Path Coupling Using Stopping Times and Counting Independent Sets and Colourings in Hypergraphs

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Abstract

We give a new method for analysing the mixing time of a Markov chain using path coupling with stopping times. We apply this approach to two hypergraph problems. We show that the Glauber dynamics for independent sets in a hypergraph mixes rapidly as long as the maximum degree Δ of a vertex and the minimum size m of an edge satisfy $m \geq 2\Delta + 1$. We also show that the Glauber dynamics for proper q-colourings of a hypergraph mixes rapidly if $m \geq 4$ and $q > \Delta$, and if $m = 3$ and $q \geq 1.65\Delta$. We give related results on the hardness of exact and approximate counting for both problems.

1 Introduction

We develop a new approach to using stopping times in conjunction with path coupling to bound the convergence of time of Markov chains. Our main interest is in applying these results to randomised approximate counting. For an introduction, see [20]. To illustrate our methods, we consider approximation of the numbers of independent sets and q-colourings in hypergraphs with upper-bounded degree, and lower-bounded edge size. These problems in hypergraphs are of interest in their own right but, while approximate optimisation has received attention [6, 7, 18, 21], there has been surprisingly little work on approximate counting.

Our results are achieved by considering, in the path coupling setting, the stopping time at which the distance between two coupled chains first changes. Theorem 2.1, the main technical result of the paper, shows that if the expected distance between the two chains has decreased at this stopping time, then the chain mixes rapidly. The first application of stopping times to path coupling was by Dyer, Goldberg, Greenhill, Jerrum and Mitzenmacher [10]. Their analysis was later improved by Hayes and Vigoda [17], using a method closely related to that developed this paper. Here we give a further improvement, yet with a simpler proof than that of [17]. Technically, our method does not assume a bounded stopping time, as does [17]. Instead, we couple over a fixed time unrelated to a single stopping time. Our Theorem 2.1 may be compared with [17, Corollary 3], the applicable result from [17]. Theorem 2.1 will usually give a tighter bound on mixing time than [17, Corollary 3]. See Remark 2.3 below.

The problem of approximately counting independent sets in graphs has been widely studied, see for example [9, 12, 22, 23, 27], but the only previous work on the approximate counting of independent

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sets in *hypergraphs* seems to that of Dyer and Greenhill [12]. They showed rapid mixing to the uniform distribution of a simple Markov chain on independent sets in a hypergraph with maximum degree 3 and maximum edge size 3. However, this was the only interesting case resolved. Their results imply rapid mixing only for $m \leq \Delta/(\Delta - 2)$, which gives $m \leq 3$ when $\Delta = 3$ and $m \leq 2$ when $\Delta \geq 4$. In Theorem 3.1 we prove rapid mixing of the *Glauber dynamics* for any hypergraph such that $m \geq 2\Delta + 1$, where m is the smallest edge size and Δ is the maximum degree. This is a marked improvement for large m. More generally, we consider the *hardcore distribution* on independent sets with *fugacity* λ. (See, for example, [12, 22, 27].) In [12], it is proved that rapid mixing occurs if $\lambda \leq m/((m-1)\Delta - m)$. Here we improve this considerably for larger values of m, to $\lambda \leq (m-1)/2\Delta$. We also give proofs that computing the number of independent sets in hypergraphs is #P-complete except in trivial cases, and that there can be no approximation for the number of independent sets in a hypergraphs if the minimum edge size is at most logarithmic in Δ. It may be noted that our upper and lower bounds are exponentially different. We have no strong belief that either is close to the threshold at which approximate counting is possible, if such a threshold exists.

Counting q-colourings of hypergraphs was considered by Bubley [2], who showed that the Glauber dynamics was rapidly mixing if $q \geq 2\Delta$, generalising a result of Jerrum [19] and Salas and Sokal [25] for graphs. Much work has been done on improving this result for graph colourings, see [8] and its references, but little attention appears to have been given to the hypergraph case. Here we prove rapid mixing of Glauber dynamics for proper colourings of hypergraphs if $m \geq 4$, $q > \Delta$, and if $m = 3$, $q \geq 1.65\Delta$. For a precise statement of our result see Theorem 5.3. Again we give proofs that computing the number of colourings in hypergraphs is #P-complete except in trivial cases, and that there can be no approximation for the number of colourings of hypergraphs if $q \leq (1 - 1/m)\Delta^{1/(m-1)}$. Again, there is a considerable discrepancy between the upper and lower bounds for large m.

The paper is organised as follows. Section 1.1 gives an intuitive motivation for the stopping time approach of the paper. Section 2 contains the full description and proof of Theorem 2.1 for path coupling with stopping times. We apply this to hypergraph independent sets in Section 3. Section 4 contains the hardness proofs. Section 5 contains analysis of the Glauber dynamics for hypergraph colouring. Finally, Section 6 contains the hardness results for counting colourings in hypergraphs.

1.1 Intuition

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph of maximum degree Δ and minimum edge size m. A subset $S \subseteq \mathcal{V}$ of the vertices is *independent* if no edge is a subset of S. Let $\Omega(\mathcal{H})$ be the set of all independent sets of $\mathcal{H}$. Let λ be the *fugacity*, which weights independent sets. (See [12].) The most important case is $\lambda = 1$, which weights all independent sets equally and gives rise to the uniform distribution on all independent sets. We define the Markov chain $\mathcal{M}(\mathcal{H})$ with state space $\Omega(\mathcal{H})$ by the following transition process (*Glauber dynamics*). If the state of $\mathcal{M}$ at time t is X_t, the state at $t+1$ is determined by the following procedure.

   (i) Select a vertex $v \in \mathcal{V}$ uniformly at random,

  (ii)   (a) if $v \in X_t$ let $X_{t+1} = X_t \backslash \{v\}$ with probability $1/(1+\lambda)$,

         (b) if $v \notin X_t$ and $X_t \cup \{v\}$ is independent, let $X_{t+1} = X_t \cup \{v\}$ with probability $\lambda/(1+\lambda)$,

         (c) otherwise let $X_{t+1} = X_t$.

This chain is easily shown to be ergodic with stationary probability proportional to $\lambda^{|I|}$ for each independent set $I \subseteq \mathcal{V}$. In particular, $\lambda = 1$ gives the uniform distribution. The natural coupling for this chain is the "identity" coupling, the same transition is attempted in both copies of the chain. If we try to apply standard path coupling to this chain, we immediately run into difficulties. Consider two chains X_t and Y_t such that $Y_t = X_t \cup \{w\}$, where $w \notin X_t$ (the *change vertex*) is of degree Δ. An edge $e \in \mathcal{E}$ is *critical* in Y_t if it has only one vertex $z \in \mathcal{V}$ which is not in Y_t, and we call z *critical for e*. If each of the edges through w is critical for Y_t, then there are Δ choices of v in the transition which can be added in X_t but not in Y_t. Thus, if $\lambda = 1$, the change in the expected Hamming distance between X_t and Y_t after one step could be as high as $\frac{\Delta}{2n} - \frac{1}{n}$. Thus we obtain rapid mixing only in the case $\Delta = 2$. This case has some intrinsic interest, since the complement of an independent set corresponds, under hypergraph duality, to an *edge cover* [15] in a graph. Thus we may uniformly generate edge covers, but the scope for unmodified path coupling is obviously severely limited.

The insight on which this paper is based is as follows. Although in one step it could be more likely that a *bad vertex* (increasing Hamming distance) is chosen than a *good vertex* (decreasing Hamming distance), it is even more likely that one of the other vertices in an edge containing w is chosen and removed from the independent set. Once the edge has two unoccupied vertices other than w, then any vertex in that edge can be added in both chains. This observation enables us to show that, if T is defined to be the stopping time at which the distance between X_t and Y_t first changes, the expected distance between X_T and Y_T will be less than 1. Theorem 2.1 below shows that under these circumstances path coupling can easily be adapted to prove rapid mixing.

Having established this general result, we use it to prove that $\mathcal{M}(\mathcal{H})$ is rapidly mixing for hypergraphs with $m \geq 2\lambda\Delta + 1$. Note that, though all the results in this paper will be proved for uniform hypergraphs of edge size m, they carry through trivially for hypergraphs of minimum edge size m.

2 Path coupling using a stopping time

First we prove the main result discussed above.

Theorem 2.1. *Let $\mathcal{M}$ be a Markov chain on state space Ω. Let (X_t, Y_t) be a path coupling for $\mathcal{M}$ and let d be an integer valued metric on $\Omega \times \Omega$. For any initial states X_0 and Y_0 such that $\mathrm{d}(X_0, Y_0) = 1$, let T be the stopping time given by the minimum t such that $\mathrm{d}(X_t, Y_t) \neq 1$. Suppose, for some $p > 0$, that*

(i) $\Pr(T = t \,|\, T \geq t) \geq p$, independently for each t,

(ii) $\mathbf{E}[\mathrm{d}(X_T, Y_T)] \leq \alpha < 1$.

Then $\mathcal{M}$ mixes rapidly. In particular the mixing time $\tau(\varepsilon)$ of $\mathcal{M}$ satisfies

$$\tau(\varepsilon) \;\leq\; \frac{1}{p} \frac{16}{1 - \alpha} \left(\ln \frac{2D_2}{1 - \alpha} \right)^2 \ln(D_1 \varepsilon^{-1})$$

where D_1 is the maximum of $\mathrm{d}(X, Y)$ over all $(X, Y) \in \Omega \times \Omega$, and D_2 is the maximum value possible for $\mathrm{d}(X_T, Y_T)$.

Proof. Consider the following game. In each round a gambler either wins £1, loses some amount £$(l - 1)$ or continues to the next round. If he loses £$(l - 1)$ in a game, he starts l separate (but

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possibly dependent) games simultaneously in an effort to win back his money. If he has several games going and loses one at a certain time, he starts l more games, while continuing with the others that did not conclude. We know that the probability he finishes a game in a given step is at least p, and the expected winnings in each game is at most $1 - \alpha$. The question is: does his return have positive expectation at any fixed time? We will show that it does. But first a justification for our interest in this game.

Each game represents a single step on the path between two states of the coupled Markov chain. We start with X_0 and Y_0 differing at a single vertex. The first game is won if the first time the distance between the coupled chains changes is in convergence. The game is lost if the distance increases to l. At that point we consider the distance l path X_t to Y_t, and the l games played represent the l steps in the path. Although these games are clearly dependent, they each satisfy the conditions given. The gambler's return at time t is one minus the length of the path at time t, so a positive expected return corresponds to an expected path length less than one.

First note that the gambler's return at time t is one minus the number of games active at time t. For the initial game we define the *level* to be zero, for any other possible game we define the level to be one greater than the level of the game whose loss precipitated it. We define the random variables M_k, l_{jk} and $I_{jk(t)}$ as follows. M_k is the number of games at level k that are played, l_{jk}, for $j = 1 \ldots M_k$, is the number of games in level $k + 1$ which are started as a result of the outcome of game j in level k, and $I_{jk}(t)$ is an indicator function which takes the value 1 if game j in level k is active at time t, and 0 otherwise. Let $N(t)$ be the number of games active at time t. Then, by linearity of expectations,

$$\mathbf{E}[N(t)] = \sum_{k=0}^{\infty} \mathbf{E}\left[\sum_{j=1}^{M_k} I_{jk}(t)\right]. \tag{1}$$

We will bound this sum in two parts, splitting it at a point $k = K$ to be determined. For $k \le K$ we observe that $M_k \le D_2^k$. Since $\Pr(I_{jk}(t) = 1)$ is at most the probability that exactly $k - 1$ games of a sequence are complete at time t, regardless of outcome, we have

$$\mathbf{E}\left[\sum_{j=1}^{M_k} I_{jk}(t)\right] \le D_2^k \max_j \mathbf{E}[I_{jk}(t)] \le D_2^k \Pr(\text{exactly } k - 1 \text{ games complete by time } t).$$

So that

$$\sum_{k=0}^{K} \mathbf{E}\left[\sum_{j=1}^{M_k} I_{jk}(t)\right] \le \sum_{k=0}^{K} D_2^k \Pr(\text{exactly } k - 1 \text{ games complete by time } t)$$

$$\le D_2^{K} \Pr(\text{at most } K \text{ games complete by } t). \tag{2}$$

On the other hand, for $k > K$ we observe that

$$\mathbf{E}\left[\sum_{j=1}^{M_k} I_{jk}(t)\right] \le \mathbf{E}[M_k] = \mathbf{E}_{M_{k-1}}\left[\mathbf{E}[M_k | M_{k-1}]\right] = \mathbf{E}_{M_{k-1}}\left[\mathbf{E}[\sum_{j=1}^{M_{k-1}} l_{jk-1} | M_{k-1}]\right]$$

Since $\mathbf{E}[l_{jk-1}] \le \alpha$ for any starting conditions, we may apply this bound even when conditioning on M_{k-1}. So

$$\mathbf{E}\left[\sum_{j=1}^{M_k} I_{jk}(t)\right] \le \mathbf{E}[\alpha M_{k-1}] \le \alpha^k, \tag{3}$$

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using linearity of expectation, induction and $\mathbf{E}[M_1] \leq \alpha$. Putting (2) and (3) together we get

$$\mathbf{E}[N(t)] \leq \; D_2{}^K \Pr(\text{at most } K \text{ games complete by } t) \; + \; \sum_{k=K+1}^{\infty} \alpha^k$$

$$= \; D_2{}^K \Pr(\text{at most } K \text{ games complete by } t) \; + \; \frac{\alpha^{K+1}}{1-\alpha}. \tag{4}$$

We now set $K = \lfloor (\ln\alpha)^{-1} \ln(\frac{1-\alpha}{2}) \rfloor$, hence the final term is at most $\frac{1}{2}$. The probability that a game completes in any given step is at least p. If we select the time $\tau = c(1-p)/p$ then for $c \geq K+1 \geq 1$, the probability that at most K games are complete is clearly maximised by taking this probability to be exactly p in all games. Hence,

$$\mathbf{E}[N(\tau)] \leq \; D_2{}^K \sum_{k=0}^{K} \binom{\tau}{k} p^k (1-p)^{\tau-k} + \tfrac{1}{2}$$

$$\leq \; D_2{}^K \sum_{k=0}^{K} \frac{e^{-c} c^k}{k!} + \tfrac{1}{2}$$

$$\leq \; \frac{D_2{}^K c^{K+1} e^{-c}}{(c-K)K!} + \tfrac{1}{2}$$

$$\leq \; D_2{}^K c^{K+1} e^{-c} + \tfrac{1}{2} \; = \; \beta.$$

In order that $\beta \leq \frac{3}{4}$, we choose c such that $D_2{}^K c^{K+1} e^{-c} < \frac{1}{4}$, which follows from

$$\frac{c}{\ln(2D_2 c)} > K + 1 = \left\lfloor (\ln\alpha)^{-1} \ln\left(\tfrac{1-\alpha}{2}\right) \right\rfloor + 1.$$

This is true for $c = \frac{4}{1-\alpha}\left(\ln\frac{2D_2}{1-\alpha}\right)^2$. We conclude that the gambler's expected return at time τ is $1 - \mathbf{E}[N(\tau)] \geq 1 - \beta \geq \frac{1}{4}$. More importantly, we can now apply path coupling to the Markov chain $\mathcal{M}^\tau$ obtained by taking τ steps of the chain $\mathcal{M}$ at a time. The coupling within each interval of τ steps is the global coupling implied by path coupling. By the usual analysis, we obtain the mixing time claimed. $\qquad\square$

Remark 2.2. The assumption that the stopping time occurs when the distance changes is not essential. We clearly cannot dispense with assumption (ii), or we cannot bound mixing time. Assumption (i) may appear a restriction, but appears to be naturally satisfied in most applications. It seems more natural than the assumption of bounded stopping time, used in [17].

Remark 2.3. If $\gamma = 1/(1-\alpha)$, our mixing time bound can be written $O(p^{-1}\log(D_1/\varepsilon)\gamma \log^2(\gamma D_2))$. Note that condition (i) of Theorem 2.1 implies $\mathbf{E}[T] \leq 1/p$. We may compare this with the bound from [17, Corollary 3], which may be written in similar form as $O(\mathbf{E}[T]\log(D_1/\varepsilon)\gamma^2 D_2)$. Since it would usually be necessary to estimate $\mathbf{E}[T]$ by $1/p$, our result compares favourably with that of [17]. In the applications below, for example, we have $D_2 = \Theta(1)$ and $\gamma = \Omega(\Delta)$, so the improvement is $\Omega(\Delta/\log^2\Delta)$.

3 Hypergraph independent sets

We now use the approach of path coupling via stopping times to prove that the chain discussed in Section 1.1 is rapidly mixing. The metric used in path coupling analyses throughout the paper will be Hamming distance between the coupled chains. We prove the following theorem.

Theorem 3.1. *Let λ, Δ be fixed, and let $\mathcal{H}$ be a hypergraph such that $m \geq 2\lambda\Delta + 1$. Then the Markov chain $\mathcal{M}(\mathcal{H})$ has mixing time $O(n \log n)$.*

Before commencing the proof itself, we analyse the stopping time T for this problem.

3.1 Edge Process

Let X_t and Y_t be copies of $\mathcal{M}$ which we wish to couple, with $Y_0 = X_0 \cup \{w\}$. Let e be any edge containing w, with $m = |e|$. We consider only the times at which some vertex in e is chosen. The progress of the coupling on e can then be modelled by the following "game". We will call the number of unoccupied vertices in e (excluding w) *units*. At a typical step of the game we have k units, and we either win the game, win a unit, keep the same state or lose a unit. These events happen with the following probabilities: we win the game with probability $1/m$, win a unit with probability at least $(m - k - 1)/(1 + \lambda)m$, lose a unit with probability at most $\lambda k / (1 + \lambda)m$ and stay in the same state otherwise. If ever $k = 0$, we are bankrupt and we lose the game. Winning the game models the "good event" that the vertex v is chosen and the two chains couple. Losing the game models the "bad event" that the coupling increases the distance to 2. We wish to know the probability that the game ends in bankruptcy. We are most interested in the case where $k = 1$ initially, which models e being critical. Note that the value of k in the process on hypergraph independent sets dominates the value in our model, since we can always delete (win in the game), but we may not be able to insert (lose in the game) because the chosen vertex is critical in some other edge.

Let p_k denote the probability that a game is lost, given that we start with k units. We have the following system of simultaneous equations.

$$
\begin{aligned}
(m - 1 + 2\lambda)p_1 - (m - 2)p_2 &= \lambda \\
-k\lambda p_{k-1} + (m - k + (k+1)\lambda)p_k - (m - k - 1)p_{k+1} &= 0 \qquad (k = 2, 3, \ldots, m-1) \qquad (5)
\end{aligned}
$$

Adding the equations in (5) from the kth onwards gives

$$
\begin{aligned}
(m - 1)p_1 + m\lambda p_{m-1} &= \lambda \\
-k\lambda p_{k-1} + (m - k)p_k + m\lambda p_{m-1} &= 0 \qquad (k = 2, 3, \ldots, m-1). \qquad (6)
\end{aligned}
$$

Now (6) is equivalent to (5), since we have simply multiplied the coefficient matrix of (5) by an upper triangular matrix with all entries 1. This transformation is clearly nonsingular. We will show by induction that (6) has solution

$$
p_k = \frac{\lambda^k - \sum_{i=1}^{k} \binom{m}{i} p_{m-1} \lambda^{k-i+1}}{\binom{m-1}{k}} \qquad (k = 1, 2, \ldots, m-1). \qquad (7)
$$

When $k = 1$, the first equation in (6) is clearly satisfied by (7). Assume by induction that (7) is true for p_{k-1}, with $k \geq 2$. Then

$$\begin{aligned} p_k &= \frac{\lambda k}{m-k} p_{k-1} - \frac{\lambda m}{m-k} p_{m-1} \\ &= \frac{\lambda k}{m-k} \frac{\lambda^{k-1} - \sum_{i=1}^{k-1} \binom{m}{i} \lambda^{k-i} p_{m-1}}{\binom{m-1}{k-1}} - \frac{\lambda m}{m-k} p_{m-1} \\ &= \frac{\lambda^k - \sum_{i=1}^{k-1} \binom{m}{i} \lambda^{k-i+1} p_{m-1}}{\binom{m-1}{k}} - \frac{\binom{m}{k}}{\binom{m-1}{k}} \lambda p_{m-1} \\ &= \frac{\lambda^k - \sum_{i=1}^{k} \binom{m}{i} \lambda^{k-i+1} p_{m-1}}{\binom{m-1}{k}}, \end{aligned}$$

continuing the induction. For consistency, we must clearly have

$$p_{m-1} = \frac{\lambda^{m-1} - \sum_{i=1}^{m-1} \binom{m}{i} \lambda^{m-i} p_{m-1}}{\binom{m-1}{m-1}} = \lambda^{m-1} - \left((1+\lambda)^m - 1 - \lambda^m \right) p_{m-1},$$

$$\text{i.e.} \quad p_{m-1} = \frac{\lambda^{m-1}}{(1+\lambda)^m - \lambda^m}. \tag{8}$$

Using (8), (7) can be rewritten

$$p_k = \frac{1}{\binom{m-1}{k}} \left(\lambda^k - \frac{\sum_{i=1}^{k} \binom{m}{i} \lambda^{m+k-i}}{(1+\lambda)^m - \lambda^m} \right) = \frac{\sum_{i=k+1}^{m} \binom{m}{i} \lambda^{m+k-i}}{\left((1+\lambda)^m - \lambda^m \right) \binom{m-1}{k}} \qquad (k = 1, 2, \ldots, m-1). \tag{9}$$

In particular

$$p_1 = \frac{\lambda}{m-1} \left(1 - \frac{m \lambda^{m-1}}{(1+\lambda)^m - \lambda^m} \right). \tag{10}$$

3.2   The expected distance between X_T and Y_T

The stopping time for the pair of chains X_t and Y_t will be when the distance between them changes, in other words either a good or bad event occurs. The probability that we observe the bad event on a particular edge e with $w \in e$ is at most p_k as calculated above. Let ξ_t denote the number of empty vertices in e at time t when the process is started with $\xi_0 = k$. Now ξ_t can never reach 0 without first reaching $k-1$ and, since the process is Markovian, it follows that

$$p_k = \Pr(\exists t \, \xi_t = 0 | \xi_0 = k) = \Pr(\exists t \, \xi_t = 0 | \xi_s = k-1) \Pr(\exists s \, \xi_s = k-1 | \xi_0 = k) < p_{k-1}.$$

Since w is in at most Δ edges, the probability that we observe the bad event on any edge is at most Δp_1. The probability that the stopping time ends with the good event is therefore at least $1 - \Delta p_1$. The path coupling calculation is then

$$\mathbf{E}[\mathrm{d}(X_T, Y_T)] \le 2\Delta p_1.$$

This is required to be less than 1 in ordered to apply Theorem 2.1. If $m \ge 2\lambda\Delta + 1$, then by (10)

$$2\Delta p_1 = 1 - \frac{(2\lambda\Delta + 1)\lambda^{2\lambda\Delta}}{(1+\lambda)^{2\lambda\Delta+1} - \lambda^{2\lambda\Delta+1}}.$$

Proof of Theorem 3.1. The above work puts us in a position to apply Theorem 2.1. Let $m \geq 2\lambda\Delta + 1$. Then for $\mathcal{M}(\mathcal{H})$ we have

(i) $\Pr(\mathrm{d}(X_t, Y_t) \neq 1 | \mathrm{d}(X_{t-1}, Y_{t-1}) = 1) \geq \frac{1}{n}$ for all t, and

(ii) $\mathbf{E}[\mathrm{d}(X_T, Y_T)] < 1 - \dfrac{(2\lambda\Delta + 1)\lambda^{2\lambda\Delta}}{(1+\lambda)^{2\lambda\Delta+1} - \lambda^{2\lambda\Delta+1}}.$

Also for $\mathcal{M}(\mathcal{H})$ we have $D_1 = n$ and $D_2 = 2$. Hence by Theorem 2.1, $\mathcal{M}(\mathcal{H})$ mixes in time

$$\tau(\varepsilon) \leq 16n\frac{(1+\lambda)^{2\lambda\Delta+1} - \lambda^{2\lambda\Delta+1}}{(2\lambda\Delta+1)\lambda^{2\lambda\Delta}}\left(\ln \frac{4\big((1+\lambda)^{2\lambda\Delta+1} - \lambda^{2\lambda\Delta+1}\big)}{(2\lambda\Delta+1)\lambda^{2\lambda\Delta}}\right)^2 \ln(n\varepsilon^{-1}).$$

This is $O(n \log n)$ for fixed λ, Δ. $\qquad\square$

Remark 3.2. In the most important case, $\lambda = 1$, we require $m \geq 2\Delta + 1$. This does not include the case $m = 3$, $\Delta = 3$ considered in [12]. We have attempted to improve the bound by employing the chain proposed by Dyer and Greenhill in [12, Section 4]. However, this gives only a marginal improvement. For large $\lambda\Delta$, we obtain convergence for $m \geq 2\lambda\Delta + \frac{1}{2} + o(1)$. For $\lambda = 1$, this gives a better bound on mixing time for $m = 2\Delta + 1$, with dependence on Δ similar to Remark 3.3 below, but does not even achieve mixing for $m = 2\Delta$. We omit the details in order to deal with the Glauber dynamics, and to simplify the analysis.

Remark 3.3. The terms in the running time which are exponential in λ, Δ would disappear if we instead took graphs for which $m \geq 2\lambda\Delta + 2$. In this case the running time would be

$$\tau(\varepsilon) \leq 16(2\lambda\Delta + 1)(\ln 4(2\lambda\Delta + 1))^2 n \ln(n\varepsilon^{-1}).$$

Furthermore, if we took graphs such that $m > (2 + \delta)\lambda\Delta$, for some $\delta > 0$, then the running time would no longer depend on λ, Δ at all, but would be $\tau(\varepsilon) \leq c_\delta n \ln(n\varepsilon^{-1})$ for some constant c_δ.

Remark 3.4. It seems that path coupling cannot show anything better than m linear in $\lambda\Delta$. Suppose the initial configuration has edges $\{w, v_1, \ldots, v_{m-2}, x_i\}$ for $i = 1, \ldots, \Delta$, with $w, v_1, \ldots, v_{m-2} \in X_0$, $x_1, \ldots, x_\Delta \notin X_0$ and w the change vertex. Consider the first step where any vertex changes state. Let $\mu = (1+\lambda)(m-1+\Delta)$. The good event occurs with probability $(1+\lambda)/\mu$, insertion of a critical vertex with probability $\lambda\Delta/\mu$, and deletion of a non-critical vertex with probability $(m-1)/\mu$. We therefore need $(m-1) + (1+\lambda) \geq \lambda\Delta$, i.e. $m \geq \lambda(\Delta - 1)$, to show convergence by path coupling.

Remark 3.5. It seems we could improve our bound $m \geq 2\lambda\Delta + 1$ for rapid mixing of the Glauber dynamics somewhat if we could analyse the process on all edges simultaneously. Examination of the extreme cases, where all edges adjacent to w are otherwise independent, or where they are dependent except for one vertex (as in Remark 3.4), suggests that improvement to $(1 + o(1))\lambda\Delta$ may be possible, where the $o(1)$ is relative to $\lambda\Delta$. However, the analysis in the general case seems difficult, since edges can intersect arbitrarily.

4 Hardness results for independent sets

We have established that the number of independent sets of a hypergraph can be approximated efficiently using the Markov Chain Monte Carlo technique for hypergraphs with edge size linear in Δ. We show next that exact counting is unlikely to be possible, and that our approximation scheme cannot be extended to cover all hypergraphs with edge size $\Omega(\log \Delta)$.

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4.1 #P-completeness

We show that the exact counting problem is #P-Complete except in trivial cases.

Theorem 4.1. *Let $\mathcal{G}(m, \Delta)$ be the class of uniform hypergraphs with minimum edge size m and maximum degree Δ. Computing the number of independent sets of hypergraphs in $\mathcal{G}(m, \Delta)$ is #P-complete if $\Delta \geq 2$. If $\Delta \leq 1$, it is in P.*

Proof. Since m is the minimum edge size, we will assume $m \geq 3$. The cases $\Delta = 0, 1$ are trivially in P. As discussed in Section 1.1, independent sets in a hypergraph with $\Delta = 2$ correspond to edge covers in a graph. Counting these is #P-complete, even for graphs with arbitrarily large minimum degree. This is stated in [3] but without proof, so we provide a proof in Appendix A. We now consider $\Delta \geq 3$. (The case $m = \Delta = 3$ is discussed in [12].) Take a graph $G = (V, E)$, and construct a hypergraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ by "extending" each edge $e = \{v_1, v_2\} \in E$ to an edge $e^+ = \{v_1, u_1^e, \ldots, u_{m-2}^e, v_2\} \in \mathcal{E}$. Observe that, for each independent set I of G and edge $e \in E$, there are $2^{m-2} - 1$ independent assignments to $u_1^e, \ldots, u_{m-2}^e$ if $v_1, v_2 \in I$ and 2^{m-2} otherwise. This is equivalent to evaluating the partition function of a *weighted H-colouring* problem [5, 13] on G, with weight matrix

$$A = \begin{bmatrix} 2^{m-2} & 2^{m-2} \\ 2^{m-2} & 2^{m-2} - 1 \end{bmatrix}.$$

The #P-completeness of H-colouring with this weight matrix follows either directly from [5] or indirectly from [13, Corollary 3.2]. The degree bound $\Delta = 3$ follows from [13, Theorem 5.1], on noting that A is nonsingular. $\square$

4.2 Approximation hardness

We now show that unless NP=RP, there can be no *fpras* for the number of independent sets of all hypergraphs with edge size $\Omega(\log \Delta)$.

Let $G = (V, E)$, with $|V| = n$, be a graph with maximum degree Δ and N_i independent sets of size i ($i = 0, 2, \ldots, n$). For $\lambda > 0$ let $Z_G(\lambda) = \sum_{i=0}^n N_i \lambda^i$ define the *hard core partition function*. The following is a combination of results in Luby and Vigoda [22] and Berman and Karpinski [1].

Theorem 4.2. *If $\lambda > 694/\Delta$, there is no* fpras *for $Z_G(\lambda)$ unless NP=RP.*

Proof. Let ε be a constant such that the size of the largest independent set in a graph of maximum degree 4 cannot be approximated to within a ratio $(1 + \varepsilon)$ unless P = NP. Berman and Karpinski [1] show that $\varepsilon \geq 1/49$. Luby and Vigoda [22, Theorem 4] prove the hardness of approximating $Z_G(\lambda)$ if $\lambda > c/\Delta$ for any $c > 20 \ln 2 \, (1 + \varepsilon)/\varepsilon$.[1] Together, these two results give the theorem. $\square$

We note that Theorem 4.2 could probably be strengthened using the approach of [9]. However, this has yet to be done.

Theorem 4.3. *Unless NP=RP, there is no* fpras *for counting independent sets in hypergraphs with maximum degree Δ and minimum edge size $m < 2 \lg(1 + \Delta/694) - 1 = \Omega(\log \Delta)$.*

---

[1] The expression in [22] omits the $\ln 2$ term

Proof. Given a graph $G = (V, E)$ with maximum degree Δ, we construct a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ as follows. Let $k = \lceil m/2 \rceil$. For each $v \in V$, let $W_v = \{w_{v1}, w_{v2}, \ldots, w_{vk}\}$ and $\mathcal{V} = \bigcup_{v \in V} W_v$. For each edge $e = \{u, v\} \in E$, let $S_e = W_u \cup W_v$, and let $\mathcal{E} = \{S_e : e \in E\}$. It is clear that $\mathcal{H}$ has maximum vertex degree Δ and every edge has size $2k \geq m$.

An independent set $\mathcal{I}$ in $\mathcal{H}$ corresponds to a unique independent set I in G as follows. If $S_v \subseteq \mathcal{I}$, then $v \in I$, otherwise $v \notin I$. Clearly $\mathcal{I}$ independent in $\mathcal{H}$ implies I independent in G. Note that for each $v \notin I$, there are $(2^k - 1)$ possible subsets of S_v which may be in $\mathcal{I}$. Thus, if $\mathcal{N}$ is the number of independent sets in $\mathcal{H}$,

$$\mathcal{N} = \sum_{i=0}^{n} N_i (2^k - 1)^{n-i} = (2^k - 1)^n \sum_{i=0}^{n} N_i (2^k - 1)^{-i} = (2^k - 1)^n Z_G(1/(2^k - 1)).$$

Thus approximating $\mathcal{N}$ is equivalent to approximating $Z_G(\lambda)$ with $\lambda = 1/(2^k - 1)$. But, by Theorem 4.2, this will be hard if $1/(2^k - 1) > 694/\Delta$. This gives $k < \lg(1 + \Delta/694)$, which holds whenever $m < 2\lg(1 + \Delta/694) - 1$. $\qquad\square$

5   Hypergraph colouring

We now consider Glauber dynamics on the set of proper colourings of a hypergraph. Again our hypergraph $\mathcal{H}$ will have maximum degree Δ, minimum edge size m, and we will have a set of q colours. A colouring of the vertices of $\mathcal{H}$ is proper if no edge is monochromatic. Let $\Omega'(\mathcal{H})$ be the set of all proper q-colourings of $\mathcal{H}$. We define the Markov chain $\mathcal{C}(\mathcal{H})$ with state space $\Omega'(\mathcal{H})$ by the following transition process. If the state of $\mathcal{C}$ at time t is X_t, the state at $t+1$ is determined by

  (i) selecting a vertex $v \in \mathcal{V}$ and a colour $k \in \{1, 2, \ldots, q\}$ uniformly at random,

  (ii) let X_t' be the colouring obtained by recolouring v colour k

  (iii) if X_t' is a proper colouring let $X_{t+1} = X_t'$
         otherwise let $X_{t+1} = X_t$.

This chain is easily shown to be ergodic with the uniform stationary distribution. Again we will use Theorem 2.1 to prove rapid mixing of this chain under certain conditions, however first we will examine the chain using standard path coupling techniques.

Theorem 5.1. *For $m \geq 4$, $q > \Delta$, the Markov chain $\mathcal{C}(\mathcal{H})$ mixes in time $O(n \log n)$.*

Proof. Suppose that two copies of $\mathcal{C}(\mathcal{H})$, X_0 and Y_0 say, start at distance one apart, i.e. they differ in only one vertex w. Suppose that the number of colours available for recolouring w is $q - k$, then the probability of the two copies of the chain coupling in one step is $\frac{q-k}{nq}$. The distance between the two chains can only increase (to 2) if we select a vertex v and recolour it with a colour that is permitted in one copy of the chain only. For this to happen, there must be an edge containing v and w such that the other vertices in this edge are all either red and we have chosen red for v, or blue and we have chosen blue for v. Hence there can be at most one vertex on each edge, and one colour for that vertex, such that the chains diverge if we select that vertex and colour. Furthermore, for each of the k unavailable colours there must be an edge containing w which, apart from w itself, is monochromatic in the forbidden colour, so on these edges there are no vertices whose selection

can cause the chains to diverge. Hence the probability that the distance increases to 2 in one step is at most $\frac{\Delta-k}{nq}$. The path coupling calculation is therefore

$$\mathbf{E}[\mathrm{d}(X_1,Y_1)] \leq 1 - \frac{q-k}{nq} + \frac{\Delta-k}{nq}.$$

If $q \geq \Delta + 1$ then $\mathbf{E}[\mathrm{d}(X_1,Y_1)] \leq 1 - 1/nq$, and therefore by the path coupling theorem the mixing time is

$$\tau(\varepsilon) \leq nq\ln(n\varepsilon^{-1}). \qquad \square$$

This analysis leaves little room for improvement in the case $m \geq 4$, indeed it is not clear whether the Markov chain described is even ergodic for $q \leq \Delta$. The following simple construction does show that the chain is not in general ergodic if $q \leq \frac{\Delta}{m} + 1$. Let $q = \frac{\Delta}{m} + 1$, and take a hypergraph $\mathcal{H}$ on $q(m-1)$ vertices. We will group the vertices into q groups $\mathcal{V} = \mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_q$, each of size $m - 1$. Then the edge set of $\mathcal{H}$ is $E = \{\{v\} \cup \mathcal{V}_j : v \in \mathcal{V}, v \notin \mathcal{V}_j\}$. The degree of each vertex is $(q-1)+(q-1)(m-1) = \Delta$. If we now colour each group $\mathcal{V}_j$ a different colour, we obtain $q!$ distinct colourings, but for each of these the Markov chain is frozen (no transition is valid).

The case $m = 2$ is graph colouring and has been extensively studied. See, for example, [8]. This leaves the case $m = 3$, hypergraphs with 3 vertices in each edge. The standard path coupling argument, as in Theorem 5.1, only shows rapid mixing for $q \geq 2\Delta$, since there may be two vertices in each edge that can be selected and lead to a divergence of the two chains. This occurs if, of the two vertices in an edge which are not w, one is coloured red and the other blue. However, we can do better using Theorem 2.1. We will need the following technical Lemma.

Lemma 5.2. *Let* $\varphi(d) = 1 - d(1 - e^{-(q-\Delta+d)t/Mq})/(q - \Delta + d)$. *For all* $t \geq 0$ *and all* $d \geq 1$, $\varphi(d) \geq \varphi(1)^d$.

Proof. Let $\kappa = q - \Delta > 0$, $x = t/Mq \geq 0$. We wish to show that

$$\psi(x) = \varphi(d) - \varphi(1)^d = 1 - d(1 - e^{-(\kappa+d)x})/(\kappa + d) - \left(1 - (1 - e^{-(\kappa+1)x})/(\kappa + 1)\right)^d \geq 0.$$

Since $\psi(0) = 0$, it suffices to show that $\psi(x)$ is increasing for all $x \geq 0$. But

$$\psi'(x) = -de^{-(\kappa+d)x} + de^{-(\kappa+1)x}\left(1 - (1 - e^{-(\kappa+1)x})/(\kappa + 1)\right)^{d-1}$$

$$= de^{-(\kappa+1)x}\left(\left(1 - (1 - e^{-(\kappa+1)x})/(\kappa + 1)\right)^{d-1} - e^{-(d-1)x}\right),$$

so it suffices to show $1 - (1 - e^{-(\kappa+1)x})/(\kappa + 1) \geq e^{-x}$. Let $\zeta(x) = 1 - e^{-x} - (1 - e^{-(\kappa+1)x})/(\kappa + 1)$. Then $\zeta(0) = 0$, so we need only show that $\zeta(x)$ is increasing for all $x \in (0, \infty)$. But

$$\zeta'(x) = e^{-x} - e^{-(\kappa+1)x} = e^{-x}(1 - e^{-\kappa x}) \geq 0,$$

for $x \geq 0$. $\qquad \square$

Theorem 5.3. *There exists* Δ_0 *such that, if* $\mathcal{H}$ *is a 3-uniform hypergraph with maximum degree* $\Delta > \Delta_0$ *and* $q \geq 1.65\Delta$, *the Markov chain* $\mathcal{C}(\mathcal{H})$ *mixes rapidly.*

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Proof. We choose Δ_0 large enough that all the approximations below are valid. We couple two copies of this chain using the identity coupling. Let X and Y be two copies of $\mathcal{C}(\mathcal{H})$ such that X_0 and Y_0 differ only at a single vertex w. As before, we will examine the stopping time T at which $\mathrm{d}(X_T, Y_T) \neq 1$ for the first time, and show that $\mathbf{E}[\mathrm{d}(X_T, Y_T)] < 1$. We assume that w is coloured blue in X_0 and red in Y_0. We will call any other colour *neutral*. Let $\Gamma(w)$ denote the set of vertices of $\mathcal{H}$ that share an edge with w. We will only consider transitions in which either w or a vertex in $\Gamma(w)$ is selected, since any transition which involves any other vertex will not change the distance between X and Y. Let $M = |\Gamma(w)| + 1$. We will first assume that none of the edges containing w is otherwise monochromatic, and hence that all colours are available for recolouring w. We will deal with other cases later. Let S_t denote the event that $T = t$ and $\mathrm{d}(X_T, Y_T) = 0$, which we will call *success*. The bad event we will call *failure*.

The probability that the two chains couple in any one step is $\frac{q}{Mq}$. For each $v \in \Gamma(w)$, let $\beta_{v,t}$ be an indicator variable which takes value 1 if v is either red or blue after t steps of the chain, and takes value 0 otherwise. We describe a choice of vertex $v \in \Gamma(w)$ and colour $c \in \{\text{red,blue}\}$ at step t as 'bad' if there is an edge containing v and w whose other vertex is currently coloured c, and let B_t denote the number of bad choices at time t. The probability of failure in step t is therefore $\frac{B_t}{Mq}$. For each $v \in \Gamma(w)$ let d_v be the number of edges which contain both v and w. Then $B_t \leq \sum_{v \in \Gamma(w)} d_v \beta_{v,t}$. Now, using $\approx$ to imply equality up to a factor $1 + o_\Delta(1)$,

$$
\begin{aligned}
\Pr(S_t) = \; & \mathbf{E}\left[\prod_{j=0}^{t-1} \left(1 - \frac{1}{M} - \frac{B_t}{Mq} \right) \frac{1}{M} \right] \quad \approx \quad \frac{1}{M} \mathbf{E}\left[e^{-\sum_{j=0}^{t-1} (\frac{1}{M} + \frac{B_t}{Mq})} \right] \\
\geq \; & \frac{1}{M} \mathbf{E}\left[e^{-\sum_{j=0}^{t-1} (\frac{1}{M} + \frac{\sum_{w \in \Gamma(v)} d_w \beta_{w,t}}{Mq})} \right] \quad = \quad \frac{e^{-t/M}}{M} \mathbf{E}\left[\prod_{w \in \Gamma(v)} e^{-\frac{d_w}{Mq} \sum_{j=0}^{t-1} \beta_{w,t}} \right]. \quad (11)
\end{aligned}
$$

We will now study the properties of $\beta_{v,t}$, with a view to analysing $\mathbf{E}[e^{-\frac{d_v}{Mq} \sum_{j=0}^{t-1} \beta_{v,t}}]$. Note that the probability has not coupled or diverged by time 40Δ is at most

$$
(1 - 1/M)^{20\Delta} \leq (1 - 1/2\Delta)^{40\Delta} \leq e^{-20} < 10^{-8},
$$

so we consider times only up to 40Δ. Let $t_v = \max\{t : t < 40\Delta \text{ and } \beta_{v,t} = 1\}$. If v starts out either red or blue, the probability that it is recoloured to a neutral colour in each step is at least $(q - \Delta - 2)/Mq$. Also, the probability that it becomes red or blue before time 40Δ is at most $80\Delta/Mq$. Hence

$$
\begin{aligned}
\Pr(t_w > t) \leq \; & \left(1 - \frac{q - \Delta - 2}{Mq} \right)^t + \frac{80\Delta}{Mq} \\
\approx \; & e^{-\frac{q-\Delta}{Mq} t},
\end{aligned}
$$

since the second term is $O(1/\Delta)$ and small compared to the first, which is $\Omega(1)$ for $t \leq 40\Delta$. Now we can bound $\sum_{j=0}^{t-1} \beta_{v,t}$ by the minimum of t and t_v, an exponentially distributed random variable

with parameter $\frac{q-\Delta}{Mq}$. We are in a position to bound $\mathbf{E}[e^{-\frac{d_v}{Mq}\sum_{j=0}^{t-1}\beta_{v,t}}]$ as follows.

$$\mathbf{E}[e^{-\frac{d_v}{Mq}\sum_{j=0}^{t-1}\beta_{v,t}}] \geq \sum_{j=0}^{t}\Pr(t_v=j)e^{-\frac{d_v}{Mq}j}+\Pr(t_v>t)e^{-\frac{d_v}{Mq}t}$$

$$\approx \int_0^t \frac{q-\Delta}{Mq}e^{-\frac{q-\Delta}{Mq}x}e^{-\frac{d_v}{Mq}x}dx+e^{-\frac{q-\Delta+d_v}{Mq}t}$$

$$= 1-\frac{d_v}{q-\Delta+d_v}\left(1-e^{-\frac{q-\Delta+d_v}{Mq}t}\right).$$

Inserting this into Equation (11), we get

$$\mathbf{E}[\Pr(S_t)] \geq \frac{e^{-t/M}}{M}\prod_{v\in\Gamma(w)}\left(1-\frac{d_v}{q-\Delta+d_v}\left(1-e^{-\frac{q-\Delta+d_v}{Mq}t}\right)\right) = \frac{e^{-t/M}}{M}\prod_{v\in\Gamma(w)}\varphi(d_v),$$

where $\varphi(d)$ was defined in Lemma 5.2. Since $\sum_{v\in\Gamma(w)}d_v=2\Delta$, Lemma 5.2 implies that for all $t\geq 0$,

$$\prod_{v\in\Gamma(w)}\varphi(d_v)\geq\varphi(1)^{2\Delta}.$$

Hence, for $t\leq 40\Delta$,

$$\mathbf{E}[\Pr(S_t)] \geq \frac{e^{-t/M}}{M}\left(1-\frac{1}{q-\Delta+1}\left(1-e^{-\frac{q-\Delta+1}{Mq}t}\right)\right)^{2\Delta}$$

$$\approx \frac{1}{M}e^{-\frac{t}{M}-\frac{2\Delta}{q-\Delta}(1-e^{-\frac{q-\Delta}{Mq}t})}.$$

Finally, noting that $\Pr(\mathrm{d}(X_T,Y_T)=0)=\sum_{t=0}^{\infty}\Pr(S_t)$ by linearity of expectation, we have

$$\Pr(\mathrm{d}(X_T,Y_T)=0) \geq \int_0^{40\Delta}\frac{1}{M}e^{-\frac{t}{M}-\frac{2\Delta}{q-\Delta}(1-e^{-\frac{q-\Delta}{Mq}t})}dt$$

$$= \int_0^{\frac{40\Delta}{M}}e^{-z-\frac{2\Delta}{q-\Delta}(1-e^{-\frac{q-\Delta}{q}z})}dz$$

If we now substitute $q=1.65\Delta$ and $M\leq 2\Delta$, we see that

$$\Pr(\mathrm{d}(X_T,Y_T)=0) \geq \int_0^{20}e^{-z-3.077(1-e^{-0.3941z})}dz > 0.5003.$$

Since $\mathrm{d}(X_T,Y_T)\in\{1,2\}$, it follows that $\mathbf{E}[\mathrm{d}(X_T,Y_T)]<0.9994$ and we can apply Theorem 2.1. This yields the claimed result.

We have assumed that all colours are available for recolouring w at every step. This will not be the case if there is any edge e adjacent to w for which $e\setminus\{w\}$ is monochromatic. Let us call such an edge *blocking*, and suppose there are ρ_t blocking edges at time t. Note that the failure cannot occur on a blocking edge. The total number ρ' of blocking edges created during time 40Δ is at most $\ln\Delta$ since

$$\Pr(\rho'\geq\ln\Delta) \leq \binom{40\Delta}{\ln\Delta}\left(\frac{1}{q}\right)^{\ln\Delta} \leq \left(\frac{40}{\ln\Delta}\right)^{\ln\Delta} = O\left(\frac{1}{\Delta^\gamma}\right)$$

for every constant $\gamma > 0$. Since $\ln \Delta$ is negligible in comparison with q and M, these do not affect the probability estimates in the proof above. Thus we may assume that all blocking edges exist initially. We may further assume these persist until termination, so $\rho_t = \rho_0 = \rho$ for all t. This can only decrease the probability of success. We now observe that this is no worse in our analysis than taking $\Delta' = \Delta - \rho$ and $q' = q - \rho$. Let us formally define $M' = Mq/q'$. Then the conditional success probability is $(q - \rho)/Mq = 1/M'$, and the recolouring probability at each step is at least $(q - \Delta)/Mq = (q' - \Delta')/M'q'$. The analysis now proceeds as before. Since M' plays no part in the final condition, we finally require $q' \geq 1.65\Delta'$, i.e. $q \geq 1.65\Delta - 0.65\rho$. This is clearly a weaker condition than $q \geq 1.65\Delta$. $\qquad\square$

Remark 5.4. If we let $\beta = (q - \Delta)/q$ then, as $\Delta_0 \to \infty$, the analysis can be tightened slightly to work for $\beta > \beta^*$, where β^* is the root of the equation

$$\int_0^\infty e^{-z - \frac{2(1-\beta)}{\beta}(1 - e^{-\beta z})} dz \;=\; \tfrac{1}{2}.$$

The integral can be expanded, by parts integration, as an infinite series to give an alternative equation

$$\sum_{i=0}^\infty \frac{(-2)^i (1-\beta)^i}{\prod_{j=0}^i (1 + j\beta)} \;=\; \tfrac{1}{2}.$$

This has root $\beta^* = 0.392729$, giving $q > 1.64671$.

Remark 5.5. A route to improving our bound on q would be to consider the changes in the numbers of colours available at each vertex of $\Gamma(w)$ during the process. We make the pessimistic assumption that this is always $q - \Delta$ but, while this could be true initially, we would expect more colours to become available later on. A proper analysis of this effect seems more difficult, however, because $\Theta(\Delta^2)$ vertices are now involved, and the edges containing them may intersect.

6 Hardness results for colouring

6.1 #P-completeness

Again we show that exact counting is #P-complete except in the few cases where it is clearly in P. Let $\mathcal{G}(m, \Delta)$ be as in Theorem 4.1.

Theorem 6.1. *Computing the number of q-colourings of hypergraphs in $\mathcal{G}(m, \Delta)$ is #P-complete if $\Delta, q > 1$. If $\Delta \leq 1$ or $q \leq 1$ it is in P.*

Proof. Again we assume $m \geq 3$. The cases $\Delta \leq 1$, $q \leq 1$ are trivially in P. The case $\Delta = 2$ corresponds to counting edge q-colourings of graphs in which no vertex is monochromatic. We call an edge colouring with no monochromatic vertex a *weak edge colouring*. Counting weak edge colourings is #P-complete for graphs of arbitrarily large minimum degree. We give a proof in Appendix B.

For $\Delta \geq 3$, $q \geq 2$, we use the construction from the proof of Theorem 4.1. For each colouring $X : V \to \{1, 2, \ldots, q\}$ of G and edge $e \in E$, there are $q^{m-2} - 1$ permitted colourings of $u_1^e, \ldots, u_{m-2}^e$ if $X(v_1) = X(v_2)$ and q^{m-2} otherwise. The corresponding H-colouring problem has the following

$q \times q$ weight matrix:

$$A = \begin{bmatrix} q^{m-2}-1 & q^{m-2} & \cdots & q^{m-2} \\ q^{m-2} & q^{m-2}-1 & \cdots & q^{m-2} \\ \vdots & \ddots & & \vdots \\ q^{m-2} & q^{m-2} & \cdots & q^{m-2}-1 \end{bmatrix}.$$

The #P-completeness of H-colouring with this weight matrix, and the bound $\Delta = 3$, follow as in Theorem 4.1, since A is again nonsingular. $\square$

6.2 Hardness of Approximation

Again let $\mathcal{G}(m, \Delta)$ be as defined in Theorem 4.1. Our result, Corollary 6.3, follows directly from the following NP-completeness proof.

Theorem 6.2. *Determining whether a hypergraph in $\mathcal{G}(m, \Delta)$ has any q-colouring is NP-complete for any $m > 1$ and $2 < q \le (1 - 1/m)\Delta^{1/(m-1)}$.*

Proof. If $m = 2$, this is graph colouring, and the result follows from [14, Theorem 1.4]. (See also [24].) For $m \ge 3$, we use the following reduction from graph colouring. Let $G = (V, E)$ be a graph with degree Δ_G, and $2 < q \le 3\Delta_G/4$. Without loss, we may assume $\Delta_G = \lceil 4q/3 \rceil$. Colouring G with q colours is NP-complete [14]. For each edge $e = \{v_1, v_2\} \in E$, let $S_i^e = \{u_{i1}^e, u_{i2}^e, \ldots, u_{im}^e\}$ $(i = 1, 2, \ldots, q)$ and $\mathcal{V}_0^e = \bigcup_{i=1}^q S_i^e$. Let $\mathcal{E}_0^e$ comprise all subsets of $\mathcal{V}_0^e$ of size m other than S_i^e $(i = 1, 2, \ldots, q)$. We claim that any proper q-colouring of the hypergraph $\mathcal{H}_0^e = (\mathcal{V}_0^e, \mathcal{E}_0^e)$ must assign the same colour to all $u_{ij}^e \in S_i^e$ $(j = 1, 2, \ldots, m)$, and a different colour for each $i = 1, \ldots, q$. The claim holds since there must be some colour class of size at least m, since there are q colours and mq vertices. If there was a colour class of size greater than m, at least one of its subsets of size m would be a monochromatic edge. Thus there must be exactly q colour classes, each of size m. If these are not the S_i^e $(i = 1, 2, \ldots, q)$, again there is a monochromatic subset of size m which is an edge. Clearly, by symmetry, any assignment of the q colours to the q classes S_i^e is permissible.

Let $\mathcal{V}^e = \mathcal{V}_0^e \cup \{v_1, v_2\}$, and add the edges $\{v_1, u_{i2}^e, \ldots, u_{im}^e\}$ $(i = 1, \ldots, \lfloor q/2 \rfloor)$ and $\{v_2, u_{i2}^e, \ldots, u_{im}^e\}$ $(i = \lfloor q/2 \rfloor + 1, \ldots, q)$ to $\mathcal{E}_0^e$ to give $\mathcal{E}^e$. We claim that, in any proper q-colouring of the hypergraph $\mathcal{H}^e = (\mathcal{V}^e, \mathcal{E}^e)$, v_1 and v_2 must receive different colours. The claim holds since v_1 can have any colour different from all S_i^e $(i = 1, 2, \ldots, \lfloor q/2 \rfloor)$, and v_2 any colour different from all S_i^e $(i = \lfloor q/2 \rfloor + 1, \ldots, q)$. But these permitted colour sets for v_1 and v_2 are disjoint. Also, given any colours for v_1 and v_2, there are $\lfloor q/2 \rfloor \lceil q/2 \rceil (q-2)! > 0$ colourings of $\mathcal{H}^e$. Thus we may use $\mathcal{H}^e$ to simulate the edge $e \in E$. Thus we set $\mathcal{V} = \bigcup_{e \in E} \mathcal{V}^e$, $\mathcal{E} = \bigcup_{e \in E} \mathcal{E}^e$ and consider the hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. Then $\mathcal{H}$ is q-colourable if and only if G is q-colourable.

The maximum degree in $\mathcal{H}$ of any u_{ij} is $\binom{mq}{m-1} \ge eq^{m-1}$. The degree in $\mathcal{H}$ of each $v \in V$ is at most $\Delta_G \lceil q/2 \rceil \le (4q + 2)(q + 1)/6 < 2q^2$. Thus $\Delta = \binom{mq}{m-1} \ge (mq/(m-1))^{m-1}$, and hence $q \le (1 - 1/m)\Delta^{1/(m-1)}$. $\square$

Corollary 6.3. *Unless NP=RP, there is no* fpras *for counting q-colourings of a hypergraphs with maximum degree Δ and minimum edge size m if $2 < q \le (1 - 1/m)\Delta^{1/(m-1)}$.*

Proof. We cannot tell if there is *any* colouring for q in this range, so there can be no *fpras*. $\square$

Remark 6.4. It is clearly a weakness that our lower bound for approximate counting is based entirely on an NP-completeness result. However, we note that the same situation pertains for graph colouring, which has been the subject of more intensive study.

7 Conclusions

We have presented an approach to the analysis of path coupling with stopping times which improves on the method of [17] in most applications. Our method may itself permit further development.

We apply the method to independent sets and q-colourings in hypergraphs with maximum degree Δ and minimum edge size m. In the case of independent sets, there seems scope for improving the bound $m \geq 2\Delta + 1$, but anything better than $m \geq \Delta + o(\Delta)$ would seem to require new methods. For colourings, there is probably little improvement possible in our result $q > \Delta$ for $m \geq 4$, but many questions remain for $m \leq \Delta$. For example, even the ergodicity of the Glauber (or any other) dynamics is not clearly established. For the most interesting case, $m = 3$, the bound $q > 1.65\Delta$ (for large Δ) can almost certainly be reduced, but substantial improvement may prove difficult.

Our #P-completeness results seem best possible for both of the problems we consider. On the other hand, our lower bounds for hardness of approximate counting seem very weak in both cases, and are far from our upper bounds. These lower bounds can probably be improved, but we have no plausible conjecture as to what may be the truth.

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Appendices

A Edge cover is #P-complete

Proof. We prove this by reduction from counting independent sets, using methods similar to Bubley and Dyer [3], where this result was claimed without proof. Let $\mathcal{G}$ be a class of 3-regular graphs for which counting independent sets is #P-complete [16, Theorem 3.1]. Let $G = (V, E)$, with $I_j(G)$ independent sets of size j ($j = 0, 1, \ldots, n$). Form G' by subdividing each edge $e \in E$ with a new vertex u_e. Let $U = \{u_e : e \in E\}$. Let $N_i(G')$ be the number of edge sets in G' which leave exactly i vertices in V uncovered, but no vertex in U. In particular, $N_0(G')$ is the number of edge covers of G', and we assume an oracle computing this quantity. Observe that the uncovered vertices in G' must form an independent set in G. Then it follows, similarly to [3], that

$$2^{3(n-2j)} I_j(G) = \sum_{i=j}^{n} \binom{i}{j} N_i(G').$$

Thus, if we can determine the $N_i(G')$, we can determine the number of independent sets of all sizes in G. Let $N_{ij}(G')$ be the number of edge sets of G' in which i vertices in V and j in U are uncovered ($i = 0, \ldots, n$, $j = 0, \ldots, 3n/2$). Then $N_i(G') = N_{i0}(G')$. We attach a copy K_m^v of K_m to each vertex $v \in V$ and a copy K_k^u of K_k to each vertex $u \in U$. Call the resulting graph G_{mk}. Let M_m be the number of edge covers of K_m, then M_{m-1} is the number of edge sets in K_m which leave a fixed vertex uncovered. We can show by inclusion-exclusion that

$$M_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} 2^{\binom{m-i}{2}}.$$

(See [26].) It is easy to show that that M_m/M_{m-1} is a rapidly increasing sequence (in fact $M_m/M_{m-1} \approx 2^{m-1}$ for large m), and hence has a different value for every value of m. We have

$$N_0(G'_{mk}) = \sum_{i=0}^{n} \sum_{j=0}^{3n/2} M_m^i (M_m + M_{m-1})^{n-i} M_k^j (M_k + M_{k-1})^{3n/2-j} N_{ij}(G'). \tag{12}$$

$$= M_m^n M_k^{3n/2} \sum_{i=0}^{n} \left(1 + \frac{M_{m-1}}{M_m}\right)^{n-i} \sum_{j=0}^{3n/2} \left(1 + \frac{M_{k-1}}{M_k}\right)^{3n/2-j} N_{ij}(G').$$

By choosing any $(n + 1)$ values of m and any $(3n/2 + 1)$ values of k, we can determine all the $N_{ij}(G')$ by interpolation, and hence all the $N_i(G')$. From these, we can determine all the $I_j(G)$, and hence $\sum_{j=1}^n I_j(G)$, the total number of independent sets in G.

Note that the minimum degree of G'_{mk}, $\min\{m, k\} - 1$, can be made as large as we wish. $\qquad\square$

B Weak edge colouring is #P-complete

Proof. We use the same notation and construction as in Appendix A. Now $\mathcal{G}$ is a class of 3-regular graphs for which vertex q-colouring is #P-complete [16, Theorem 2.2]. Let $N_i(G')$ denote the number of edge colourings of G' with i monochromatic vertices, so $N_0(G')$ is the number of weak edge colourings of G', and we assume an oracle for this. Let $N_{ij}(G')$ be the number edge colourings of G' in which i vertices in V and j in U are monochromatic ($i = 0, \ldots, n$, $j = 0, \ldots, 3n/2$). Now observe that $N_{n0}(G')$ is equal to the number of proper *vertex* q-colourings of G, $Q_q(G)$ say. In every colouring counted in $N_{n0}(G')$, every vertex is monochromatic and adjacent vertices receive different colours. Again we attach a copy K_m^v of K_m to each vertex $v \in V$, and a copy K_k^u of K_k to each vertex $u \in U$, to give G_{mk}. Let M_m be the number of weak colourings of K_m, and M'_m the number of edge colourings of K_m with a given monochromatic vertex. Now we have

$$M_m = q^{\binom{m}{2}} + q\sum_{i=1}^m (-1)^i \binom{m}{i} q^{\binom{m-i}{2}}, \qquad M'_m = q\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} q^{\binom{m-i-1}{2}}.$$

Again the sequence M_m/M'_m increases rapidly ($M_m/M'_m \approx q^{m-2}$ for large m), and takes a different value for every m when $q \geq 2$. Now, as in (12),

$$N_0(G'_{mk}) = M_m^n M_k^{3n/2} \sum_{i=0}^n \left(1 + \frac{M'_m}{M_m}\right)^{n-i} \sum_{j=0}^{3n/2} \left(1 + \frac{M'_k}{M_k}\right)^{3n/2-j} N_{ij}(G').$$

Hence, choosing $(n + 1)$ values of m and $(3n/2 + 1)$ values of k, we can determine all the $N_{ij}(G')$ by interpolation. In particular, we can determine $N_{n0}(G') = Q_q(G)$.

Again the minimum degree of G'_{mk}, $\min\{m, k\} - 1$, can be made arbitrarily large. $\qquad\square$