$  separately. Since  $G$  has bounded-size orbits, and since the iterated commutator  $[h_1, \dots, h_n]$  is a formula over the  $h_i$ 's, we can evaluate  $[h_1, \dots, h_n]$  restricted to each orbit  $\Omega_i$  in  $\text{NC}^1$  (and hence deterministic logspace). The claim follows.

Finally, we claim that  $B_k$  can be computed in deterministic logspace from  $g_k$ . This is possible because it suffices to carry out the normal closure restricted to one of orbits on which  $g_k$  is nontrivial. However, notice that the computation will obtain the elements of  $B_k$  as permutations on the whole set  $\Omega$ . There will be only a constant number of elements to be included in each  $B_k$ . Thus, we have computed  $\cup_{k=1}^l B_k$  which is an SGS of  $G \cap L \text{ rel } G \cap M$ .

We now explain how to compute  $\text{Sift}(x)$  for any  $x \in G \cap L$  w.r.t. this SGS:

Let  $x \in G \cap L$  to be sifted. We apply  $\phi_{i_k}(x)$  which is in  $T_{i_k}$  for each  $k$ , corresponding to the vertices  $T_{i_k}$  picked from each connected component  $C_k$  of  $\mathcal{G}$ ,  $1 \leq k \leq l$ . Since  $\phi_{i_k}(B_k) = T_{i_k}$ , for each  $k$ , in logspace we can find an  $x_k$  in the SGS such that  $\phi_{i_k}(x_k) = (\phi_{i_k}(x))^{-1}$ . It is now easy to see that  $\text{Sift}(x) = x \prod_{k=1}^l x_k$ .

This completes the proof for the nonabelian  $T$ .

### **$T$ is abelian.**

As  $T$  is abelian and simple, for some prime  $p \leq b$ ,  $T$  is isomorphic to the additive group  $\mathbb{Z}_p$ . Since  $L/M$  to be  $T$ -semisimple, we have  $L/M = T_1 \times T_2 \times \dots \times T_s$  where  $T_i \cong \mathbb{Z}_p$  for each  $i$ , and  $s = O(m)$ . Thus,  $L/M$  is essentially  $\prod_{i=1}^m \mathbb{Z}_p$ . But  $\prod_{i=1}^m \mathbb{Z}_p$  has more structure because it is an  $m$ -dimensional vector space over the finite field  $\mathbb{F}_p$ . Furthermore, since  $G \cap L/G \cap M$  can be faithfully embedded in  $\prod_{i=1}^m L_i/M_i$ , we know that  $G \cap L/G \cap M$  is essentially a subspace of  $\prod_{i=1}^m \mathbb{Z}_p$ .

Notice that the vector space structure of  $L/M = \prod_{i=1}^m \mathbb{Z}_p$  is effectively obtained: we have the generating set of  $L/M = \prod L_i/M_i$ , say  $\{s_1M, \dots, s_rM\}$ , in terms of the generating sets for  $L_i/M_i$ , where we know the permutations  $s_i$  on the entire set  $\Omega$ . Thus, in deterministic logspace we can convert an element  $sM$  of  $L/M$  into the corresponding vector  $\mathbf{v}_s \in \prod_{i=1}^m \mathbb{Z}_p$ . Although the computation of SGS for  $G \cap L \text{ rel } G \cap M$  and the Sift operation will be solved as a linear algebra problem modulo  $p$ , alongside we need to compute the SGS for  $G \cap L \text{ rel } G \cap M$  in as elements of  $\text{Sym}(\Omega)$ . Similarly, we need to do the Sift operation on elements of  $\text{Sym}(\Omega)$ . Thus, it is important for the algorithm to keep track of the permutation  $\pi$  corresponding to each vector  $\mathbf{v}_\pi$

that is computed. However, notice that this is easy to do because for  $\mathbf{v}_\pi$  expressed as an  $\mathbb{F}_p$  linear combination of the  $\mathbf{v}_{s_i}$ 's. More precisely, we know that if  $s$  and  $s'$  correspond to  $\mathbf{v}_s$  and  $\mathbf{v}_{s'}$ , then  $ss'$  corresponds to  $\mathbf{v}_s + \mathbf{v}_{s'}$  and  $s^k$  corresponds to  $k\mathbf{v}_s$  for  $k \in \mathbb{Z}_p$ .

Now, given  $S \subset G$  such that  $\text{NCL}_G(S) = G \cap L$ , we will show how to compute an SGS for  $G \cap L$  rel  $G \cap M$  in  $\text{FL}^{\text{Mod}_p L}$ . We need the following observation from [LM88] about the normal closure that will convert the problem to a simple linear algebra problem over  $\mathbb{F}_p$  that can be solved in  $\text{FL}^{\text{Mod}_p L}$  [ABO99, BDHM92].

Define  $\tau_g : L/M \rightarrow L/M$  as  $\tau_g(h) = g^{-1}hg$  where  $g$  is a generator of  $G$  and  $h \in L/M$ . The mapping  $\tau_g$  is clearly an automorphism of the group  $L/M$ . Since  $L/M$  is the vector space  $\prod_{i=1}^m \mathbb{Z}_p$ ,  $\tau_g$  is an invertible linear transformation for each  $g \in G$ . Furthermore, given  $g \in G$  in deterministic logspace we can write down the  $m \times m$  matrix  $t_g$  corresponding to  $\tau_g$ . It is easy to see that the matrix  $t_g$  will actually be a block diagonal matrix, with one block for each  $L_i/M_i$ ,  $1 \leq i \leq s$ . The logspace algorithm just needs to figure out the block corresponding to  $L_i/M_i$  for each  $i$ . Since  $L_i/M_i$  are constant sized, this can be easily done by exhaustive search.

Thus, if  $G = \langle g_1, \dots, g_q \rangle$ , then in logspace we can compute the list of matrices  $t_{g_1}, \dots, t_{g_q}$ . Each matrix  $t_{g_i}$  here is block diagonal, with each block being a square matrix of size at most  $c$  (where  $c$  is a constant depending on  $b$ ). The following claim is from [LM88].<sup>2</sup>

**Claim.**  $\text{NCL}_G(S)/G \cap M$  is the smallest subspace of  $L/M$  that contains the set of vectors  $S$  and is closed under the linear transformations  $\tau_{g_1}, \dots, \tau_{g_q}$ .

Thus, the problem of computing the SGS for  $G \cap L$  rel  $G \cap M$  boils down to finding the smallest subspace of  $\prod_{i=1}^m \mathbb{Z}_p$  containing the vectors  $\{\mathbf{v}_s \mid s \in S\}$  and closed under the linear maps defined by  $t_{g_1}, \dots, t_{g_q}$ . Since each matrix  $t_{g_i}$  is block diagonal with each block of size bounded by the constant  $c$ , the matrix algebra  $\mathcal{A}$  that is generated by  $t_{g_1}, \dots, t_{g_q}$  can be characterized by the following claim.

**Claim.** Let  $\mathcal{M}$  consist of the set of products of all choices of  $j$  matrices from  $\{t_{g_1}, \dots, t_{g_q}\}$ , for each  $j$  such that  $1 \leq j \leq c^2$ . Then the matrix algebra  $\mathcal{A}$  is the  $\mathbb{F}_p$  linear span of the set  $\mathcal{M}$ .

The above claim follows easily from the fact that the subalgebra generated by each block can have dimension at most  $c^2$ . Thus, it suffices to consider at most  $c^2$  many matrix products for all the blocks to obtain an  $\mathbb{F}_p$  linear basis for  $\mathcal{A}$ .

With the above claim the  $\text{FL}^{\text{Mod}_p L}$  algorithm is now easy to describe. First, a deterministic logspace algorithm can output the set  $\mathcal{M}$ , since multiplying a constant number of matrices can be done in logspace. For each matrix  $t \in \mathcal{M}$ , notice that the logspace algorithm can also keep track of the corresponding element  $g \in G$ , by carrying out the corresponding multiplication.

Next, a deterministic logspace algorithm can list out the vectors  $t(\mathbf{v}_s) \mid s \in S, t \in \mathcal{M}$ . The corresponding elements as permutations of the form  $g^{-1}sg$  can be kept track of again. To obtain the SGS from this set of vectors, we need to pick a maximal linearly independent subset. Since testing feasibility of linear equations over  $\mathbb{F}_p$  can be done in  $\text{Mod}_p L$  [ABO99], it follows that a basis (and hence an SGS) can be computed in  $\text{FL}^{\text{Mod}_p L}$ . The corresponding permutations are also available. Thus, we have the SGS both as permutations and in vector form. Let the SGS as permutations be denoted by the list  $\{\pi_1, \dots, \pi_t\}$ .

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<sup>2</sup>In [LM88] this is used for showing that the membership testing problem over permutation groups is in NC. We have adapted this to the BCGI setting where we crucially use the fact that the matrices are block diagonal with constant-size blocks to derive our upper bound.

Finally, we need to describe  $\text{Sift}(g)$  for any  $g \in G$  using this SGS. We first write  $g$  as a vector  $\mathbf{v}_g$  in  $L/M$  in logspace (by examining the action of  $g$  on each orbit). We write down a system of linear equations  $AX = \mathbf{v}_g$ , where the  $i$ th column  $A_i$  of  $A$  is  $\mathbf{v}_{\pi_i}$  for  $1 \leq i \leq t$ , and  $\pi_i$  are the elements of the SGS. We compute the unique solution  $x_1, \dots, x_r$  to this in  $\text{FL}^{\text{Mod}_p L}$ . Notice that  $\text{Sift}(g) = g \prod_{i=1}^r g_i^{-x_i}$ .  $\square$

Now we are ready to give the overall algorithm for computing the SGS of a permutation group with bounded orbits via its residue series.

**Theorem 5.4.** *Let  $G \leq \text{Sym}(\Omega)$  be a permutation group with orbits  $\Omega_i$  of size bounded by a constant such that  $\text{pr}_i(G)$  is primitive on  $\Omega_i$  for each  $i$ . Then an SGS (strong generating set) for  $G$  w.r.t. its residue series can be computed in the  $\text{Mod}_k L$  hierarchy.*

*Proof.* Let  $G_i$  be the projection  $\text{pr}_i(G)$  of  $G$  onto the  $i$ th orbit  $\Omega_i$ . Recall from Equation 2 that the residue series of  $G$  is

$$G = N_0 \triangleright \dots \triangleright N_k = 1,$$

which the groups  $N_i$  are defined w.r.t. the other series given by Equation 1:

$$G_1 \times \dots \times G_m = R_0(G) \triangleright R_1(G) \triangleright \dots \triangleright R_k(G) = 1.$$

Here,  $N_i = G \cap R_i(G)$  for each  $i$ .

We assume as inductive hypothesis that we already have the strong generating set of  $G$  relative to  $N_i$ , and we have a sifting procedure for elements of  $G$  with respect to this strong generating set. Furthermore, we assume inductively that computing this SGS and the sifting procedure can be carried out in the  $\text{Mod}_k L$  hierarchy of some level  $j$ . Here  $k$  is suitably chosen constant depending only on the orbit bound  $b$ .

We show using the above inductive assumption and Theorem 5.3 that the SGS of  $G$  rel  $N_{i+1}$  and can be computed in the  $j + 1$ st level of the  $\text{Mod}_k L$  hierarchy, and the corresponding sifting procedure can be carried out in the  $j + 1$ st level of the  $\text{Mod}_k L$  hierarchy.

Using the already computed the SGS of  $G$  relative to  $N_i$  and the corresponding sifting procedure, we can compute (in the  $j$ th level of the  $\text{Mod}_k L$  hierarchy) a subset  $S \subseteq G$  such that  $\text{NCL}_G(S) = N_i$  (Lemma 5.2). Now, using this set  $S$ , notice that  $N_i = G \cap R_i(G)$  and  $N_{i+1} = G \cap R_{i+1}(G)$  are groups to which we can apply Theorem 5.3, with  $L = R_i(G)$  and  $M = R_{i+1}(G)$ , to find both the SGS of  $N_i$  rel  $N_{i+1}$  and be able to sift w.r.t. this SGS in  $\text{FL}^{\text{Mod}_k L}$ . The union of the SGS of  $G$  rel  $N_i$  with the SGS of  $N_i$  rel  $N_{i+1}$  will give the SGS of  $G$  rel  $N_{i+1}$ . Clearly, the computation can be done in the  $j + 1$ st level of the  $\text{Mod}_k L$  hierarchy.

We next need to show how to sift w.r.t. this SGS of  $G$  rel  $N_{i+1}$ . Given an element  $g \in G$  we first compute its sift  $g'$  w.r.t the SGS of  $G$  rel  $N_i$ . By induction hypothesis, this computation can be done in  $j$ th level of the  $\text{Mod}_k L$  hierarchy. Next, since the element  $g' \in G \cap N_i$ , we can apply the  $\text{FL}^{\text{Mod}_k L}$  sift procedure of Theorem 5.3 w.r.t. to the SGS of  $N_i$  rel  $N_{i+1}$  to finally obtain the sift of  $g$  w.r.t. the SGS of  $G$  rel  $N_{i+1}$ . Again, the overall complexity lies in the  $j + 1$ st level of the  $\text{Mod}_k L$  hierarchy.

Continuing this for the  $l$  levels of the residue series, we obtain the required SGS for  $G$ . The computation can be carried out in the  $l$ th level of the  $\text{Mod}_k L$  hierarchy. It is easy to see that the constant  $k$  can be chosen as the product of all distinct primes  $p$  such that  $N_i/N_{i+1}$  is  $\mathbb{Z}_p$ -semisimple. This completes the proof.  $\square$

## 6 The Pointwise Stabilizer problem

The goal of this section is to prove Theorem 4.12 which would complete the upper bound proof for primitive-POINTSET<sub>b</sub>, and hence for BCGI.

Let  $(G, \Delta)$  be an instance of primitive-POINTSET<sub>b</sub>. For each orbit  $\Omega_i$  consider the projection of the residual tower given by Equation 1 to the orbit  $\Omega_i$  resulting in the following series for  $G_i$ :

$$G_i = pr_i(R_0(G)) \triangleright pr_i(R_1(G)) \triangleright \dots \triangleright pr_i(R_k(G)) = 1.$$

Now, since each  $G_i$  is primitive, from Lemma 4.8, it follows that the smallest nontrivial group in the above series is  $\text{Soc}(G_i)$ , for each  $i$ . Consequently, in the residue series  $G = N_0 \triangleright \dots \triangleright N_l$  of  $G$ , for each  $\Omega_i$  there is some  $N_s$  such that  $pr_i(N_s) = \text{Soc}(G_i)$  and  $pr_i(N_{s+1}) = \{1\}$ . Given an index  $s$ , let  $X$  be the set of target orbit indices  $i$  such that  $pr_i(N_s) = \text{Soc}(G_i)$  and  $pr_i(N_{s+1}) = \{1\}$ .

Assume, as inductive hypothesis, that we have computed a generating set for the group  $H_s$  in the sequence of groups  $G = H_0, H_1, \dots, H_l$  claimed in Theorem 4.12 with the desired properties. Namely,  $H_s$  contains  $G_{\{\Delta\}}$  as subgroup and  $|pr_i(H_s)| \leq |G_i|/2$  for all target orbit indices  $i$  such that  $pr_i(N_s) = \{1\}$ . We will find  $H_{s+1}$  as a subgroup of  $H_s N_s$  such that  $|pr_i(H_{s+1})| \leq |G_i|/2$  for all  $i \in X$ . Clearly,  $H_{s+1}$  will satisfy Theorem 4.12.

We first give a broad outline of the FL<sup>Mod<sub>t</sub>L</sup> algorithm for finding  $H_{s+1}$ . The actual details of the algorithm varies depends upon whether the socle is abelian or not. It will be taken up in the two subsections after the outline.

The algorithm will pick a *critical subset*  $Y$  of  $X$  and compute the following: For each generator  $x$  of  $H_s$ , it computes an element  $x' \in G$  obtained as  $xy \in H_s N_s$  for a suitably chosen  $y \in N_s$  such that  $x'$  fixes  $\Delta \cap \Omega_i$  for each  $i \in Y$ . Next, the algorithm computes the subgroup  $N$  of  $N_s$  which pointwise fixes  $\Delta \cap \Omega_i$  for each  $i \in Y$ . Let  $H = \langle \{x' \mid x \text{ generator of } H_s\} \rangle$ . By construction,  $HN$  will pointwise fix  $\Delta \cap \Omega_i$  for each  $i \in Y$ . Furthermore, the algorithm will choose  $Y$  suitably (which will be explained in the details) so that  $|pr_i(HN)| \leq |pr_i(G)|/2$  for all  $i \in X$ . The subgroup  $H_{s+1}$  is defined to be  $HN$ .

Since the residue series for  $G$  has the property that for each  $i$  there is an  $s$  such that  $pr_i(N_s)$  is the socle of  $G_i$ , it follows that  $|pr_i(H_i)| \leq |G_i|/2$  for all the target orbits  $\Omega_i$ .

### When the socle is nonabelian

Suppose  $N_s/N_{s+1}$  is  $T$ -semisimple for a nonabelian simple group  $T$ . Based on the O'Nan Scott Theorem 4.10, we can partition  $X$  into  $X_1 \cup X_2$ , where for each  $i \in X_1$ ,  $\text{Soc}(G_i)$  is of type (ii), and for each  $i \in X_2$ ,  $\text{Soc}(G_i)$  is of type (iii). For  $i \in X_1$ ,  $\text{Soc}(G_i)$  is the unique minimal normal subgroup of  $G_i$ . For  $i \in X_2$ ,  $\text{Soc}(G_i)$  is of the form  $K_1 \times K_2$ , where the two *socle parts*  $K_1$  and  $K_2$  are the only two minimal normal subgroups of  $G_i$ .

Furthermore, if  $K$  is a socle for some  $i \in X_1$ , or  $K$  is a socle part for for some  $i \in X_2$ , then  $K$  is  $T$ -semisimple. More precisely, it is of the form  $T_1 \times \dots \times T_d$  for a constant  $d$  with each  $T_j \cong T$ , and  $G$  acts transitively by conjugation on  $\{T_1, \dots, T_d\}$ .

In the following lemma we summarize observations from [Lu86] (that follow from the O'Nan Scott Theorem 4.10 and Scott's Lemma 4.11). This lemma identifies the group  $N_s$  projected on orbits in  $X$  using the socle parts of  $\text{Soc}(G_i)$  for  $i \in X$ . More precisely, if  $K$  and  $K'$  are two socle parts corresponding to any two indices  $i, j \in X$ , then  $N_s$  projected on  $\Omega_i \cup \Omega_j$  either induces the group  $\text{Diag}(K \times K')$  or  $K \times K'$ . In the former case we say that the socle parts  $K$  and  $K'$  are *linked* in  $N_s$ .

**Lemma 6.1.** *Let  $G = N_0 \triangleright \dots \triangleright N_l$  be a residue series for the permutation group  $G$ , where  $(G, \Delta)$  is an instance of primitive-POINTSET<sub>b</sub>. Let  $X$  be the set of indices  $i$  such that  $\text{pr}_i(N_s) = \text{Soc}(G_i)$  and  $\text{pr}_i(N_{s+1}) = \{1\}$  where  $N_s/N_{s+1}$  is  $T$ -semisimple for a nonabelian simple group  $T$ . For any pair of indices  $i, j \in X$  exactly one of the following holds:*

- (a)  $N_s$  projected on  $\Omega_i \cup \Omega_j$  is of the form  $\text{Soc}(G_i) \times \text{Soc}(G_j)$ .
- (b) If  $i, j \in X_1$  and  $N_s$  projected on  $\Omega_i \cup \Omega_j$  is of the form  $\text{Diag}(K \times K')$ , where  $K = \text{Soc}(G_i)$  and  $K' = \text{Soc}(G_j)$ .
- (c) If  $i, j \in X_2$ , and  $\text{Soc}(G_i) = K_1 \times K_2$  and  $\text{Soc}(G_j) = K'_1 \times K'_2$  then  $N_s$  projected on  $\Omega_i \cup \Omega_j$  is either of the form  $\text{Diag}(K_1 \times K'_1) \times \text{Diag}(K_2 \times K'_2)$  or of the form  $\text{Diag}(K_1 \times K'_1) \times K_2 \times K'_2$ .
- (d) If  $i \in X_1$  and  $j \in X_2$ ,  $\text{Soc}(G_i) = K$  and  $\text{Soc}(G_j) = K_1 \times K_2$ , then  $N_s$  projected on  $\Omega_i \cup \Omega_j$  is of the form  $\text{Diag}(K \times K_1) \times K_2$ .

Furthermore, observe that if  $K = T_1 \times \dots \times T_d$  and  $K' = T'_1 \times \dots \times T'_e$  are two linked socle parts, then  $e = d$  and in  $\text{Diag}(K \times K')$ ,  $T_j$  is linked to  $T'_{\pi(j)}$  for some permutation  $\pi$  of the indices.

In deterministic logspace we first construct an undirected graph  $\mathcal{G}$  with vertex set  $V = \{K \mid K \text{ is a socle part of } \text{Soc}(G_i), i \in X\}$ . The edge set  $E$  is partitioned into RED and BLUE edges defined as follows:  $\text{RED} = \{(K, K') \mid K \text{ and } K' \text{ are linked}\}$  and  $\text{BLUE} = \{(K, K') \mid \exists i \in X : \text{Soc}(G_i) = K \times K'\}$ .

Next, in deterministic logspace the algorithm will find the connected components in the red subgraph  $(V, \text{RED})$ . Notice that by Scott's Lemma 4.11 the red subgraph is transitive. Thus, the connected components are just cliques and can be easily determined in deterministic logspace.

**Finding  $Y$**  We partition the cliques of the red subgraph of  $\mathcal{G}$  into two sets  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , where a red clique  $C$  is put in  $\mathcal{T}_1$  if  $C$  contains an element  $K = \text{Soc}(G_i)$  for some  $i \in X$ . The remaining cliques are put in  $\mathcal{T}_2$ .

We will find the critical subset  $Y$  of  $X$  in two stages. In the first stage we pick  $Y_1$  defined as follows: for every clique  $C \in \mathcal{T}_1$  pick a  $K = \text{Soc}(G_i) \in C$  and include the index  $i$  in  $Y_1$ .

We now take the graph  $\mathcal{G}$  and delete all the cliques in  $\mathcal{T}_1$  (and the incident edges). Call the new graph  $\mathcal{G}'$ . In  $\mathcal{G}'$ , considering the red cliques as vertices, we compute the lexicographically first spanning forest of blue edges in  $\text{L}^{\text{SL}} = \text{L}$ .<sup>3</sup>

Let  $E'$  be the blue edges in the spanning forest. Recall that each  $e = (K, K') \in E'$  corresponds to the orbit  $\Omega_i$  where  $\text{Soc}(G_i) = K \times K'$ . The remaining part  $Y_2$  of the critical subset  $Y$  consist of such indices  $i$  corresponding to edges in  $E'$ .

Thus, we have determined  $Y$  in  $\text{L}^{\text{SL}} = \text{L}$ .

**Finding  $N$  and  $H$**  The subgroup  $N$  of  $N_s$  pointwise fixes  $\Delta \cap \Omega_i$  for each  $i \in Y$ . I.e.  $N = \{g \in N_s \mid \delta^g = \delta, \forall \delta \in \cup_{i \in Y} (\Delta \cap \Omega_i)\}$ . Notice that, since  $N_{s+1}$  fixes  $\Delta \cap \Omega_i$  for each  $i \in X$ , we have  $N_s \geq N \geq N_{s+1}$ .

We find  $N$  in two stages corresponding to  $Y_1$  and  $Y_2$ .

Let  $N' = \{g \in N_s \mid \delta^g = \delta, \forall \delta \in \cup_{i \in Y_1} (\Delta \cap \Omega_i)\}$ .

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<sup>3</sup>Alternatively, we can shrink the cliques in  $\mathcal{G}'$  into vertices and find a spanning forest of blue edges.



Let the set  $S$  denote the SGS for  $N_s$  rel  $N_{s+1}$ . By the construction of the SGS described in Theorem 5.3 for the nonabelian case, it is easy to see that for every clique  $C \in \mathcal{T}_1$ , there is a subset  $S_C$  of the SGS  $S$  with the following property:

For a  $K \in C$  let the corresponding orbit index be  $i$  (note that  $i \in Y_1$ ). Either  $\text{Soc}(G_i) = K$  or  $\text{Soc}(G_i) = K \times K'$ , and we have  $pr_i(S_C) = K$  (note that  $K'$  has to be in a different clique). Furthermore, for each index  $j \neq i$  we have  $pr_j(S_C) = \{1\}$ .

Now, from the elements of  $S_C$  for each  $C \in \mathcal{T}_1$ , we can compute the subgroup  $N'$  of  $N_s$  in deterministic logspace. Let  $S'_C \subset S_C$  be those elements that fix  $\Delta \cap \Omega_i$  for each  $i \in Y_1$  corresponding to some  $K \in C$ . Then  $N'$  is generated by the union of the subsets  $S'_C$  of  $S_C$  for different  $C \in \mathcal{T}_1$ .

Furthermore, for each generator  $x$  of  $H_s$ , we can also compute an element  $y \in S$  such that  $xy = x'$  pointwise fixes  $\cup_{i \in Y_1} (\Delta \cap \Omega_i)$ . More specifically,  $y$  can be chosen by picking appropriate elements from different  $S_C$ ,  $C \in \mathcal{T}_1$  and multiplying them. Denote by  $H'$  the group generated by these  $x'$ 's computed as above.

Next we explain how to handle the critical orbit indices in  $Y_2$ . Consider the spanning forest with vertex set  $V' = V(\mathcal{G}')$  and edges  $E'$  (already defined, consisting of blue edges). Each tree in this forest can be considered as a rooted tree, with its root at the lexicographically least vertex. Thus, each edge in the forest is directed blue edge.

Consider any  $x \in G$ . We explain an  $\text{FL}^{\text{SL}} = \text{L}$  algorithm for computing an element  $y \in N'$  such that  $xy$  fixes all the elements in  $\cup_{i \in Y_2} (\Delta \cap \Omega_i)$ . In order to do this we define an  $\text{FL}^{\text{SL}} = \text{L}$  subroutine that does the following:

- Step 1. It takes as input an edge  $\{u, v\}$  in the spanning forest  $(V', E')$ . Using an SL oracle it determines the the unique path  $s = u_0, u_1, \dots, u_{r-1} = u, v = u_r$  in the forest from a root  $s$ , this gives the edge  $(u, v)$  an orientation (in the direction of the path from  $s$ ).
- Step 2. Let the actual socle parts determining the edges in this path be  $(K_i, L_i)$  for each edge  $(u_i, u_{i+1})$ . These can be identified easily in logspace. Now, suppose we have picked elements  $y_0, \dots, y_{i-1}$  of the SGS of  $N_s$  rel  $N_{s+1}$  for the first  $i$  edges of this path. Let the orbit corresponding to  $(K_i, L_i)$  be  $\Omega_j$  and let  $\{\delta\} = \Delta \cap \Omega_j$ . If  $\delta^x = \mu$  and  $\mu^{y_{i-1}} = \nu$ , then for the edge  $(u_i, u_{i+1})$  pick the lexicographically least element  $y_i$  from the SGS of  $N_s$  rel  $N_{s+1}$  such that  $\nu^{y_i} = \delta$ . This is clearly possible since  $L_i$  is transitive and it is not linked to any socle part associated to an edge at distance smaller than  $i + 1$ .
- Step 3. Output the elements  $y_i$  corresponding to the edges of  $G$  in increasing order of distance from their corresponding roots. This can also be done in  $\text{FL}^{\text{SL}} = \text{L}$ .

Define  $y$  as a product of the  $y_i$ 's computed in the order output by the above subroutine. It is easy to see by construction that  $x' = xy$  will fix each point in  $\cup_{i \in Y_2} (\Delta \cap \Omega_i)$ .

Now, we can define the group  $H''$  as the subgroup generated by these elements  $x'$  for each generator  $x$  of  $H'$ .

Likewise, let  $N''$  be the subgroup of  $N'$  obtained by computing  $z'$  for each generator  $z$  of  $N'$  using the above subroutine. It is easy to see that  $N''$  is the pointwise stabilizer of  $\cup_{i \in Y} (\Delta \cap \Omega_i)$ .

We define  $H_{s+1} = H''N''$ . The construction of  $H_{s+1}$  enforces that if  $K$  and  $K'$  are any two socle parts (vertices in  $V$ ) that are connected by a path in  $(V, \text{RED} \cup \text{BLUE})$  then in  $H_{s+1}$  the socle parts  $K$  and  $K'$  are diagonally linked. We first argue that in  $H_{s+1}$  for each  $i \in X$ ,  $|pr_i(H_{s+1})| \leq |G_i|/2$ . This is already true for each  $i \in Y$  by construction. Let  $i \in X \setminus Y$ .

If  $\text{Soc}(G_i)$  is of type (ii), then the corresponding socle part  $K$  is in a red clique. Furthermore, since the red clique has a representative  $L$  in  $Y$  on which  $H_{s+1}$  is cut down, due to the linking of  $K$  and  $L$ ,  $H_{s+1}$  is cut down in  $K$  (and hence  $G_i$ ) as well.

If  $\text{Soc}(G_i)$  is of type (iii), let  $\text{Soc}(G_i) = K_1 \times K_2$ . Now,  $pr_i(H_{s+1}) = \text{Diag}(K_1 \times K_2)$ , implying that  $H_{s+1}$  is cut down on this orbit as well.

Let  $\Sigma = \cup_{i \in Y} (\Delta \cap \Omega_i)$ . By construction we have  $H_s \leq H_{s+1}N_s$ . Now, inductively assuming that  $G_{\{\Delta\}} \leq H_s$ , notice that

$$G_{\{\Delta\}} \leq (H_s)_{\{\Sigma\}} \leq (H_{s+1}N_s)_{\{\Sigma\}} = H_{s+1}N'' = H_{s+1}.$$

This completes the proof for the case when the socle is nonabelian.

### When the socle is abelian

We now turn to the case when  $\text{Soc}(G_i)$  is abelian for each  $i \in X$ , where  $X$  is the set of orbit indices for which  $pr_i(N_s)$  is abelian. Assume that the group  $H_s$  is already computed.

**Finding  $Y$**  Suppose  $N_s/N_{s+1}$  is  $T$ -semisimple. Then  $T$  is  $\mathbb{Z}_p$  for some prime  $p$ . By the O’Nan Scott Theorem 4.10, for each  $i$   $\text{Soc}(G_i)$  is a subgroup of  $\mathbb{Z}_p^{c_i}$ , where the number of copies of  $\mathbb{Z}_p$  is a constant  $c_i$  (that depends on  $b$ ). It follows that  $N_s$  projected on the set of orbits  $\Omega_i$  for  $i \in X$  is isomorphic to a subgroup of  $\mathbb{Z}_p^r$  where  $r = \sum_{i \in X} c_i = O(m)$ . Thus,  $N_s$  projected on the set of orbits  $\Omega_i$  for  $i \in X$  is actually a subspace  $U$  of  $\mathbb{Z}_p^r$ . Here, in  $\mathbb{Z}_p^r$  we assume that the coordinates corresponding each orbit  $i \in X$  are adjacent and appear in the increasing order of  $i \in X$ . Furthermore, we can compute a basis for this projected subspace  $U$  from the generators of  $N_s$  in deterministic logspace. Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$  be the computed basis. In  $\text{FL}^{\text{Mod}_pL}$  we can compute a special basis for the vector space  $U$  as follows: for each index  $j$ ,  $1 \leq j \leq r$  form a column vector  $y^{(j)} = (0, \dots, 0, 1, y_{j+1}, \dots, y_r)^T$ , where the 1 occurs in the  $j$ th position, the first  $j-1$  positions have 0, and  $y_{j+1}, \dots, y_r$  are indeterminates. Let  $A$  denote the matrix with columns as  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$ . In  $\text{FL}^{\text{Mod}_pL}$  the feasibility of  $Ax = \mathbf{y}^{(j)}$  can be tested, and if feasible then a solution  $\mathbf{y}^{(j)}$  to  $y^{(j)}$  can be computed. These vectors  $\mathbf{y}^{(j)}$  clearly form a basis  $B$  for  $U$ .

Define the subset of orbit indices  $Y \subset X$  as follows: include index  $i$  in  $Y$  if for some index  $j$  corresponding to  $\Omega_i$ , there is a solution  $y^{(j)} \in B$ .

**Finding  $N$  and  $H$**  Now, projecting the vectors in  $B$  only on the orbits of  $Y$ , we again apply an  $\text{FL}^{\text{Mod}_pL}$  computation to obtain a new set of vectors  $B'$  such that  $B'$  spans  $U$  projected on the orbits of  $Y$ , and for each orbit index  $i \in Y$  there is a vector  $z_i \in B'$  such that  $z_i$  is nontrivial on all orbits of  $Y$  except  $\Omega_i$ . As mentioned before, notice here that the algorithm can keep track of the corresponding entire permutation in  $G$  while working with a vector at each stage. Thus, we also have a  $\pi_i \in N_s$  which coincides with  $z_i$  on the orbits  $Y$ .

Now, let  $x$  be a generator of  $H_s$ . By the above construction we can obtain an element  $y \in N_s$  as a product of the form  $\prod_{i \in Y} \pi_i^{k_i}$  such that  $x' = xy \in H_s N_s$  fixes  $\Delta \cap \Omega_i$  for each  $i \in Y$ .

Let  $H$  be the subgroup generated by the above elements  $x'$ .

Next, notice that the subgroup  $N \leq N_s$  consisting of elements that fix  $\cup_{i \in Y} \Delta \cap \Omega_i$  is trivial on all the orbits in  $Y$  (by O’Nan Scott Theorem). Furthermore, the definition of  $Y$  forces that  $N$  is trivial on all  $\Omega_i$ ,  $i \in X$ .

Hence, notice that  $N_{s+1} \triangleleft N \triangleleft N_s$ . Thus,  $N_s/N$  is also  $\mathbb{Z}_p$ -semisimple. To find a generating set for  $N$  we will apply the algorithm of sifting and normal closure from Theorem 5.3. To sift we proceed as follows: for each generator  $x$  of  $N_s$ , find an element  $y \in N_s$  as a product of the form  $\prod_{i \in Y} \pi_i^{k_i}$  such that  $x' = xy$  fixes  $\Delta \cap \Omega_i$  for each  $i \in Y$ . Let  $S'$  be this set of elements  $x'$ , which is the sift of  $x$  in  $N_s$  rel  $N$ . It remains to compute  $\text{NCL}_{N_s}(S')$ . This can be easily done in  $\text{FL}^{\text{Mod}_p \text{L}}$  by applying the algorithm described in the abelian case of Theorem 5.3 to the set  $S'$ .

Now, consider the group  $H_{s+1}$  defined as  $HN$ . Since for each  $i \in X$ ,  $\text{pr}_i(H \cap N_s)$  is trivial, by Lemma 4.9 it follows that  $|\text{pr}_i(H_{s+1})| \leq |G_i|/2$  for each orbit  $i \in X$ .

Notice that by construction of  $H_{s+1}$  we have  $H_s \leq H_{s+1}N_s$ . Let  $\Sigma = \cup_{i \in Y} (\Delta \cap \Omega_i)$ . Now, inductively assuming that  $G_{\{\Delta\}} \leq H_s$ , notice that

$$G_{\{\Delta\}} \leq (H_s)_{\{\Sigma\}} \leq (H_{s+1}N_s)_{\{\Sigma\}} = H_{s+1}N = HNN = HN = H_{s+1}.$$

This completes the proof of the abelian case, and hence the proof of Theorem 4.12.

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## A Appendix

**Theorem A.1.** *Given  $k, j \in \mathcal{N}$  there is a constant  $b$  such that  $\text{BCGI}_b$  is hard for the  $j$ th level of the  $\text{Mod}_k\text{L}$  under logspace many-one reductions.*

*Proof.* The proof is entirely based on ideas and the gadgets in [Tor04] (specifically [Tor04, Lemma 4.7]). However, we include an outline to make the present paper self-contained.

We will prove the result by induction on the level  $j$  of the hierarchy.

For  $j = 1$ , it is shown in [Tor04, Theorem 3.3] that Graph Isomorphism is hard for  $\text{Mod}_k\text{L}$  (even under  $\text{AC}^0$  many-one reductions), where the size of the color classes is bounded by  $k^2$ . I.e.  $\text{BCGI}_{k^2}$  is hard for  $\text{Mod}_k\text{L}$  for every  $k \in \mathcal{N}$ . In fact, a stronger reduction is shown: given  $L \in \text{Mod}_k\text{L}$  and an instance  $x$  of  $L$  the reduction produces two pairs of graphs  $(G, H)$  and  $(L, M)$  such that

$$\begin{aligned} x \in L & \text{ implies } G \cong H \text{ and } L \not\cong M \\ x \notin L & \text{ implies } G \not\cong H \text{ and } L \cong M \end{aligned}$$

We shall inductively show a similar stronger claim.

*Induction hypothesis:* Suppose  $\text{BCGI}_b$ , for  $b = k^{2(j-1)}$  is logspace many-one hard for the  $j - 1$ th level of the  $\text{Mod}_k\text{L}$  hierarchy, in the following sense: given a language  $A$  in the  $j - 1$ th level of the  $\text{Mod}_k\text{L}$  hierarchy, suppose there is a logspace reduction that on input  $x$  produces  $(G, H)$  and  $(L, M)$  such that

$$\begin{aligned} x \in A & \text{ implies } G \cong H \text{ and } L \not\cong M \\ x \notin A & \text{ implies } G \not\cong H \text{ and } L \cong M, \end{aligned}$$

where the graphs  $G, H, L$ , and  $M$  have color classes of size at most  $k^{2(j-1)}$ .

Now consider a language  $L$  in the  $j$ th level of the  $\text{Mod}_k\text{L}$  hierarchy. Then there is a language  $A$  in the  $j - 1$ th level of the  $\text{Mod}_k\text{L}$  hierarchy such that  $L \in \text{Mod}_k\text{L}^A$ . In other words there is a nondeterministic logspace oracle machine  $M$  that defines the following  $\#\text{L}^A$  function  $f$ :  $f(x)$  is the number of accepting paths of  $M^A$  on input  $x$ , for each  $x \in \Sigma^*$  such that

$$x \in L \text{ if and only if } f(x) \not\equiv 0 \pmod{k}.$$

Here we recall that the oracle access made by  $M$  is using the Russo-Simon-Tompa oracle access mechanism (c.f. [ABO99] for details on its relevance to logspace counting classes). In other words, the oracle queries are written *deterministically* on the query tape. Since  $M$  is logspace-bounded, there are only polynomially many configurations (and hence only polynomially many possible queries that  $M$  can make on an input  $x$ ).

Given an input  $x$  for  $M^A$ , a logspace machine can generate (from  $M$  and  $x$ ) the following configuration graph  $G$ : the nodes of the graph  $G$  are tuples of the kind  $(s, c, p_1, p_2, q)$ , where  $s$  is a state of  $M$ ,  $c$  denotes the possible contents of the worktape,  $p_1$  is the input head position and  $p_2$  the work-tape head position. Finally,  $q$ , which stands for query answer, takes values in the set  $\{\#, 0, 1\}$ . Here  $q$  takes the value  $\#$  if the state  $s$  is *not* an oracle query state. On the other hand if  $s$  is a query state, then the variable  $q$  takes the value 0 or 1, and the correct value of  $q$  is 1 if and only if the oracle query generated from the configuration  $(s, c, p_1, p_2)$  is in the set  $A$ .

Now, we describe the edges in the graph  $G$ . Consider a node  $(s, c, p_1, p_2, q)$  of  $G$ . The following cases arise

- For a node  $(s, c, p_1, p_2, \#)$  of  $G$ , if the configuration  $(s', c', p'_1, p'_2)$  can be reached by  $M$  from  $(s, c, p_1, p_2)$  in one step. Then if  $s'$  is also not a query state, we put a directed edge from  $(s, c, p_1, p_2, \#)$  to  $(s', c', p'_1, p'_2, \#)$ . If  $s'$  is a query state then we include a directed edge from  $(s, c, p_1, p_2, \#)$  to  $(s', c', p'_1, p'_2, 0)$  and another from  $(s, c, p_1, p_2, \#)$  to  $(s', c', p'_1, p'_2, 1)$ .
- Now consider a node  $(s, c, p_1, p_2, q)$  of  $G$ , where  $s$  is a query state (and  $q$  is either 0 or 1). Suppose the configuration  $(s', c', p'_1, p'_2)$  can be reached by  $M$  from  $(s, c, p_1, p_2)$  in one step *assuming that the correct query answer is  $q$* . If  $s'$  is not a query state, we put a directed edge from  $(s, c, p_1, p_2, q)$  to  $(s', c', p'_1, p'_2, \#)$ . If  $s'$  is a query state then we include a directed edge from  $(s, c, p_1, p_2, q)$  to  $(s', c', p'_1, p'_2, 0)$  and another from  $(s, c, p_1, p_2, q)$  to  $(s', c', p'_1, p'_2, 1)$ .

Notice that we can add a time-stamp to each node of the graph  $G$  and make it a directed acyclic graph.

In order to check if  $f(x) \pmod k$  is a specific value  $m$ , it amounts to counting the number of paths modulo  $k$  in the configuration graph from the start node to the accepting nodes.

This can be done in the configuration graph (actually by first making it into a DAG by putting the time-stamps as part of the nodes) and by converting this graph into a circuit with every node replaced by a mod  $k$  gate. These mod  $k$  gates can then be simulated using Graph Isomorphism as in [Tor04, Lemma 3.2, Theorem 3.3]. However, we must note that for query states  $s$  we have nodes of the kind  $(s, c, p_1, p_2, 0)$  and  $(s, c, p_1, p_2, 1)$ , exactly one of which is valid and the other is spurious depending on the correct oracle answer. Thus, a direct conversion to a circuit as described above is incorrect due to the spurious nodes.

At this point we can follow exactly what is done in the proof of [Tor04, Lemm 4.7]:

By induction hypothesis recall that for each oracle query  $Q_i$  that is made to  $A$  we can, in logspace, obtain two pairs of graphs  $(G_i, H_i)$  and  $(L_i, M_i)$  such that either  $G_i \cong H_i$  or  $L_i \cong M_i$  and  $Q_i \in A$  if and only if  $G_i \cong H_i$ . By induction hypothesis, these graphs  $G_i, H_i, L_i$ , and  $M_i$  have color class size bounded by  $b = 2^{2^{j-1}}$ .

We will insert the gadget  $Gad_k$  (see [Tor04, Lemma 4.8]) in every wire carrying the output of a gate of the form  $(s, c, p_1, p_2, q)$  into some other gate.  $Gad_k$  is a graph gadget made out of  $(G_i, H_i)$  and  $(L_i, M_i)$ , and ensures the following: the input contribution of the gate  $(s, c, p_1, p_2, q)$ , where  $s$  is a query state and  $q$  is an incorrect answer, is made 0 (mod  $k$ ). On the other hand, if  $q$  is the correct answer, then the output of gate  $(s, c, p_1, p_2, q)$  is propagated along the wire. We note here that the size of the color classes in  $Gad_k$  will increase by a factor of  $k^2$ , as it uses upto  $k^2$  copies of the input graphs.

Having made this modification, the construction of [Tor04, Lemma 3.2] will work. Thus, following [Tor04, Lemma 3.2], we can obtain two pairs of graphs  $(G, H)$  and  $(L, M)$  such that if  $x$  is accepted by  $M^A$  then  $G \cong H$  and  $L \not\cong M$ , and if  $x$  is rejected by  $M^A$  then  $G \not\cong H$  and  $L \cong M$ .

Since the color class size of  $Gad_k$  is at most  $k^{2^j}$ , the graphs  $G, H, L$ , and  $M$  have color class size bounded by  $k^{2^j}$ . This completes the proof sketch.  $\square$