Electronic Colloquium on Computational Complexity, Comment 1 on Report No. 6 (2005)

A stronger version of Corollary 1

Comment to ECCC Report 05-006

Edward A. Hirsch   Sergey I. Nikolenko

February 13, 2005

Abstract

In ECCC Report 05-006 we have shown how to simulate Cutting Plane proofs with degree of falsity bounded by $d(n) \le n/2$ by resolution proofs at the cost of a $\binom{n}{d(n)} 64^{d(n)}$ factor (n is the number of variables). This simulation and Urquhart's [Urq87] lower bound for resolution imply immediately a superpolynomial bound for this version of Cutting Plane proof system for $d(n) = o(n/\log n)$, which gives a simplified proof of a result by Goerdt [Goe91].

In this comment we notice that our main theorem that gives this simulation actually yields an *exponential* lower bound for $d(n) \le cn$ for an appropriate constant $c > 0$.

We recall the main theorem and refer the reader to the original report for all necessary definitions.

Theorem 1. *A Cutting Plane proof* Π *with* $\max_{\iota\in\Pi} \mathtt{DGF}(\iota) \le d \le n/2$ *of a formula in CNF with* n *variables can be transformed into a resolution proof of size at most* $\binom{n}{d-1}|\Pi|2^{6d}$.

Corollary 0. *If formulas* F_n *(where* F_n *contains* n *variables) have no resolution proofs containing less than* $2^{c_{\mathrm{res}}n}$ *clauses (*$c_{\mathrm{res}} > 0$ *is a constant), then these formulas do not have Cutting Plane proofs of size less than* $2^{c_{\mathrm{CP}}n}$ *and degree of falsity bounded by* $c_{\mathtt{DGF}}n$ *for every choice of positive constants* $c_{\mathrm{CP}} < c_{\mathrm{res}}$ *and* $c_{\mathtt{DGF}} \le \frac{1}{2}$ *such that*

$$c_{\mathrm{CP}} + 6c_{\mathtt{DGF}} - c_{\mathtt{DGF}}\log_2 c_{\mathtt{DGF}} - (1-c_{\mathtt{DGF}})\log_2(1-c_{\mathtt{DGF}}) \le c_{\mathrm{res}}. \tag{1}$$

In particular, formulas F_n *have only exponential-size Cutting Plane proofs of degree of falsity bounded by an appropriate linear function of* n.

Proof. By Theorem 1 Cutting Plane proofs of size less than $2^{c_{\mathtt{DGF}}n}$ could be converted into resolution proofs of size less than

$$\binom{n}{c_{\mathtt{DGF}}n-1}2^{c_{\mathrm{CP}}n+6c_{\mathtt{DGF}}n} = o\big(2^{(c_{\mathrm{CP}}+6c_{\mathtt{DGF}}-c_{\mathtt{DGF}}\log_2 c_{\mathtt{DGF}}-(1-c_{\mathtt{DGF}})\log_2(1-c_{\mathtt{DGF}}))n}\big) = o\big(2^{c_{\mathrm{res}}n}\big)$$

(the first equality uses Stirling's formula).

Finally, note that $f(c) = 6c - c\log_2 c - (1-c)\log_2(1-c)$ decreases to 0 as c decreases from $\frac{1}{2}$ to 0. Therefore, for every $c_{\mathrm{CP}} < c_{\mathrm{res}}$ there is $c_{\mathtt{DGF}}$ that satisfies (1). □

Corollary 1 — a stronger version. *There exists a positive constant* δ *such that Tseitin-Urquhart formulas of* n *variables (described in [Urq87]) have only* $2^{\Omega(n)}$*-size Cutting Plane proofs with degree of falsity bounded by* δn.

Proof. Follows immediately from Corollary 0 and Urquhart's theorem:

Theorem ([Urq87, Theorem 5.7]) *There is a constant* $c > 1$ *such that for sufficiently large* m*, any resolution refutation of* S_m *contains* c^n *distinct clauses, where* S_m *is of length* $O(n)$*,* $n = m^2$. □

References

[Goe91] Andreas Goerdt. The Cutting Plane proof system with bounded degree of falsity. In *Proceedings of CSL 1991*, volume 626 of *Lecture Notes in Computer Science*, pages 119–133. Springer, 1991.

[Urq87] Alasdair Urquhart. Hard examples for resolution. *JACM*, 34(1):209–219, 1987.

ISSN 1433-8092