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Addendum to "Approximation Algorithms for Unique Games"

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Abstract

We present a polynomial time algorithm that, given a unique game of value $1 - c/\log n$, satisfies a constant fraction of constraints, where n is the number of variables.

This improves an algorithm of Trevisan (ECCC TR05-34), that satisfies a constant fraction of constraints in unique games of value $1 - c/(\log n)^3$ and, for sufficiently large alphabets, it improves an algorithm of Khot (STOC'02) that satisfies a constant fraction of constraints in unique games of value $1 - c/(k^{10}(\log k)^5)$, where k is the size of the alphabet.

Our algorithm is based on semidefinite programming.

The result presented in this note will be incorporated in a later version of ECCC TR05-34.

1 The Semidefinite Program

A unique game [FL92, Kho02] is presented as a graph $G = (V, E)$, a set S and a permutation $\pi_e : S \to S$ for every edge $e \in E$. We think of a unique game as a constraint satisfaction problem where there is a variable for every vertex u, variables take values in the set S, and every edge $e = (u, v)$ with associated permutation $\pi_e : S \to S$ defines the constraint $v = \pi_{(u,v)}(u)$. The goal is to find an assignment that satisfies as many constraints as possible. The value of a unique game is the fraction of constraints satisfied by an optimal assignment.

We consider the following integer programming formulation. We assume without loss of generality that the set S equals $\{1, \dots, |S|\}$. For every variable u of the unique game we have $k := |S|$ boolean variables $u_1, \dots, u_k$ in the integer program, with the intended meaning that if $u = i$ then $u_i = 1$ and $u_j = 0$ for $j \neq i$. Each constraint $v = f(u)$ contributes $\sum_{i \in S} v_{f(i)} u_i$ to the objective function. The integer program, therefore, looks like this:

$$
\begin{array}{ll}
\max \sum_{(u,v)\in E} \sum_{i\in S} v_{\pi_{u,v}(i)} u_i & \\
\text{Subjet to} & \\
\quad u_i \cdot u_j = 0 & (\forall u \in V, \forall i, j \in [k], i \neq j) \\
\quad \sum_{i \in S} u_i = 1 & (\forall u \in V) \\
\quad u_i \in \{0, 1\} & (\forall u \in V)
\end{array}
$$

In the semidefinite relaxation, each variable u_i is replaced by a vector $\mathbf{u}_i$.

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$$
\begin{array}{ll}
\max \sum_{(u,v)\in E} \sum_{i\in S} \mathbf{v}_{\pi_{u,v}(i)} \mathbf{u}_i & \\
\text{Subjet to} & \\
\quad \mathbf{u}_i \cdot \mathbf{u}_j = 0 & (\forall u \in V, \forall i,j \in [k], i \neq j) \\
\quad \sum_{i\in S} ||\mathbf{u}_i||^2 = 1 & (\forall u \in V)
\end{array}
$$

We will work with an equivalent formulation of the objective function, and add "triangle inequalities."

$$
\begin{array}{ll}
\max \sum_{(u,v)\in E} \left(1 - \sum_{i\in S} \frac{1}{2} ||\mathbf{v}_{\pi_{u,v}(i)} - \mathbf{u}_i||^2\right) & \\
\text{Subject to} & \\
\quad \mathbf{u}_i \cdot \mathbf{u}_j = 0 & (\forall u \in V, \forall i,j \in [k], i \neq j) \\
\quad \sum_{i\in S} ||\mathbf{u}_i||^2 = 1 & (\forall u \in V) \\
\quad ||\mathbf{w}_h - \mathbf{u}_i||^2 \leq ||\mathbf{w}_h - \mathbf{v}_j||^2 + ||\mathbf{v}_j - \mathbf{u}_i||^2 & (\forall u,v,w \in V, \forall i,j,h \in [k]) \\
\quad ||\mathbf{v}_j - \mathbf{u}_i||^2 \geq ||\mathbf{v}_j||^2 - ||\mathbf{u}_i||^2 & (\forall u,v \in V, \forall i,j \in [k])
\end{array}
\tag{1}
$$

Feige and Lovasz [FL92] and Khot [Kho02] add different inequalities, and it is not clear if the SDP (1) is equivalent or not to the ones in [FL92, Kho02].

2 Analysis

In this section we prove the following result.

Theorem 1 *Suppose that the SDP (1) has a solution of cost at least $(1 - c\varepsilon^3/(\log n))|E|$. Then it is possible to find in polynomial time a solution for the unique game that satisfies at least a $1 - \varepsilon$ fraction of constraints. (c is an absolute constant.)*

Suppose that the SDP relaxation of a unique game $(G = (V,E), S, \{\pi_e\}_{e\in E})$ has a solution of cost at least $(1-\gamma)\cdot|E|$, where $\gamma = c\varepsilon^3/\log n$ (we will fix c later). Then for all but an $\varepsilon/3$ fraction of constraints their contribution to the objective function is at least $1 - 3\gamma/\varepsilon = 1 - 3c\varepsilon^2/\log n$. The algorithm of Theorem 1 is as follows:

1. Remove the constraints whose contribution is smaller than $1 - 3\gamma/\varepsilon$.

   This step removes at most $|E|\varepsilon/3$ edges and, in the residual graph, every edge contributes at least $1 - 3c\varepsilon^2/\log n$ to the objective function.

2. Apply the Leighton-Rao decomposition of Lemma 4 below with $t = 1/(1-\varepsilon/3))$ to the residual graph.

   This step removes at most $|E|\varepsilon/3$ edges, and the residual graph breaks up into connected components of diameter at most $d = O((\log n)/\varepsilon)$.

3. Use Lemma 2 below to satisfy at least $1 - \varepsilon/3$ fraction of constraints in each connected component of the residual graph that we obtain after steps (1) and (2).

   (The constant c will have to be set so that $1 - 3c\varepsilon^2/(\log n) \geq 1 - \varepsilon/24(d+1)$, that is, $c < (\log n)/(72 \cdot (d+1) \cdot \varepsilon)$.)

This step finds a solution that satisfies all but at most $\varepsilon|E|/3$ constraints of the residual graph obtained after steps (1) and (2).

It remains to state and prove Lemma 2 (see below) and to state the Leighton-Rao decomposition result (see the Appendix).

Lemma 2 *Suppose we are given a unique game $(G, [k], \{\pi_e\})$ such that the SDP (1) has a feasible solution in which every edge contributes at least $1 - \varepsilon/8(d+1)$, where d is the diameter of the graph. Then it is possible, in polynomial time, to find a solution that satisfies a $1 - \varepsilon$ fraction of the constraints.*

PROOF:[Of Lemma 2] We fix a spanning tree of diameter d of G and we let r be the root of the tree. We pick at random a value $i \in [k]$ with probability $||\mathbf{r}_i||^2$, and we assign i to r. For every other variable v, we assign to v the value j that minimizes the "distance" $||\mathbf{v}_j - \mathbf{r}_i||^2$. Let A be the random variable corresponding to the above described distribution of assignments. We claim that every constraint has a probability at least $1 - \varepsilon$ of being satisfied by such an assignment.

Let (u, v) be a constraint in G, and let $r = u^0, u^1, \ldots, u^t = u$ be a path from r to u of length $t \leq d$ in G. Let π_u be the composition of the permutations $\pi_{(r,u^1)}, \ldots, \pi_{(u^{t-1},u)}$ corresponding to the path. Let $\pi_v() := \pi_{(u,v)}(\pi_u())$.

We will show that there is a probability at least $1 - \varepsilon/2$ that $A(u) = \pi_u(A(r))$ and a probability at least $1 - \varepsilon/2$ that $A(v) = \pi_v(A(r))$. (We only prove the former statement, since the latter has an identical proof.) By a union bound, it will follow that there is a probability at least $1 - \varepsilon$ that $A(v) = \pi_{(u,v)}(A(u))$.

By the triangle inequality, we have

$$\sum_{i \in S} ||\mathbf{r}_i - \mathbf{u}_{\pi_u(i)}||^2 \leq \varepsilon/4$$

Let B be the set of indices i such that $\pi_u(i)$ is not the j that minimizes $||\mathbf{r}_i - \mathbf{u}_j||^2$. Then we have

$$\mathbf{Pr}[A(u) \neq \pi_u(A(r))] = \sum_{i \in B} ||\mathbf{r}_i||^2$$

We claim that for every $i \in B$, $||\mathbf{r}_i - \mathbf{u}_{\pi_u(i)}||^2 \geq \frac{1}{2}||\mathbf{r}_i||^2$. This follows from the following simple fact (substitute $\mathbf{r} \leftarrow \mathbf{r}_i$, $\mathbf{u} \leftarrow \mathbf{u}_{\pi_u(i)}$ and $\mathbf{v} \leftarrow \mathbf{u}_j$, where j is the minimizer of $||\mathbf{r}_i - \mathbf{u}_j||^2$).

Claim 3 *Let $\mathbf{r}, \mathbf{u}, \mathbf{v}$ be vectors such that: (i) $\mathbf{u} \cdot \mathbf{v} = 0$, (ii) $||\mathbf{r} - \mathbf{u}||^2 \geq ||\mathbf{r} - \mathbf{v}||^2$, and (iii) the vectors $\mathbf{r}, \mathbf{u}, \mathbf{v}$ satisfy the "triangle inequality" constraints of (1).*
Then $||\mathbf{r} - \mathbf{u}||^2 \geq ||\mathbf{r}||^2$.

PROOF: We consider three cases:

1. If $||\mathbf{u}||^2 \leq \frac{1}{2}||\mathbf{r}||^2$, then

$$||\mathbf{r} - \mathbf{u}||^2 \geq ||\mathbf{r}||^2 - ||\mathbf{u}||^2 \geq \frac{1}{2}||\mathbf{r}||^2$$

2. If $||\mathbf{v}||^2 \leq \frac{1}{2}||\mathbf{r}||^2$, then

$$||\mathbf{r}-\mathbf{u}||^2 \geq ||\mathbf{r}-\mathbf{v}||^2 \geq ||\mathbf{r}||^2 - ||\mathbf{v}||^2 \geq \frac{1}{2}||\mathbf{r}||^2$$

3. If $||\mathbf{u}||^2, ||\mathbf{v}||^2 \geq \frac{1}{2}||\mathbf{r}||^2$, then from the triangle inequality and from assumption (ii) we have

$$||\mathbf{v}-\mathbf{u}||^2 \leq ||\mathbf{v}-\mathbf{r}||^2 + ||\mathbf{r}-\mathbf{u}||^2 \leq 2||\mathbf{r}-\mathbf{u}||^2$$

and by Pythagoras theorem and the orthogonality of $\mathbf{v}$ and $\mathbf{u}$ we have

$$||\mathbf{v}-\mathbf{u}||^2 = ||\mathbf{v}||^2 + ||\mathbf{u}||^2$$

so that

$$||\mathbf{r}-\mathbf{u}||^2 \geq \frac{1}{2}||\mathbf{v}-\mathbf{u}||^2 = \frac{1}{2}||\mathbf{v}||^2 + \frac{1}{2}||\mathbf{u}||^2 \geq \frac{1}{2}||\mathbf{r}||^2$$

□

We can now estimate the probability of an inconsistent assignment to $\mathbf{r}$ and $\mathbf{u}$ as

$$\mathbf{Pr}[A(u) \neq \pi_u(A(r))] = \sum_{i\in B} ||\mathbf{r}_i||^2 \leq 2\sum_{i\in B} ||\mathbf{r}_i - \mathbf{u}_{\pi_u(i)}||^2 \leq 2\sum_{i\in S} ||\mathbf{r}_i - \mathbf{u}_{\pi_u(i)}||^2 \leq \frac{\varepsilon}{2}$$

□

3 Extensions

By choosing parameters differently in the decomposition step, we can, for example, satisfy a $O(\frac{\log\log n}{\log n})$ fraction of constraints in unique games of value $1 - O(\frac{\log\log n}{\log n})$, or satisfy a $1/n^{O(\gamma)}$ fraction of constraints in a unique game of value $\geq 1-\gamma$.

Khot and Vishnoi [KV05] construct instances of unique games such that the SDP optimum is $(1-\gamma)|E|$ but the value of the game is at most $1/(\log n)^\gamma$. Their result applies to our formulation as well. It would be very interesting to close this exponential gap.

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A Appendix

A.1 Statement of the Leighton Rao Decomposition Theorem

Lemma 4 (Leighton and Rao [LR99]) *There is a polynomial time algorithm that, on input a graph $G = (V, E)$ and a parameter $t > 1$, returns a subset of edges $E' \subseteq E$ such that $|E|' \geq |E|/t$ and such that every connected component of the graph $G' = (V, E')$ has diameter at most $(1 + \log|E|)/(\log t)$.*

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