Sub-Constant Error Low Degree Test of Almost Linear Size

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Abstract

Given a function \( f : \mathbb{F}^m \rightarrow \mathbb{F} \) over a finite field \( \mathbb{F} \), a low degree tester tests its proximity to an \( m \)-variate polynomial of total degree at most \( d \) over \( \mathbb{F} \). The tester is usually given access to an oracle \( A \) providing the supposed restrictions of \( f \) to affine subspaces of constant dimension (e.g., lines, planes, etc.). The tester makes very few (probabilistic) queries to \( f \) and to \( A \) (say, one query to \( f \) and one query to \( A \)), and decides whether to accept or reject based on the replies.

We wish to minimize two parameters of a tester: its error and its size. The error bounds the probability that the tester accepts although the function is far from a low degree polynomial. The size is the number of bits required to write the oracle replies on all possible tester’s queries.

Low degree testing is a central ingredient in most constructions of probabilistically checkable proofs (PCPs) and locally testable codes (LTCs). The error of the low degree tester is related to the soundness of the PCP and its size is related to the size of the PCP (or the length of the LTC).

We design and analyze new low degree testers that have both sub-constant error \( o(1) \) and almost-linear size \( n^{1+o(1)} \) (where \( n = |\mathbb{F}|^m \)). Previous constructions of sub-constant error testers had polynomial size [3, 17]. These testers enabled the construction of PCPs with sub-constant soundness, but polynomial size [3, 17, 11]. Previous constructions of almost-linear size testers obtained only constant error [15, 8]. These testers were used to construct almost-linear size LTCs and almost-linear size PCPs with constant soundness [15, 8, 6, 7, 10].

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1 Introduction

1.1 Low Degree Testing

Let \( F \) be a finite field, let \( m \) be a dimension and let \( d \) be a degree. [A particular setting of parameters to have in mind is the one used in construction of PCPs and LTCs: a large field \( F \), and a fairly large degree \( d \), which is, nonetheless, considerably smaller than \( |F| \); specifically, \( \frac{md}{|F|} \leq o(1) \), but \( md \geq |F|^{1-o(1)} \).]

Define \( \mathcal{P} \) to be the set of all \( m \)-variate polynomials of total degree at most \( d \) over \( F \). The proximity of a function \( f : F^m \to F \) to a low degree polynomial is

\[
\Gamma(f, \mathcal{P}) = \max_{Q \in \mathcal{P}} \left\{ \Pr_{\bar{x} \in F^m} [f(\bar{x}) = Q(\bar{x})] \right\}
\]

Note that \( \Gamma(f, \mathcal{P}) \) is simply \( 1 - \Delta(f, \mathcal{P}) \), where \( \Delta \) denotes the (normalized) Hamming distance.

A low degree tester is a probabilistic procedure \( M \) that is meant to check the proximity of a function \( f \) to a low degree polynomial by making as few queries to \( f \) as possible. If \( f \in \mathcal{P} \), \( M \) should always accept, while if \( f \) is far from \( \mathcal{P} \) (i.e., \( \Gamma(f, \mathcal{P}) \) is small) \( M \) should be likely to reject.

It is easy to see that when having oracle access only to \( f \), any low degree tester must make more than \( d \) queries. To break this degree barrier, the low degree tester is usually given access to an additional oracle \( \mathcal{A} \) providing the supposed restrictions of \( f \) to affine subspaces of constant dimension (e.g., lines, planes, etc.). The convention is that these restrictions in themselves are polynomials of total degree at most \( d \) over the subspaces. The tester is required to satisfy:

- **Completeness:** If \( f \in \mathcal{P} \) there is an oracle \( \mathcal{A} \) that makes the tester accept with prob. 1.
- **Soundness:** If \( \Gamma(f, \mathcal{P}) \) is small then for every oracle \( \mathcal{A} \), the tester is not likely to accept.

Rubinfeld and Sudan [18] designed the Line vs. Point tester that makes only two probabilistic queries. This tester picks independently at random a line \( l \) in \( F^m \) and a point \( \bar{x} \in l \), queries the oracle \( \mathcal{A} \) for the (supposed) restriction of \( f \) to \( l \) (which is simply a univariate polynomial of degree at most \( d \) over \( F \)), queries \( f \) at \( \bar{x} \), and checks whether the two restrictions are consistent on \( \bar{x} \), i.e., \( \mathcal{A}(l)(\bar{x}) = f(\bar{x}) \).

Low degree testers enabled the construction of Probabilistically Checkable Proofs (PCPs) [5, 12, 2, 1] and Locally Testable Codes (LTCs) [13, 15], hence their great importance. These applications motivated further improvements to low degree testing.

Specifically, the following parameters were of interest:

1. **Queries:** How many queries does the tester make? (should be a constant; preferably 2).
2. **Error:** How sound is the tester?
3. **Size:** How many bits are needed to write the oracle replies on all possible queries?
1.1.1 Error

To prove that a low degree tester is sound, most results address contrapositive arguments of the following type: assume that the tester accepts with probability \( \gamma \geq \gamma_0 \) and show the existence of a low degree polynomial that identifies with \( f \) on at least \( \approx \gamma \) of the points. In this case, we say that \( \gamma_0 \) bounds the error of the tester, since the probability that the tester accepts although the function is very far from a low degree polynomial is at most \( \gamma_0 \).

The first analyses of the Line vs. Point tester [18, 2, 13] only showed that the error of the tester is bounded away from 1. The error can be amplified to any constant, by a constant number of repetitions. Nevertheless, to keep the total number of queries constant, one cannot perform more than a constant number of repetitions.

Only a later, more careful, inspection [3, 17] revealed that there are low degree testers with a sub-constant error. Specifically, [3, 17] proved claims of the following type for various low degree testers: there exist (large enough) constants \( C \geq 1, a, b \geq 0 \), and a (small enough) constant \( 0 < c \leq 1 \), such that the error is at most \( C m^a d^b \| \mathbb{F} \|. \) In other words, the error can be made arbitrarily small by taking \( m \) and \( d \) to be small enough with respect to \( |\mathbb{F}| \). The number of queries remains 2.

Arora and Sudan [3] proved that the error of the Line vs. Point tester is in fact sub-constant. Their proof was very algebraic in nature. Raz and Safra [17] proved a sub-constant error for a slightly different tester, considering planes that intersect by a line, or a plane and a point within it. Their proof was more combinatorial in nature. The two proofs led to the construction of PCPs with sub-constant soundness [3, 17, 11].

1.1.2 Size

Let us represent the set of honest oracles by a code. That is, for every polynomial \( Q : \mathbb{F}^m \to \mathbb{F} \) of degree at most \( d \), we have a codeword. The codeword has an entry for every affine subspace \( s \) that the tester may query. This entry contains the oracle’s reply when it is queried regarding \( s \), i.e., the restriction of \( Q \) to \( s \). The size of a tester is the length (in bits) of a codeword. For instance, the size of Rubinfeld and Sudans’ Line vs. Point tester [18] is roughly \( |\mathbb{F}|^{2m} (d + 1) \log |\mathbb{F}| \). For every line (defined by two points), the oracle should provide a univariate polynomial of degree at most \( d \) over \( \mathbb{F} \).

Alternatively, we refer to the randomness of the tester, which is the amount of random bits that the tester requires. For instance, to pick a random line and a random point within it, we merely have to pick a random point and a random direction in \( \mathbb{F}^m \). Hence, the randomness of the Line vs. Point tester [18] is \( 2m \log |\mathbb{F}| \).

The size of a tester is measured with respect to \( n = |\mathbb{F}|^m \). For instance, the size of the Line vs. Point tester [18] is quadratic \( n^{2+o(1)} \). The size of a tester is related to the size of probabilistically checkable proofs and locally testable codes constructed using it. Hence, Goldreich and Sudan [15] suggested to improve the Line vs. Point tester by considering a relatively small subset of lines (instead of all lines). Goldreich and Sudan achieved non-explicit constant error tester of almost-linear size \( n^{1+o(1)} \), instead of quadratic size \( n^{2+o(1)} \). Shortly afterwards, Ben-Sasson, Sudan, Vadhan and Wigderson [8] gave an explicit construction of a constant error Line vs. Point tester of almost-linear size. Their idea was to choose a line by picking a uniformly distributed point over \( \mathbb{F}^m \) (as before), and a direction that is uniformly distributed over a small \( \epsilon \)-biased set within \( \mathbb{F}^m \). They showed that the error of this tester is bounded away from 1. Unfortunately, their elegant analysis is inherently applicable only for acceptance probability \( \gamma \geq \frac{1}{2} \).
The work of [15, 8] gave rise to explicit constructions of almost-linear size LTCs and PCPs with a constant soundness [15, 8, 6]. The recent work of Dinur [10] also depicts almost-linear size LTCs and PCPs with constant soundness, based on the work of Ben-Sasson and Sudan [7], which in turn uses a different low degree tester with constant error due to Polishchuk and Spielman [16]. Dinur’s work [10] also gives new constructions of PCPs and LTCs without low degree testers. However, at this point, these constructions achieve neither sub-constant error nor almost-linear size.

1.2 Our Contribution: Randomness-Efficient Sub-Constant Error Testers

We design and analyze two low degree testers that have both sub-constant error and almost-linear size. Using known methods [17, 3, 11] our constructions seem to lead to PCPs with sub-constant soundness and almost-linear size (and a constant number of queries), but we have not verified the details.

Our key idea is to consider a subfield $\mathbb{H} \subseteq \mathbb{F}$, and generate subspaces by picking directions uniformly over $\mathbb{H}^m$, instead of over $\mathbb{F}^m$. The field structure of $\mathbb{H}$ allows us to apply symmetry-based arguments. It allows us to prove sampling properties (see subsection 1.3). It allows us to use the combinatorial approach of Raz and Safra [17]. And, most importantly, it allows us to use induction: the structure of the problem in dimension $m$ and checks that they are consistent:

1. Pick uniformly and independently at random $\vec{z} \in \mathbb{F}^m$, $\vec{y}_1, \vec{y}_2 \in \mathbb{H}^m$.

2. Accept if either $\vec{y}_1, \vec{y}_2$ are linearly dependent, or if the plane $p$ through $\vec{z}$ in directions $\vec{y}_1, \vec{y}_2$ satisfies $A(p)(\vec{z}) = f(\vec{z})$.

Figure 1: Randomness-Efficient Plane vs. Point Tester

When $\mathbb{H} = \mathbb{F}$, this tester is exactly the Plane vs. Point tester of Raz and Safra [17]. However, in our work the more interesting case is $|\mathbb{H}| \leq |\mathbb{F}|^{o(1)}$. In this case, the tester requires only $m \log |\mathbb{F}| + 2m \log |\mathbb{H}| = m \log |\mathbb{F}| (1 + o(1))$ bits of randomness. This corresponds to an almost linear size $n^{1+o(1)}$ (recall that $n = |\mathbb{F}^m|$). The tester is randomness efficient in comparison to all known testers with sub-constant error, such as the tester of Arora and Sudan [3] that requires $2m \log |\mathbb{F}|$ bits of randomness and the tester of Raz and Safra [17] that requires $3m \log |\mathbb{F}|$ bits of randomness. As to testers with constant error: that of Ben-Sasson, Sudan, Vadhan and Wigderson [8] requires $m \log |\mathbb{F}| + \text{polylog}(m, \log |\mathbb{F}|)$ bits of randomness, which is (usually) less than the randomness of our tester, but the difference is only in the dependence of the low order term in $m$.

The tester is clearly complete, namely, if there exists a polynomial $Q : \mathbb{F}^m \rightarrow \mathbb{F}$ of degree at most $d$, such that for every $\vec{x} \in \mathbb{F}^m$, $f(\vec{x}) = Q(\vec{x})$, and for every affine subspace $s$, the oracle $A$ replies $A(s) = Q_s$, then the tester accepts with probability 1. We show that the tester is also sound: if the tester accepts with probability $\gamma$ then $f$ identifies with a polynomial of total degree at most $md$ on at least $\gamma - \varepsilon$ of the points in $\mathbb{F}^m$, where $\varepsilon \leq \text{const} \cdot m \left( \frac{1}{|\mathbb{H}|} + \sqrt{\frac{md}{|\mathbb{F}|}} \right)$. Note that the analysis works for any acceptance probability $\gamma$. In particular, this means that
when $\gamma$ is significantly larger than $\varepsilon$, say $\gamma \geq 100\varepsilon$, $f$ identifies with a polynomial of total degree at most $md$ on at least $\approx \gamma$ of the points. [Even if $\mathbb{H} = \mathbb{F}$, the constants 4 and 8 in the error expression appear to improve on the results of [3, 17], where unspecified constants were given].

The downside of the Randomness-Efficient Plane vs. Point tester is that it only allows us to argue something about the proximity of the oracle to a polynomial of a relatively low degree $md$, rather than $d$. Hence, we design another tester that has essentially the same parameters, but ensures proximity to a polynomial of degree at most $d$.

The additional consideration that comes into play when designing the new tester is degree preservation. We want the total degree of a polynomial not to decrease when restricted to most of the subspaces queried by the tester. We achieve this by picking one of the directions for the subspace (rather than the base-point) uniformly from $\mathbb{F}^m$. In order to keep the size almost linear, this tester considers linear subspaces (i.e., affine subspaces through the origin), rather than general affine subspaces. A related technique was previously used by [8].

Specifically, given access to $f$ and to an oracle $\mathcal{A}$, the Randomness-Efficient Subspace vs. Point tester chooses a three dimensional subspace and a point within it and checks that they are consistent:

1. Pick uniformly and independently at random $\tilde{z} \in \mathbb{F}^m$, $\tilde{y}_1, \tilde{y}_2 \in \mathbb{H}^m$.
2. Accept if either $\tilde{z}, \tilde{y}_1, \tilde{y}_2$ are linearly dependent, or if the linear subspace $s$ spanned by $\tilde{z}, \tilde{y}_1, \tilde{y}_2$ satisfies $\mathcal{A}(s)(\tilde{z}) = f(\tilde{z})$.

Figure 2: Randomness-Efficient Subspace vs. Point Tester

This tester uses the same number of random bits as the Randomness-Efficient Plane vs. Point tester $m \log |\mathbb{F}| + 2m \log |\mathbb{H}|$, and its size is only slightly larger (as the answer size is larger: the oracle should provide polynomials over three-dimensional subspaces rather than two-dimensional subspaces). For this small price, we manage to prove a stronger soundness claim: if the Randomness-Efficient Subspace vs. Point tester accepts with probability $\gamma$, then $f$ identifies with a polynomial of total degree at most $d$ (rather than $md$) on at least $\gamma - \varepsilon$ of the points in $\mathbb{F}^m$, where $\varepsilon \leq \text{const} \cdot m \left( \sqrt{\frac{1}{|\mathbb{H}|}} + \sqrt[4]{\frac{md}{|\mathbb{F}|}} \right)$. This follows rather easily from the soundness of the Randomness-Efficient Plane vs. Point tester together with an argument showing that the degree of the recovered polynomials must in fact be at most $d$.

There is a tradeoff between the size of the testers and their error. To make the size as small as possible, one wishes to minimize $|\mathbb{H}|$. In particular, to get an almost-linear size, one needs to take $|\mathbb{H}| \leq |\mathbb{F}|^{o(1)}$. On the other hand, to make the error as small as possible, one wishes to maximize $|\mathbb{H}|$. In particular, to get a sub-constant error, one needs to take $|\mathbb{H}| \geq \omega(m^8)$.

Note that not every finite field $\mathbb{F}$ has a subfield $\mathbb{H}$ of an appropriate size. The size of a finite field may only be of the form $p^k$ for some prime number $p$ and a natural number $k$. Moreover, the only possible sizes of subfields are $p^r$ for $r|k$. Nevertheless, in applications (e.g., when constructing a PCP or designing a LTC), one usually has the freedom to choose the field and tune the parameters.

1.3 Sampling

A basic step in our proof is the analysis of the sampling properties of affine subspaces with directions over a subfield. This analysis may be of independent interest.
By sampling we refer to assertions of the following nature: if one colors a large enough fraction of the points in $\mathbb{F}^m$ green then a subspace (e.g., a line) picked at random is likely to hit the green points in almost their true fraction.

First, let us consider the non-randomness-efficient setting. For instance, consider choosing a line by picking a point and a direction independently at random from $\mathbb{F}^m$. The indicator variables “is the $i$'th point on the line green?” for $i = 1, \ldots, |\mathbb{F}|$ are pairwise independent. Thus, one can easily bound the variance of the number of green points on a line. This yields a sampling property by Chebyshev’s inequality (see, e.g., [3]).

In the randomness-efficient setting, more subtle arguments are needed. For instance, consider the work of Ben-Sasson, Sudan, Vadhan and Wigderson [8]. They use an $\epsilon$-biased set $S \subseteq \mathbb{F}^m$, and choose a line by independently picking a uniformly distributed base-point in $\mathbb{F}^m$ and a uniformly distributed direction in $S$. They show that almost pairwise independence still holds, and this allows them to bound the variance, by bounding the covariances.

Our approach is different. We observe that we can directly bound the variance of the number of green points on a line by analyzing the convolution of two relatively simple functions. We do this by means of Fourier analysis. The difference between the previous approaches and our approach is that instead of giving one bound for the probability that two points $i \neq j$ on a line are green for every $i \neq j$, we directly bound the average probability over all pairs $i \neq j$.

The extension to higher dimensional subspaces is a relatively simple consequence of the analysis for lines.

1.4 Proof Outline

We first prove the soundness of the Randomness-Efficient Plane vs. Point tester, and then deduce the soundness of the Randomness-Efficient Subspace vs. Point tester from it. For the first, we assume that the Randomness-Efficient Plane vs. Point tester accepts with probability $\gamma$, and find a polynomial over $\mathbb{F}^m$ of degree at most $md$ that identifies with $f$ on at least $\gamma - \epsilon$ of the points, for $\epsilon \leq \text{const} \cdot m \left( \sqrt{\frac{1}{|\mathbb{F}|}} + \sqrt{\frac{md}{|\mathbb{F}|}} \right)$. This task is referred to as decoding.

A $\gamma$ acceptance probability of the tester immediately implies some degenerated sort of decoding: a decoding for planes which is good on average. The planes we consider are those that are spanned by directions over $\mathbb{H}$. A $\gamma$ acceptance probability of the tester implies the existence of low degree polynomials $Q_p$ over such planes $p$, such that

$$\mathbb{E}_p \left[ \Pr_{\bar{x} \in p} [Q_p(\bar{x}) = f(\bar{x})] \right] \approx \gamma$$

Note that if we could assert a similar claim for the $m$-dimensional space, instead of for planes, we would be done.

Thus, we gradually work our way up to construct low degree polynomials for affine subspaces from dimension 2 to dimension $m$. For every dimension $2 \leq k \leq m$ we would like to argue the existence of polynomials $Q_s$ of degree at most $kd$ for affine subspaces $s$ of dimension $k$ spanned by directions over the subfield $\mathbb{H}$, such that

$$\mathbb{E}_s \left[ \Pr_{\bar{x} \in s} [Q_s(\bar{x}) = f(\bar{x})] \right] \geq \gamma - \frac{k}{m} \cdot \epsilon$$

The crucial observation is that every such subspace $s$ has the same structure as $\mathbb{F}^k$ with respect to subspaces spanned by vectors over $\mathbb{H}$ within it. Hence, to work our way up, we only
need to show how we climb up one dimension, i.e., how to construct a polynomial for $\mathbb{F}^k$ from low degree polynomials to the hyperplanes (i.e., subspaces of dimension $k - 1$).

To go up one dimension we use the idea of Raz and Safra [17], and reduce the problem to a combinatorial problem. The combinatorial problem centers around a construction of a consistency graph. The graph represents hyperplanes and consistency between low degree polynomials assigned to them. It has interesting combinatorial properties due to its underlying algebraic structure. They allow us to use a graph-theoretic lemma originally proven in [17], and go up one dimension.

The reduction to the combinatorial realm introduces a certain deterioration of the degree and consistency parameters. Hence, after going up one dimension, we apply steps of what we call consolidation to retain the desired parameters.

1.5 Organization

We state the main theorems regarding the completeness and soundness of our testers in section 2. The rest of the paper is devoted to proving these theorems. We start with some preliminary definitions and propositions in section 3. We discuss basic properties of affine subspace with directions over a subfield in section 4. We prove sampling properties in section 5. This allows us to prove consolidation claims in section 6. We present and analyze the consistency graph in section 7 and use it for going up one dimension in section 8. The soundness of the Randomness-Efficient Plane vs. Point tester is proven via induction in section 9. We show that the soundness of the Randomness-Efficient Subspace vs. Point tester follows in section 10. For the sake of self-containment, we give the proof of the combinatorial lemma of [17] in the appendix.

2 Our Results

2.1 Notations

In all that follows, we consider a finite field $\mathbb{F}$, a subfield $\mathbb{H} \subseteq \mathbb{F}$, a dimension $m$, and a degree $d$.

Given vectors $\vec{y}_1, \ldots, \vec{y}_k \in \mathbb{F}^m$, we define the linear subspace they span by $\text{span}\{\vec{y}_1, \ldots, \vec{y}_k\} = \{a_1\vec{y}_1 + \ldots + a_k\vec{y}_k \mid a_1, \ldots, a_k \in \mathbb{F}\}$. We say that $\vec{y}_1, \ldots, \vec{y}_k$ are linearly independent, and denote $\text{ind}(\vec{y}_1, \ldots, \vec{y}_k)$, if for every $a_1, \ldots, a_k \in \mathbb{F}$, if $\sum_{i=1}^{k} a_i \vec{y}_i = 0$ then $a_1 = \cdots = a_k = 0$. Throughout the paper we will refer to span over $\mathbb{F}$ (and not over a subfield, even if the vectors are over a subfield). Note that vectors $\vec{y}_1, \ldots, \vec{y}_k \in \mathbb{H}^m$ are linearly independent over $\mathbb{H}$ if and only if $\vec{y}_1, \ldots, \vec{y}_k \in \mathbb{H}^m$ are linearly independent over $\mathbb{F}$.

Given two sets $A, B \subseteq \mathbb{F}^m$, we define $A + B = \{\vec{x} + \vec{y} \mid \vec{x} \in A, \vec{y} \in B\}$. Given a point $\vec{x} \in \mathbb{F}^m$ and a set $A \subseteq \mathbb{F}^m$, define $\vec{x} + A = \{\vec{x}\} + A$. A $k$-dimensional affine subspace in the vector space $\mathbb{F}^m$ is defined by a base-point $\vec{x} \in \mathbb{F}^m$ and $k$ linearly independent directions, $\vec{y}_1, \ldots, \vec{y}_k \in \mathbb{F}^m$, as

$$\text{affine}(\vec{x}; \vec{y}_1, \ldots, \vec{y}_k) = \vec{x} + \text{span}\{\vec{y}_1, \ldots, \vec{y}_k\}$$

Points are 0-dimensional affine subspaces. Lines are 1-dimensional affine subspaces. Planes are 2-dimensional affine subspaces. Every affine subspace can be equivalently represented by many choices of vectors $\vec{x}; \vec{y}_1, \ldots, \vec{y}_k$, but, clearly, there is a linear transformation between every two representations of the same affine subspace.

An $m$-variate polynomial over a field $\mathbb{F}$ is a function $Q : \mathbb{F}^m \to \mathbb{F}$ of the form

$$Q(x_1, \ldots, x_m) = \sum_{i_1, \ldots, i_m} a_{i_1, \ldots, i_m} x_1^{i_1} \cdots x_m^{i_m}$$
where all the coefficients \( a_{i_1, \ldots, i_m} \) are in \( F \). The degree of \( Q \) is \( \deg Q = \max \{ \sum_{j=1}^{m} i_j \mid a_{i_1, \ldots, i_m} \neq 0 \} \), where the degree of the identically zero polynomial is defined to be 0.

The restriction of a polynomial \( Q : F^m \rightarrow F \) to an affine subspace \( s \) represented as \( s = \text{affine}(\bar{x}; \bar{y}_1, \ldots, \bar{y}_k) \) is a polynomial in \( k \) variables, \( Q|_s(\alpha_1, \ldots, \alpha_k) = Q(\bar{x} + \alpha_1 \bar{y}_1 + \ldots + \alpha_k \bar{y}_k) \).

We will sometimes wish to refer to a polynomial \( Q \) defined over an affine subspace \( s \) without specifying the subspace’s representation, in which case we will use the notation \( Q(\bar{x}) \) for a point \( \bar{x} \in s \). Note that the degree of a polynomial does not depend on the representation.

2.2 Oracles

We assume an oracle \( A \) that given any affine subspace \( s \) in \( F^m \), provides a polynomial \( A(s) \) of degree at most \( d \) defined over \( s \). For the sake of simplicity, we do not refer to both an oracle \( A \) and a function \( f : F^m \rightarrow F \) as in the introduction. Instead, we assume that \( f \)’s values on points \( \bar{x} \) are given by \( A(\bar{x}) \). Our testers query \( A \) only on affine subspaces of constant dimension. However, for the analysis, it will be convenient to consider oracles queried regarding higher dimensional affine subspaces as well. Hence, an oracle \( A \) is defined to provide a value for any affine subspace.

For a polynomial \( Q : F^m \rightarrow F \), we will use the notation \( (Q \equiv A)(s) \) to indicate that \( Q \) and \( A \) identify on a subspace \( s \), i.e., for every \( \bar{x} \in s \), \( Q(\bar{x}) = A(s)(\bar{x}) \).

2.3 Low Degree Testers

Define two predicates for our two testers: for \( \bar{z} \in F^m \) and \( \bar{y}_1, \bar{y}_2 \in H^m \) chosen uniformly at random,

1. PlanePoint\(^A\)(\( \bar{z}; \bar{y}_1, \bar{y}_2 \)): \( \bar{y}_1, \bar{y}_2 \) are linearly dependent or \( A(\text{affine}(\bar{z}; \bar{y}_1, \bar{y}_2))(\bar{z}) = A(\bar{z}) \)

2. SpacePoint\(^A\)(\( \bar{z}; \bar{y}_1, \bar{y}_2 \)): \( \bar{z}, \bar{y}_1, \bar{y}_2 \) are linearly dependent or \( A(\text{affine}(\bar{z}, \bar{z}; \bar{y}_1, \bar{y}_2))(\bar{z}) = A(\bar{z}) \)

2.4 Soundness

To prove that a tester is sound we assume that it accepts with probability \( \gamma \) when given access to an oracle \( A \) and show the proximity of \( A \) to a low degree polynomial. Specifically, for a sub-constant \( \varepsilon \), we prove two claims, which we argue to be essentially equivalent:

1. (decoding) There exists a low degree polynomial that is consistent with the oracle \( A \) on at least \( \gamma - \varepsilon \) of the points.

2. (list decoding) For every \( 0 < \delta < 1 \), there exists a short list of \( t = t(\delta) \) low degree polynomials that explains all the tester’s success, but \( \delta + \varepsilon \) (explanation follows).

When saying that a list of polynomials explains almost all the success, we mean that with high probability over the random bits of the tester (i.e., over the choice of a subspace and a point within it), either the tester rejects or one of the polynomials identifies with the oracle on the subspace and on the point. There is a tradeoff between the amount of success explained and the length of the list: the more one wishes to explain – the longer the list is.

We wish \( \varepsilon \) to be as small as possible. The parameter \( \varepsilon \) we achieve depends on \( \frac{md}{|F|} \). This comes from the use of the Schwartz-Zippel lemma. It also depends on \( \frac{1}{|H|} \) which is the price we pay for considering the subfield \( H \) instead of the entire field \( F \).
The statement for the Randomness-Efficient Plane vs. Point tester is as follows. Note that we make no effort to optimize the constants.

**Theorem 1 (Plane vs. Point Soundness).** Fix a dimension $m \geq 2$, a field $F$, a subfield $\mathbb{H} \subseteq F$ and a degree $d$. Denote $\varepsilon = 2^m \left( \frac{1}{|\mathbb{H}|} + \frac{m}{|F|} \right)$. For every oracle $A$ and every success probability $0 < \gamma \leq 1$, satisfying

$$\Pr_{\bar{x} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} \left[ \text{PlanePoint}^A(\bar{x}, \bar{y}_1, \bar{y}_2) \right] = \gamma$$

The following hold:

1. **(Decoding)** There exists a polynomial $Q : \mathbb{F}^m \to \mathbb{F}$ with $\deg Q \leq md$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^m} \left[ Q(\bar{x}) = A(\bar{x}) \right] \geq \gamma - \varepsilon$$

2. **(List decoding)** For every $\delta > 2\varepsilon$, there exist $t \leq 2/\delta$ polynomials $Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F}$ with $\deg Q_i \leq md$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} \left[ \neg \text{PlanePoint}^A(\bar{x}, \bar{y}_1, \bar{y}_2) \lor \exists i \left( Q_i \equiv A \right) (\text{affine}(\bar{z}; \bar{y}_1, \bar{y}_2)) \right] \geq 1 - \delta - 2\varepsilon$$

We prove a similar theorem for the Randomness-Efficient Subspace vs. Point tester. Note that for this tester we manage to show proximity to polynomials of degree at most $d$, rather than $md$.

**Theorem 2 (Subspace vs. Point Soundness).** Fix a dimension $m \geq 3$, a field $F$, a subfield $\mathbb{H} \subseteq F$ and a degree $d$. Denote $\varepsilon = 2^m \left( \frac{1}{|\mathbb{H}|} + \frac{m}{|F|} \right)$. For every oracle $A$ and every success probability $0 < \gamma \leq 1$, satisfying

$$\Pr_{\bar{x} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} \left[ \text{SpacePoint}^A(\bar{x}, \bar{y}_1, \bar{y}_2) \right] = \gamma$$

The following hold:

1. **(Decoding)** There exists a polynomial $Q : \mathbb{F}^m \to \mathbb{F}$ with $\deg Q \leq d$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^m} \left[ Q(\bar{x}) = A(\bar{x}) \right] \geq \gamma - 3\varepsilon$$

2. **(List decoding)** For every $\delta > 3\varepsilon$, there exist $t \leq 2/\delta$ polynomials $Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F}$ with $\deg Q_i \leq d$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} \left[ \neg \text{SpacePoint}^A(\bar{x}, \bar{y}_1, \bar{y}_2) \lor \exists i \left( Q_i \equiv A \right) (\text{affine}(\bar{0}; \bar{z}; \bar{y}_1, \bar{y}_2)) \right] \geq 1 - \delta - 3\varepsilon$$

It is interesting to note that our sampling arguments also imply a converse to the above theorems: if there exists a polynomial $Q : \mathbb{F}^m \to \mathbb{F}$ with $\deg Q \leq d$, such that $\Pr_{\bar{x} \in \mathbb{F}^m} \left[ Q(\bar{x}) = A(\bar{x}) \right] \geq \gamma$, then there exists an oracle $A'$ identifying with $A$ on the points and assigning affine subspaces polynomials of degree at most $d$, such that both our testers accept with probability at least $\gamma - \varepsilon$ when given access to $A'$. 
3 Preliminaries

3.1 Orthogonality and Vector Spaces

Given a vector \( \mathbf{y} \in \mathbb{F}^m \), we write \( \mathbf{y} = (y_1, \ldots, y_m) \). For a series of vectors \( \mathbf{y}_1, \ldots, \mathbf{y}_k \), we write for every \( 1 \leq i \leq k \), \( \mathbf{y}_i = (y_{i,1}, \ldots, y_{i,m}) \).

We define an inner-product between two vectors \( \mathbf{x}, \mathbf{y} \in \mathbb{F}^m \) as \( \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{m} x_i \cdot y_i \). We say that \( \mathbf{x}, \mathbf{y} \) are orthogonal if \( \langle \mathbf{x}, \mathbf{y} \rangle = 0 \).

**Proposition 3.1.** For every \( \mathbf{y} \neq \mathbf{0} \in \mathbb{F}^m \), for every \( c \in \mathbb{F} \),

\[
\Pr_{\mathbf{z} \in \mathbb{H}^m} \left[ \langle \mathbf{z}, \mathbf{y} \rangle = c \right] \leq \frac{1}{|\mathbb{H}|}
\]

*Proof.* As \( \mathbf{y} \neq \mathbf{0} \in \mathbb{F}^m \), there exists \( 1 \leq i \leq m \) such that \( y_i \neq 0 \). For every fixing of all \( \mathbf{z} \)'s coordinates but the \( i \)'th, the condition \( \langle \mathbf{z}, \mathbf{y} \rangle = c \) uniquely determines \( z_i \) to some scalar in \( \mathbb{F} \). This scalar may or may not be in the subfield \( \mathbb{H} \), but, in any case, there exists at most one possibility for \( z_i \in \mathbb{H} \).

**Proposition 3.2.** For every \( \mathbf{y} \neq \mathbf{0} \in \mathbb{F}^m \), for every \( k < m \),

\[
\Pr_{\mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{H}^m} \left[ \mathbf{y} \in \text{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_k\} \mid \ind(\mathbf{y}_1, \ldots, \mathbf{y}_k) \right] \leq \frac{1}{|\mathbb{H}|}
\]

*Proof.* Consider uniformly distributed linearly independent \( \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{H}^m \). Pick uniformly and independently at random a vector \( \mathbf{z} \neq \mathbf{0} \in \mathbb{H}^m \) that is orthogonal to \( \mathbf{y}_1, \ldots, \mathbf{y}_k \) (there exist such vectors since \( k < m \)). Note that for every \( \mathbf{y} \in \text{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_k\} \) it holds that \( \langle \mathbf{z}, \mathbf{y} \rangle = 0 \). By proposition 3.1, since \( \mathbf{z} \) is uniformly distributed over \( \mathbb{H}^m \setminus \{\mathbf{0}\} \), this happens with probability at most \( \frac{1}{|\mathbb{H}|} \).

**Proposition 3.3.** For every subset \( A \subseteq \mathbb{F}^m \), if

\[
\Pr_{\mathbf{y} \in \mathbb{F}^m} \left[ \mathbf{y} \in A \right] > \frac{1}{|\mathbb{F}|}
\]

then there exist linearly independent \( \mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{F}^m \), such that for every \( 1 \leq i \leq m \), \( \mathbf{y}_i \in A \).

*Proof.* Let us prove by induction that for every \( 0 \leq k \leq m \) there exist linearly independent \( \mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{F}^m \), such that for every \( 1 \leq i \leq k \), \( \mathbf{y}_i \in A \). For \( k = 0 \), the claim trivially holds. Suppose that the claim holds for \( k < m \), and let \( \mathbf{y}_1, \ldots, \mathbf{y}_k \) be the appropriate vectors. Clearly,

\[
\Pr_{\mathbf{y} \in \mathbb{F}^m} \left[ \mathbf{y} \in \text{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_k\} \right] = \frac{|\mathbb{F}|^k}{|\mathbb{F}|^m} \leq \frac{1}{|\mathbb{F}|}
\]

Thus, there exists \( \mathbf{y} \in A \) such that \( \mathbf{y}_1, \ldots, \mathbf{y}_k, \mathbf{y} \) are linearly independent.

3.2 Polynomials

The Schwartz-Zippel lemma shows that different low degree polynomials differ on most points.

**Proposition 3.4 (Schwartz-Zippel).** For two different polynomials \( Q, P : \mathbb{F}^m \to \mathbb{F} \) with \( \deg Q, \deg P \leq d \),

\[
\Pr_{\mathbf{x} \in \mathbb{F}^m} \left[ Q(\mathbf{x}) = P(\mathbf{x}) \right] \leq \frac{d}{|\mathbb{F}|}
\]
The Schwartz-Zippel lemma can be viewed as showing the unique-decoding property of the Reed-Muller code. This immediately implies a list decoding property, namely, that only few polynomials can identify with a function on many of the points.

We include a simple proof of that property for the sake of self-containment.

**Proposition 3.5 (list decoding of Reed-Muller).** For every function \( f : \mathbb{F}^m \to \mathbb{F} \), if there are \( t \) different polynomials \( Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F} \) such that for every \( 1 \leq i \leq t \),

1. \( \deg Q_i \leq d \)
2. \( \Pr_{\bar{x} \in \mathbb{F}^m} [ f(\bar{x}) = Q_i(\bar{x}) ] \geq \rho \)

where \( \rho \geq 2 \sqrt{\frac{d}{|\mathbb{F}|}} \), then \( t \leq \frac{2}{\rho} \).

**Proof.** For every \( 1 \leq i \leq t \), let \( A_i = \{ \bar{x} \in \mathbb{F}^m \mid f(\bar{x}) = Q_i(\bar{x}) \} \). By inclusion-exclusion,

\[
|\mathbb{F}^m| \geq \left| \bigcup_{i=1}^{t} A_i \right| \geq \sum_{i=1}^{t} |A_i| - \sum_{i \neq j} |A_i \cap A_j|
\]

By Schwartz-Zippel, for every \( 1 \leq i \neq j \leq t \), \( |A_i \cap A_j| \leq \frac{d}{|\mathbb{F}|} \cdot |\mathbb{F}^m| \). Therefore, by the premise,

\[
|\mathbb{F}^m| \geq t \rho |\mathbb{F}^m| - \left( \binom{t}{2} \frac{d}{|\mathbb{F}|} |\mathbb{F}^m| \right)
\]

By the bound on \( \rho \), \( t \leq \frac{2}{\rho} \).


4 **Affine Subspaces With Directions Over a Subfield**

In this section we prove basic facts regarding affine subspaces in \( \mathbb{F}^m \) that are spanned by directions over a subfield \( \mathbb{H} \subseteq \mathbb{F} \). All the properties we prove for such subspaces are well known when \( \mathbb{H} = \mathbb{F} \).

For \( 0 \leq k \leq m \), consider the set of representations of affine subspaces with directions over a subfield,

\[
\mathcal{R}_k^m = \{ (\bar{z}; \bar{y}_1, \ldots, \bar{y}_k) \mid \bar{z} \in \mathbb{F}^m, \bar{y}_1, \ldots, \bar{y}_k \in \mathbb{H}^m, \text{ind}(\bar{y}_1, \ldots, \bar{y}_k) \}
\]

The corresponding set of affine subspaces is

\[
\mathcal{S}_k^m = \{ \text{affine}(r) \mid r \in \mathcal{R}_k^m \}
\]

First we would like to assert that every subspace in \( \mathcal{S}_k^m \) is associated with the same number of tuples in \( \mathcal{R}_k^m \), and that every subspace in \( \mathcal{S}_k^m \) contains the same number of subspaces in \( \mathcal{S}_{k'}^m \) for \( k' \leq k \).

**Proposition 4.1 (uniformity).** For every dimension \( k \), there is a number \( T = T(k) \), such that for every \( s \in \mathcal{S}_k^m \), \( |\{ r \in \mathcal{R}_k^m \mid s = \text{affine}(r) \}| = T \).

**Proposition 4.2 (uniformity downwards).** For every dimension \( k' \leq k \), there is a number \( T = T(k, k') \), such that for every \( s \in \mathcal{S}_k^m \), \( |\{ s' \in \mathcal{S}_{k'}^m \mid s' \subseteq s \}| = T \).
To prove both assertions we introduce an additional notation allowing us to refer to affine subspaces in $\mathcal{S}^{m}_{k}$ as isomorphic copies of $\mathbb{F}^{k}$. Fix an affine subspace together with a representation for it, $s = \text{affine}(\bar{z}; \bar{y}_{1}, \ldots, \bar{y}_{k})$. For a representation $r = (\bar{a}_{0}; \bar{a}_{1}, \ldots, \bar{a}_{k'})$ of an affine subspace within $\mathbb{F}^{k}$, we define the representation $r$ relative to the space $s$ by

$$r_{s} = \left(\bar{z} + \sum_{i=1}^{k} \bar{a}_{0,i}\bar{y}_{i}; \sum_{i=1}^{k} \bar{a}_{1,i}\bar{y}_{i}, \ldots, \sum_{i=1}^{k} \bar{a}_{k',i}\bar{y}_{i}\right)$$

Note that since $\bar{y}_{1}, \ldots, \bar{y}_{k}$ are linearly independent, if two representations $r, r'$ are the same relative to a subspace $s$, $r_{s} = r'_{s}$, then they are the same representation $r = r'$.

Denote the corresponding relative affine subspace:

$$\text{affine}_{s}(r) = \text{affine}(r_{s})$$

Note that for every $r$, $\text{affine}_{s}(r) \subseteq s$. Moreover, if $\text{affine}(r) = \text{affine}(r')$ then $\text{affine}_{s}(r) = \text{affine}_{s}(r')$. Now, the above two propositions follow from the following proposition:

**Proposition 4.3.** For every subspace $s \in \mathcal{S}^{m}_{k}$, for every dimension $k' \leq k$,

$$S_{1} = \left\{ r \in \mathcal{R}^{m}_{k'} \mid \text{affine}(r) \subseteq s \right\} = \left| \mathcal{R}^{k}_{k'} \right| \equiv S_{2}$$

**Proof.** Fix a subspace $s \in \mathcal{S}^{m}_{k}$ and fix a tuple $(\bar{z}; \bar{y}_{1}, \ldots, \bar{y}_{k}) \in \mathcal{R}^{m}_{k}$ with $s = \text{affine}(\bar{z}; \bar{y}_{1}, \ldots, \bar{y}_{k})$.

1. $S_{1} \geq S_{2}$: for every tuple $r = (\bar{a}_{0}; \bar{a}_{1}, \ldots, \bar{a}_{k'}) \in \mathcal{R}^{k}_{k'}$, the tuple $r_{s}$ satisfies $r_{s} \in \mathcal{R}^{m}_{k'}$ and $\text{affine}(r_{s}) \subseteq s$.

2. $S_{1} \leq S_{2}$: for every tuple $r \in \mathcal{R}^{m}_{k'}$ satisfying $\text{affine}(r) \subseteq s$, there exists exactly one $\alpha = (\bar{a}_{0}; \bar{a}_{1}, \ldots, \bar{a}_{k'})$, $\bar{a}_{0}, \bar{a}_{1}, \ldots, \bar{a}_{k'} \in \mathbb{F}^{k}$, $\text{ind}(\bar{a}_{1}, \ldots, \bar{a}_{k'})$, such that $r = \alpha_{s}$. Since $r \in \mathcal{R}^{k'}_{k'}$ and $\bar{y}_{1}, \ldots, \bar{y}_{k} \in \mathbb{H}^{m}$, also $\bar{a}_{1}, \ldots, \bar{a}_{k'} \in \mathbb{H}^{k}$.

Every subspace in $\mathcal{S}^{m}_{k}$ is contained in the same number of subspaces in $\mathcal{S}^{m}_{k'}$ for $k' \geq k$.

**Proposition 4.4 (uniformity upwards).** For every dimensions $k \leq k' \leq m$, there is a number $T = T(m, k, k')$, such that for every subspace $s \in \mathcal{S}^{m}_{k}$,

$$\left| \left\{ s' \in \mathcal{S}^{m}_{k'} \mid s' \supseteq s \right\} \right| = T$$

**Proof.** Let us introduce an additional piece of notation: $\mathcal{L}^{m}_{k'}$ is the set of all linear subspaces of dimension $k'$ in $\mathbb{F}^{m}$ spanned by vectors from $\mathbb{H}^{m}$.

Fix $s = \text{affine}(\bar{z}; \bar{y}_{1}, \ldots, \bar{y}_{k}) \in \mathcal{S}^{m}_{k}$. Since $\bar{y}_{1}, \ldots, \bar{y}_{k} \in \mathbb{H}^{m}$ are linearly independent, the proposition will clearly follow if we prove the following:

$$S_{1} = \left| \left\{ s' \in \mathcal{S}^{m}_{k'} \mid s' \supseteq s \right\} \right| = \left| \left\{ Y' \in \mathcal{L}^{m}_{k'} \mid Y' \supseteq \{\bar{y}_{1}, \ldots, \bar{y}_{k}\} \right\} \right| \equiv S_{2}$$

1. $S_{1} \leq S_{2}$: Let $s' = \text{affine}(\bar{z}; \bar{y}_{1}, \ldots, \bar{y}_{k'}) \in \mathcal{S}^{m}_{k}$, $(\bar{z}; \bar{y}_{1}, \ldots, \bar{y}_{k'}) \in \mathcal{R}^{m}_{k}$, $s' \supseteq s$. Let $Y' = \text{span}\{\bar{y}_{1}, \ldots, \bar{y}_{k}\}$. Clearly, $Y'$ is in $\mathcal{L}^{m}_{k}$ and $Y'$ is uniquely defined by $s'$, $s' = \bar{z} + Y'$. It holds that $\bar{z} \in s \subseteq s' = \bar{z} + Y'$, thus $\bar{z} \in \bar{z} + Y'$, and, hence, $s' = \bar{z} + Y'$. Let $1 \leq i \leq k$. It holds that $\bar{z} + \bar{y}_{i} \in s \subseteq s'$. This implies that $\bar{z} + \bar{y}_{i} \in \bar{z} + Y'$. Hence, $\bar{y}_{i} \in Y'$. Therefore, $\{\bar{y}_{1}, \ldots, \bar{y}_{k}\} \subseteq Y'$. 

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2. \( S_1 \geq S_2 \): Let \( Y' \in \mathcal{L}_k^m \), \( Y' \supseteq \{ \bar{y}_1, \ldots, \bar{y}_k \} \). Clearly, \( \bar{z} + Y' \in \mathcal{S}_k^m \) and \( s \subseteq \bar{z} + Y' \). 

Uniformity is so important because it allows us to count in several manners. A simple argument of this nature is that the fraction of affine subspaces \( s \in \mathcal{S}_k^m \) satisfying some condition is exactly the same as the fraction of \( r \in \mathbb{R}_k^m \) such that affine\( (r) \) satisfies the condition. Let us demonstrate a more sophisticated argument of this nature. Fix \( k' \leq k \). Suppose that we have a predicate \( R \) indicating whether an affine subspace \( s \in \mathcal{S}_k^m \) and an affine subspace \( s' \in \mathcal{S}_k' \) contained in it, \( s' \subseteq s \), satisfy some relation. Then,

\[
E_s \left[ \Pr_{s' \subseteq s} \left[ R(s, s') \right] \right] = E_{s'} \left[ \Pr_{s \supseteq s'} \left[ R(s, s') \right] \right]
\]

A useful representation of affine subspaces is given in the following proposition,

Proposition 4.5 (affine subspaces as solutions of linear equations). Let \( s = \text{affine}(\bar{z}; \bar{y}_1, \ldots, \bar{y}_k) \in \mathcal{S}_k^m \), let \( \bar{\alpha}_1, \ldots, \bar{\alpha}_{m-k} \in \mathbb{H}^m \) be \((m-k)\) linearly independent vectors orthogonal to \( \bar{y}_1, \ldots, \bar{y}_k \in \mathbb{H}^m \). Then,

\[ s = \{ \bar{x} \in \mathbb{F}^m \mid \forall 1 \leq j \leq m-k, \; (\bar{x}, \bar{\alpha}_j) = (\bar{z}, \bar{\alpha}_j) \} \]

Proof. Fix \( \bar{x} \in s \). Hence, there exists \( \bar{c} \in \mathbb{F}^k \) such that \( \bar{x} = \bar{z} + \sum_{i=1}^{k} c_i \bar{y}_i \). For every \( 1 \leq j \leq m-k \),

\[ (\bar{x}, \bar{\alpha}_j) = \left( \bar{z} + \sum_{i=1}^{k} c_i \bar{y}_i, \bar{\alpha}_j \right) = (\bar{z}, \bar{\alpha}_j) + \sum_{i=1}^{k} c_i \cdot (\bar{y}_i, \bar{\alpha}_j) = (\bar{z}, \bar{\alpha}_j) \]

Thus, \( s \subseteq \{ \bar{x} \in \mathbb{F}^m \mid \forall 1 \leq j \leq m-k, \; (\bar{x}, \bar{\alpha}_j) = (\bar{z}, \bar{\alpha}_j) \} \). The proposition follows noticing that, in addition, the two sets are of size \( |\mathbb{F}^k| \).

Using this dual representation, we can easily conclude closedness under intersection,

Proposition 4.6 (closedness under intersection). If \( s_1 \in \mathcal{S}_k^{m(1)} \) and \( s_2 \in \mathcal{S}_k^{m(2)} \) where \( s_1 \cap s_2 \neq \phi \), then there exists \( k^{(3)} \) such that \( s_1 \cap s_2 \in \mathcal{S}_k^{m(3)} \).

Proof. For \( i \in \{1, 2\} \), denote \( s_i = \text{affine}(\bar{z}_i^{(i)}; \bar{y}_1^{(i)}, \ldots, \bar{y}_k^{(i)}) \), where \( \bar{y}_1^{(i)}, \ldots, \bar{y}_k^{(i)} \in \mathbb{H}^m \). Let \( \bar{\alpha}_1^{(i)}, \ldots, \bar{\alpha}_{m-k}^{(i)} \in \mathbb{H}^m \) be \((m-k)^{(i)}\) linearly independent vectors orthogonal to \( \bar{y}_1^{(i)}, \ldots, \bar{y}_k^{(i)} \). Let \( \bar{z}^{(3)} \in s_1 \cap s_2 \), and let \( \bar{\alpha}_1^{(3)}, \ldots, \bar{\alpha}_{m-k}^{(3)} \in \mathbb{H}^m \) be a maximal subset of linearly independent vectors among \( \bar{\alpha}_1^{(1)}, \ldots, \bar{\alpha}_{m-k}^{(1)}, \bar{\alpha}_1^{(2)}, \ldots, \bar{\alpha}_{m-k}^{(2)} \). By proposition 4.5 applied on \( s_1 \) and \( s_2 \) and by the definition of intersection, \( s_1 \cap s_2 = \{ \bar{x} \in \mathbb{F}^m \mid \forall 1 \leq j \leq m-k^{(3)}, \; (\bar{x}, \bar{\alpha}_j^{(3)}) = (\bar{z}^{(3)}, \bar{\alpha}_j^{(3)}) \} \subseteq \mathcal{S}_k^{m(3)} \).

5 Affine Subspaces With Directions Over A Subfield Sample Well

We say that an affine subspace \( s \) in \( \mathbb{F}^m \) samples a set \( A \subseteq \mathbb{F}^m \) well if the fraction of points from \( A \) contained in it, i.e., \( \frac{|s \cap A|}{|A|} \), is approximately \( \frac{|A|}{|\mathbb{F}^m|} \). We say that a distribution \( D \) on affine subspaces in \( \mathbb{F}^m \) samples well, if no matter how one fixes a large enough subset \( A \subseteq \mathbb{F}^m \), a random subspace \( s \sim D \) samples \( A \) well with high probability. In this section we use Fourier analysis to show that the distributions induced by our testers sample well.
5.1 Fourier Transform

Let \((G, \cdot)\) be a finite Abelian group. Consider functions from the group to the complex numbers \(f : G \to \mathbb{C}\). One example for such a function is the indicator function of a subset \(A \subseteq G\),

\[
I_A(x) = \begin{cases} 
1 & x \in A \\
0 & \text{otherwise} 
\end{cases}
\]

Similarly, the indicator function of a multi-set assigns each \(x \in G\) its multiplicity.

We define an inner-product between functions \(f, g : G \to \mathbb{C}\) as

\[
\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}
\]

A character of \(G\) is a homomorphism \(\chi : G \to \mathbb{C}^\ast\), where \(\mathbb{C}^\ast\) is the multiplicative group of the complex numbers. Namely, for every \(x, y \in G\),

\[
\chi(x + y) = \chi(x) \cdot \chi(y)
\]

Every group \(G\) trivially has the identically 1 function as a character.

It can be shown that the set of all characters of \(G\) forms an orthonormal basis for the space of all functions \(f : G \to \mathbb{C}\) under the predefined inner-product. Hence, every function \(f : G \to \mathbb{C}\) can be equivalently represented as \(f(x) = \sum_\chi \hat{f}(\chi) \cdot \chi(x)\), where \(\hat{f}(\chi) = \langle f, \chi \rangle\) is called the Fourier coefficient of \(f\) corresponding to the character \(\chi\). The linear transformation from \(f\) to \(\hat{f}\) is called the Fourier transform of \(f\).

We will need two basic facts regarding the Fourier transform:

**Proposition 5.1 (Parseval’s identity).** For two functions \(f, g : G \to \mathbb{C}\),

\[
\langle f, g \rangle = |G| \cdot \langle \hat{f}, \hat{g} \rangle = \sum_\chi \hat{f}(\chi) \overline{\hat{g}(\chi)}
\]

Define the convolution of two functions, \(f, g : G \to \mathbb{C}\), denoted \((f * g) : G \to \mathbb{C}\), as \((f * g)(x) = \frac{1}{|G|} \sum_{y \in G} f(y)g(x - y)\).

**Proposition 5.2 (convolution formula).** Fix two functions, \(f, g : G \to \mathbb{C}\). For every character \(\chi\) of \(G\),

\[
(f * g)(\chi) = \hat{f}(\chi) \cdot \hat{g}(\chi)
\]

We focus on the additive group \(G = \mathbb{F}^m\) for some finite field \(\mathbb{F}\). In this context, when referring to a vector \(\vec{x} \in \mathbb{F}^m\) we usually omit the vector sign. It can be shown that for every \(\alpha \in \mathbb{F}^m\), there is a character \(\chi_\alpha : \mathbb{F}^m \to \mathbb{F}\),

\[
\chi_\alpha(x) = \omega^{(\alpha, x)}
\]

where \(\omega = e^{2\pi i / |\mathbb{F}|}\) is the \(|\mathbb{F}|\)th primitive root of unity in \(\mathbb{C}\). For a function \(f : \mathbb{F}^m \to \mathbb{C}\), we denote its Fourier coefficient corresponding to the character \(\chi_\alpha\) by \(\hat{f}(\alpha)\).
5.2 Sampling Lemma

All our sampling conclusions will be proven using the following lemma. Given \( z, y \in \mathbb{F}^k \) and a subset \( A \subseteq \mathbb{F}^k \), define \( I_{x,y} \) to be the number of \( c \in \mathbb{F} \) satisfying \( z + c \cdot y \in A \). Clearly, the expectation of \( I_{x,y} \) when picking independently at random \( z \in \mathbb{F}^k \) and \( y \in \mathbb{H}^k \) is \( |\mathbb{F}| \cdot \frac{|A|}{|\mathbb{F}^k|} \). We show that the random variable \( I_{x,y} \) is concentrated around its expectation. This is done by bounding its variance. The proof uses Fourier analysis.

**Lemma 5.3.** Fix \( k \geq 1 \). For any subset \( A \subseteq \mathbb{F}^k \) of density \( \mu = |A|/|\mathbb{F}^k| \),

\[
\Var_{z \in \mathbb{F}^k, y \in \mathbb{H}^k} [I_{x,y}] \leq |\mathbb{F}|^2 \frac{\mu}{|\mathbb{H}|}
\]

**Proof.** If we denote the indicator function of \( A \) by \( I_A \), and the indicator function of the multi-set \( \{c \cdot y \mid c \in \mathbb{F}\} \) by \( I_{\mathbb{F}y} \), we can express:

\[
I_{x,y} = \sum_{x \in \mathbb{F}^k} I_A(x) I_{\mathbb{F}y}(z - x) = |\mathbb{F}^k| \cdot (I_A * I_{\mathbb{F}y})(z)
\]

Hence, by Parseval’s identity and the convolution formula,

\[
\mathbb{E}_{z \in \mathbb{F}^k, y \in \mathbb{H}^k} [I_{x,y}^2] = \frac{1}{|\mathbb{F}^k|} \cdot \sum_{y \in \mathbb{H}^k} \sum_{z \in \mathbb{F}^k} \left( |\mathbb{F}^k| (I_A * I_{\mathbb{F}y})(z) \right)^2
\]

\[
= \frac{|\mathbb{F}^k|^2}{|\mathbb{H}^k|} \cdot \sum_{y \in \mathbb{H}^k} \sum_{\alpha \in \mathbb{F}^k} (I_A * I_{\mathbb{F}y})(\alpha)^2
\]

\[
= \frac{|\mathbb{F}^k|^2}{|\mathbb{H}^k|} \cdot \sum_{y \in \mathbb{H}^k} \sum_{\alpha \in \mathbb{F}^k} I_A(\alpha)^2 \cdot |I_{\mathbb{F}y}(\alpha)|^2
\]

By definition,

\[
\hat{I}_A(0) = \langle I_A, \chi_0 \rangle = \frac{1}{|\mathbb{F}|} \cdot \sum_{z \in \mathbb{F}^k} I_A(z) \cdot \omega^{-(0,z)} = \frac{1}{|\mathbb{F}|} \cdot \sum_{z \in \mathbb{F}^k} I_A(z) = \frac{|A|}{|\mathbb{F}|}
\]

In addition,

\[
|\hat{I}_{\mathbb{F}y}(\alpha)| = |\langle I_{\mathbb{F}y}, \chi_\alpha \rangle| = \left| \frac{1}{|\mathbb{F}|} \cdot \sum_{z \in \mathbb{F}^k} I_{\mathbb{F}y}(z) \omega^{-(\alpha,z)} \right| = \left| \frac{1}{|\mathbb{F}|} \cdot \sum_{c \in \mathbb{F}} \omega^{-c(\alpha,y)} \right| = \begin{cases} 0 \quad (\alpha, y) \neq 0 \\ \frac{|\mathbb{F}|}{|\mathbb{F}|} \quad \text{otherwise} \end{cases}
\]

Therefore,

\[
\mathbb{E}_{z \in \mathbb{F}^k, y \in \mathbb{H}^k} [I_{x,y}^2] = \frac{|\mathbb{F}^k|^2}{|\mathbb{H}^k|} \cdot \sum_{y \in \mathbb{H}^k} \left( |\hat{I}_A(0)|^2 \cdot |\hat{I}_{\mathbb{F}y}(\bar{0})|^2 + \sum_{\alpha \neq \bar{0} \in \mathbb{F}^k} |\hat{I}_A(\alpha)|^2 \cdot |\hat{I}_{\mathbb{F}y}(\alpha)|^2 \right)
\]

\[
= \left( \frac{|\mathbb{F}|}{|\mathbb{F}^k|} \cdot \frac{|A|}{|\mathbb{F}|} \right)^2 + |\mathbb{F}^k|^2 \cdot \sum_{\alpha \neq \bar{0} \in \mathbb{F}^k} \left( |\hat{I}_A(\alpha)|^2 \cdot \frac{1}{|\mathbb{F}|} \cdot \sum_{y \in \mathbb{H}^k} |I_{\mathbb{F}y}(\alpha)|^2 \right)
\]
Note that for every $\alpha \neq 0 \in \mathbb{F}^k$, if $S = \{ y \in \mathbb{H}^k \mid (\alpha, y) = 0 \}$, then, by proposition 3.1,

$$
\frac{1}{|\mathbb{H}^k|} \cdot \sum_{y \in \mathbb{H}^k} |\mathcal{I}_{Fy}(\alpha)|^2 = \frac{1}{|\mathbb{H}^k|} \cdot \left( \sum_{y \in \mathbb{H}^k \setminus S} |\mathcal{I}_{Fy}(\alpha)|^2 + \sum_{y \in S} |\mathcal{I}_{Fy}(\alpha)|^2 \right)
$$

$$
= \frac{1}{|\mathbb{H}^k|} \cdot \sum_{y \in S} |\mathcal{I}_{Fy}(\alpha)|^2
\leq \frac{1}{|\mathbb{H}|} \cdot \frac{|\mathbb{F}|^2}{|\mathbb{F}^k|^2}
$$

We can substitute and apply Parseval’s identity again to get:

$$
\mathbb{E}_{z \in \mathbb{F}^k, y \in \mathbb{H}^k} [X_{z,y}^2] \leq \left( \frac{|\mathbb{F}| |A|}{|\mathbb{F}^k|} \right)^2 + \frac{|\mathbb{F}|^2}{|\mathbb{H}|} \cdot \sum_{\alpha \neq 0 \in \mathbb{F}^k} |\mathcal{I}_A(\alpha)|^2
\leq \left( \frac{|\mathbb{F}| |A|}{|\mathbb{F}^k|} \right)^2 + \frac{|\mathbb{F}|^2}{|\mathbb{H}|} \cdot \frac{1}{|\mathbb{F}^k|} \cdot \sum_{z \in \mathbb{F}^k} |\mathcal{I}_A(z)|^2
= \left( \frac{|\mathbb{F}| |A|}{|\mathbb{F}^k|} \right)^2 + \frac{|\mathbb{F}|^2}{|\mathbb{H}|} \cdot \frac{|A|}{|\mathbb{F}^k|}
$$

On the other hand, by linearity of expectations,

$$
\mathbb{E}_{z \in \mathbb{F}^k, y \in \mathbb{H}^k} [X_{z,y}] = \frac{|\mathbb{F}| |A|}{|\mathbb{F}^k|}
$$

Therefore,

$$
\text{Var}_{z \in \mathbb{F}^k, y \in \mathbb{H}^k} [X_{z,y}] = \mathbb{E}_{z,y} [X_{z,y}^2] - \mathbb{E}_{z,y} [X_{z,y}]^2
\leq \left( \frac{|\mathbb{F}| |A|}{|\mathbb{F}^k|} \right)^2 + \frac{|\mathbb{F}|^2}{|\mathbb{H}|} \cdot \frac{|A|}{|\mathbb{F}^k|} - \left( \frac{|\mathbb{F}| |A|}{|\mathbb{F}^k|} \right)^2
= \frac{|\mathbb{F}|^2}{|\mathbb{H}|} \cdot \frac{\mu}{|\mathbb{F}^k|}
\$$

5.3 Affine Subspaces Sample Well

Using the sampling lemma (lemma 5.3), we can prove that the uniform distribution over lines in $\mathcal{S}^m_1$ samples well. Note that the sampling lemma does not show exactly this, as it considers $y$ uniformly distributed over $\mathbb{H}^m$, instead of over $\mathbb{H}^m \setminus \{0\}$.

Lemma 5.4. For any $A \subseteq \mathbb{F}^m$ of density $\mu = |A|/|\mathbb{F}^m|$,

$$
\text{Var}_{l \in \mathcal{S}^m_1} \left[ \frac{|l \cap A|}{|l|} \right] \leq \frac{\mu}{|\mathbb{H}|}
$$
Proof. Note that the probability that a random point in \( \mathbb{F}^m \) is in \( A \) is the same as the expected fraction of points in \( A \) on a random line in \( \mathbb{S}^{m-1} \),

\[
\mathbb{E}_{p \in \mathbb{S}^{m-1}} \left[ \frac{|p \cap A|}{|p|} \right] = \mathbb{E}_{l \in \mathbb{S}^{m-1}} \left[ \frac{|l \cap A|}{|l|} \right] = \mu
\]

but the variance may only decrease when considering lines rather than points,

\[
\text{Var}_{p \in \mathbb{S}^{m-1}} \left[ \frac{|p \cap A|}{|p|} \right] \geq \text{Var}_{l \in \mathbb{S}^{m-1}} \left[ \frac{|l \cap A|}{|l|} \right]
\]

Hence, since \( \text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] \) and by linearity of expectations,

\[
\begin{align*}
\text{Var}_{z \in \mathbb{F}^m, y \in \mathbb{H}^m} \left[ \frac{1}{|H|} \cdot X_{z,y} \right] & = \frac{1}{|H|^m} \cdot \text{Var}_{p \in \mathbb{S}^{m-1}} \left[ \frac{|p \cap A|}{|p|} \right] + \left(1 - \frac{1}{|H|^m}\right) \cdot \text{Var}_{l \in \mathbb{S}^{m-1}} \left[ \frac{|l \cap A|}{|l|} \right] \\
& \geq \frac{1}{|H|^m} \cdot \text{Var}_{l \in \mathbb{S}^{m-1}} \left[ \frac{|l \cap A|}{|l|} \right] + \left(1 - \frac{1}{|H|^m}\right) \cdot \text{Var}_{l \in \mathbb{S}^{m-1}} \left[ \frac{|l \cap A|}{|l|} \right] \\
& = \text{Var}_{l \in \mathbb{S}^{m-1}} \left[ \frac{|l \cap A|}{|l|} \right] \\
& \geq \mu \frac{|H|}{|H|^m}
\end{align*}
\]

The lemma follows from lemma 5.3.

We can now bound the deviation of the hitting rate from its expected value,

**Corollary 5.6 (sampling).** Fix dimensions \( k \) and \( m \), \( 1 \leq k \leq m \). For any \( A \subseteq \mathbb{F}^m \) of density \( \mu = |A|/|\mathbb{F}^m| \),

\[
\text{Var}_{s \in \mathbb{S}^m_k} \left[ \frac{|s \cap A|}{|s|} \right] \leq \frac{\mu}{|H|^m}
\]

Proof. Pick \( s \in \mathbb{S}^m_k \) and additional \( r \in \mathbb{R}^k \) independently at random. Denote by \( l = \text{affine}_s(r) \) the line within \( s \) corresponding to \( r \) (the notation \( \text{affine}_s \) was introduced in section 4). By uniformity, \( l \) is uniformly distributed in \( \mathbb{S}^{m-1} \). Hence, by lemma 5.4,

\[
\begin{align*}
\text{Var}_{s} \left[ \frac{|s \cap A|}{|s|} \right] & = \text{Var}_{s} \left[ \mathbb{E}_{r} \left[ \frac{|l \cap A|}{|l|} \right] \right] \\
& \leq \text{Var}_{s,r} \left[ \frac{|l \cap A|}{|l|} \right] \\
& \leq \frac{\mu}{|H|^m}
\end{align*}
\]

We can now bound the deviation of the hitting rate from its expected value,
5.4 Linear Subspaces Sample Well

We can similarly prove that linear subspaces with one direction chosen from $\mathbb{F}^m$ and all other directions chosen from $\mathbb{H}^m$ sample well. We will need this lemma to analyze the Randomness-Efficient Subspace vs. Point tester.

**Lemma 5.7.** Fix dimensions $k$ and $m$, $1 \leq k < m$. Fix a set $A \subseteq \mathbb{F}^m$ of density $\mu = |A|/|\mathbb{F}^m|$. Pick uniformly $\vec{z} \in \mathbb{F}^m, \vec{y}_1, \ldots, \vec{y}_k \in \mathbb{H}^m$, such that $\vec{z}, \vec{y}_1, \ldots, \vec{y}_k$ are linearly independent. Denote $s = \text{affine}(\vec{0}; \vec{z}, \vec{y}_1, \ldots, \vec{y}_k)$. Then,

$$E_{s, \alpha} \left( \left( \frac{|s \cap A|}{|s_\alpha|} - \mu \right)^2 \right) \leq \frac{\mu}{|\mathbb{H}|} + \frac{1}{|\mathbb{F}|}$$

**Proof.** Pick an additional scalar $\alpha \in \mathbb{F}$ independently at random. Let $s_\alpha = \text{affine}(\alpha \vec{z}; \vec{y}_1, \ldots, \vec{y}_k)$. Note that $s_\alpha$ is distributed in $S^m_k$ as follows: with probability $\frac{1}{|\mathbb{F}|}$, $s_\alpha$ is uniformly distributed in the set of affine subspaces in $S^m_k$ through the origin; with probability $1 - \frac{1}{|\mathbb{F}|}$, $s_\alpha$ is uniformly distributed in the set of affine subspaces in $S^m_k$ that do not contain the origin. Therefore,

$$E_{s_\alpha} \left( \left( \frac{|s_\alpha \cap A|}{|s_\alpha|} - \mu \right)^2 \right) \leq \frac{1}{|\mathbb{F}|} \cdot E_{s, \alpha} \left( \left( \frac{|s \cap A|}{|s_\alpha|} - \mu \right)^2 \right) + \frac{1}{|\mathbb{F}|} \cdot 1$$

By lemma 5.5,

$$E_{s, \alpha} \left( \left( \frac{|s_\alpha \cap A|}{|s_\alpha|} - \mu \right)^2 \right) \leq \frac{\mu}{|\mathbb{H}|} + \frac{1}{|\mathbb{F}|}$$

By Jensen inequality,

$$E_s \left( \left( \frac{|s \cap A|}{|s|} - \mu \right)^2 \right) \leq E_{s, \alpha} \left( \left( \frac{|s_\alpha \cap A|}{|s_\alpha|} - \mu \right)^2 \right)$$

The lemma follows.

We can now bound the deviation of the hitting rate from its expected value,

**Corollary 5.8 (sampling).** Fix dimensions $k$ and $m$, $1 \leq k < m$. Fix a set $A \subseteq \mathbb{F}^m$ of density $\mu = |A|/|\mathbb{F}^m|$. Pick uniformly $\vec{z} \in \mathbb{F}^m, \vec{y}_1, \ldots, \vec{y}_k \in \mathbb{H}^m$, such that $\vec{z}, \vec{y}_1, \ldots, \vec{y}_k$ are linearly independent. Denote $s = \text{affine}(\vec{0}; \vec{z}, \vec{y}_1, \ldots, \vec{y}_k)$. Then, for any $\varepsilon > 0$,

$$\Pr_s \left[ \left| \frac{|s \cap A|}{|s|} - \mu \right| \geq \varepsilon \right] \leq \frac{1}{\varepsilon^2} \cdot \left( \frac{\mu}{|\mathbb{H}|} + \frac{1}{|\mathbb{F}|} \right)$$

**Proof.** Apply Markov inequality on lemma 5.7.

6 Consolidation

In this section we show that weak low degree testing claims imply strong low degree testing claims. Specifically, we are interested in the following (for exact definitions, see the next subsections):
1. **decoding/list decoding**: by **decoding** we refer to finding a single polynomial $Q : \mathbb{F}^m \to \mathbb{F}$ identifying with the oracle on many of the points. By **list-decoding** we refer to finding a short list of polynomials $Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F}$ explaining almost all the acceptance probability of a tester.

2. **consistency**: we are able to construct polynomials $Q : \mathbb{F}^m \to \mathbb{F}$ identifying with the oracle on some fraction of the points, and wish to find polynomials identifying with the oracle on a larger fraction of the points.

3. **degree**: we are able to construct polynomials $Q : \mathbb{F}^m \to \mathbb{F}$ of degree at most $d_0$, and wish to find polynomials of degree at most $d$.

We call such arguments **consolidating arguments**. They are standard in the low degree testing literature (see, e.g., [3, 17, 11]), however, they require some adaptation to our new setting. In the following subsections we provide the statements and the proofs of the exact claims we need.

### 6.1 From Decoding to List-Decoding

If we have a way to decode, then we can list-decode by repeatedly applying decoding. In our setting, it is easy to force the decoding process to output a polynomial that differs from existing polynomials, by modifying the oracle.

**Lemma 6.1 (from decoding to list-decoding).** Assume $|\mathbb{F}| \geq 4$. Fix a distribution $\mathcal{D}$ over affine subspaces of dimension $k > 0$ in $\mathbb{F}^m$. Fix a function $f : \mathbb{R} \to \mathbb{R}$, and a degree $d'$ such that $d \leq d' \leq |\mathbb{F}| - 3$. If

**decoding:**

for every success probability $0 < \gamma \leq 1$ and oracle $\mathcal{A}$,

(much consistency)

$$\mathbb{E}_{s \sim \mathcal{D}} \left[ \Pr_{\bar{x} \in s} [\mathcal{A}(s)(\bar{x}) = \mathcal{A}(\bar{x})] \right] \geq \gamma$$

implies

(a relatively-low degree polynomial that slightly agrees with the oracle)

There exists a polynomial $Q : \mathbb{F}^m \to \mathbb{F}$, with $\deg Q \leq d'$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^m} [Q(\bar{x}) = \mathcal{A}(\bar{x})] \geq f(\gamma)$$

Then

**list-decoding:**

for every success probability $0 < \gamma \leq 1$ and oracle $\mathcal{A}$,

(much consistency)

$$\mathbb{E}_{s \sim \mathcal{D}} \left[ \Pr_{\bar{x} \in s} [\mathcal{A}(s)(\bar{x}) = \mathcal{A}(\bar{x})] \right] \geq \gamma$$

implies

(almost all consistency is explained by a relatively short list),

Fix $\epsilon_0 = \sqrt{\frac{\delta}{|\mathbb{F}|}}$. For every $\epsilon_0 < \delta < 1$, such that $\delta' = f(\delta - \epsilon_0) - \epsilon_0 \geq 2\epsilon_0$, there exists a list of $t \leq \frac{2}{\delta'}$ polynomials $Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F}$ with $\deg Q_i \leq d'$, such that
\[ \mathbf{E}_{s \sim \mathcal{D}} \left[ \Pr_{\bar{x} \sim s} \left[ A(s) (\bar{x}) \neq A(\bar{x}) \right] \vee \exists i \left( Q_i \equiv A(s) \right) \right] \geq 1 - \delta \]

**Proof.** Assume on way of contradiction that decoding holds and there exists an oracle \( A \) for which the premise of list-decoding holds as well, however there exists \( \epsilon_0 < \delta < 1 \) satisfying \( f(\delta - \epsilon_0) - \epsilon_0 \geq 2\epsilon_0 \), such that there is no list-decoding for \( \delta \).

Let \( Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F} \) be all polynomials of degree at most \( d' \) for which

\[ \Pr_{\bar{x} \in \mathbb{F}^m} \left[ Q_i(\bar{x}) = A(\bar{x}) \right] \geq \delta' \]

By proposition 3.5, \( t \leq 2/\delta' \). By our assumption, \( Q_1, \ldots, Q_t \) is not a list-decoding for \( \delta \). Note that \( t \leq 1/\epsilon_0 \).

When picking a subspace \( s \sim \mathcal{D} \) and a point \( \bar{x} \) uniformly distributed in \( s \), define the following events:

1. \( C : A(s)(\bar{x}) = A(\bar{x}) \) (consistent).
2. \( P : \exists i \in [t], A(\bar{x}) = Q_i(\bar{x}) \) (point explained).
3. \( S : \exists i \in [t], (Q_i \equiv A)(s) \) (subspace explained).

In these notations, the contradicting assumption implies that there is much consistency within unexplained subspaces,

\[ \Pr_{s,\bar{x}} [C \land \neg S] = 1 - \Pr_{s,\bar{x}} [-C \lor S] = 1 - \mathbf{E}_s \left[ \Pr_{\bar{x}} [-C \lor S] \right] > \delta \]

By the Schwartz-Zippel lemma, an unexplained subspace is rarely consistent with explained points,

\[ \Pr_{s,\bar{x}} [C \land P \land \neg S] \leq \frac{td'}{|\mathbb{F}|} \leq \frac{1}{\epsilon_0} \cdot \epsilon_0^2 = \epsilon_0 \]

Thus, there is much consistency on unexplained points,

\[ \Pr_{s,\bar{x}} [C \land \neg P] \geq \Pr_{s,\bar{x}} [C \land \neg P \land \neg S] \]

\[ = \Pr_{s,\bar{x}} [C \land \neg S] - \Pr_{s,\bar{x}} [C \land P \land \neg S] \]

\[ \geq \Pr_{s,\bar{x}} [C \land \neg S] - \Pr_{s,\bar{x}} [C \land P \land \neg S] \]

\[ > \delta - \epsilon_0 \]

Pick an arbitrary polynomial \( Q' : \mathbb{F}^m \to \mathbb{F} \) with \( \deg Q' = d' + 1 \). Define a new oracle \( A' \) as follows: \( A' \) assigns \( Q'(\bar{x}) \) to all explained points \( \bar{x} \), and identifies with \( A \) on all other affine subspaces. Hence,

\[ \mathbf{E}_{s \sim \mathcal{D}} \left[ \Pr_{\bar{x} \in s} \left[ A'(s)(\bar{x}) = A'(\bar{x}) \right] \right] \geq \mathbf{E}_{s \sim \mathcal{D}, \bar{x} \in s} \left[ A(s)(\bar{x}) = A(\bar{x}) \land A(\bar{x}) = A'(\bar{x}) \right] \]

\[ \geq \Pr_{s,\bar{x}} [C \land \neg P] \]

\[ > \delta - \epsilon_0 \]

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Thus, by decoding, there exists a polynomial $Q$, $\deg Q \leq d'$, identifying with $A'$ on many of the points

$$\Pr_{\bar{x} \in \mathbb{F}^m} [A' (\bar{x}) = Q (\bar{x})] \geq f (\delta - \epsilon_0)$$

The polynomials $Q$ and $Q'$ are necessarily distinct (they do not have the same degree). Thus, by the Schwartz-Zippel lemma,

$$\Pr_{\bar{x} \in \mathbb{F}^m} [A' (\bar{x}) = Q (\bar{x})] \leq \Pr_{\bar{x} \in \mathbb{F}^m} [Q' (\bar{x}) = Q (\bar{x})] \leq \frac{d' + 1}{|\mathbb{F}|} \leq \epsilon_0$$

Hence,

$$\Pr_{\bar{x} \in \mathbb{F}^m} [A (\bar{x}) = Q (\bar{x})] = \Pr_{\bar{x} \in \mathbb{F}^m} [A' (\bar{x}) = Q (\bar{x})] - \Pr_{\bar{x} \in \mathbb{F}^m} [A' (\bar{x}) = Q (\bar{x}) \land A' (\bar{x}) \neq A (\bar{x})] \geq f (\delta - \epsilon_0) - \epsilon_0 = \delta'$$

Therefore,

$$\Pr_{\bar{x} \in \mathbb{F}^m} [A (\bar{x}) = Q (\bar{x})] \geq \Pr_{\bar{x} \in \mathbb{F}^m} [A (\bar{x}) = Q (\bar{x})] \geq \delta'$$

Hence, there exists $i \in [t]$ such that $Q = Q_i$. However, if this is the case,

$$\delta' \leq \Pr_{\bar{x} \in \mathbb{F}^m} [A (\bar{x}) = Q_i (\bar{x}) = A' (\bar{x})] \leq \Pr_{\bar{x} \in \mathbb{F}^m} [Q' (\bar{x}) = Q (\bar{x})] \leq \epsilon_0$$

Contradiction.

We can additionally demand that each member of the list decoding identifies with the oracle on many of the subspaces, i.e., there are no non-useful members in the list,

**Lemma 6.2 (pruning the list).** Fix a distribution $\mathcal{D}$ over affine subspaces in $\mathbb{F}^m$. For every $0 < \epsilon < 1$ and oracle $A$, if $Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F}$ are $t > 0$ polynomials satisfying

(almost all consistency is explained by the list)

$$\mathbb{E} \left[ \Pr_{\bar{x} \in \mathcal{D}} [A (s) (\bar{x}) \neq A (\bar{x}) \lor \exists i (Q_i \equiv A) (s)] \right] \geq 1 - \delta$$

then there exists a sublist $T \subseteq [t]$, such that

1. (each polynomial identifies with the oracle on many of the subspaces)
   
   for every $i \in T$,
   
   $$\Pr_{s \sim \mathcal{D}} [(Q_i \equiv A) (s)] > \frac{\epsilon}{t}$$

2. (still almost all consistency is explained by the list)

   $$\mathbb{E} \left[ \Pr_{\bar{x} \in \mathcal{D}} [A (s) (\bar{x}) \neq A (\bar{x}) \lor \exists i \in T (Q_i \equiv A) (s)] \right] \geq 1 - \delta - \epsilon$$

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Proof. We prune the given list \( Q_1, \ldots, Q_t \) by throwing away any polynomial \( Q_i \), for which the first item does not hold. In other words,

\[
T \doteq \left\{ i \in [t] \mid \Pr_{s \sim D} [(Q_i \equiv A)(s)] > \frac{\epsilon}{t} \right\}
\]

By the union bound, \( \Pr_{s \sim D} [\exists i \in [t] \setminus T, \ (Q_i \equiv A)(s)] \leq t \cdot \frac{\epsilon}{t} = \epsilon \). For a subspace \( s \sim D \) and a point \( \bar{x} \) uniformly distributed in \( s \), define the following events:

1. \( C : A(s)(\bar{x}) = A(\bar{x}) \) (consistent).
2. \( B : \exists i \in [t], \ (Q_i \equiv A)(s) \) (explained before).
3. \( N : \exists i \in T, \ (Q_i \equiv A)(s) \) (explained now).

In these notations, we have (e.g., by observing the appropriate Venn diagram),

\[
\mathbb{E}_{s, x} \left[ \Pr_{s \sim D, x \sim s} [C \lor N] \right] = \Pr_{s, x} [C \lor N] \\
\geq \Pr_{s, x} [C \lor B] - \Pr_{s, x} [B \land \neg N] \\
\geq 1 - \delta - \epsilon
\]

6.2 Consistency Consolidation

In this subsection, we prove a lemma allowing us to deduce that a significant consistency \( \gamma \) together with a list-decoding for it imply that at least one of the polynomials in the list identifies with the oracle on almost \( \gamma \) fraction of the points. The lemma requires that the distribution over affine subspaces would sample well (see section 5). Together with lemma 6.1 that transforms decoding into list decoding, this lemma allows us to improve the consistency we manage to recover.

We phrase a rather general lemma addressing distributional oracles, instead of oracles. We say that \( A \) is a distributional oracle, if it assigns each affine subspace \( s \) a distribution over functions \( f : s \to \mathbb{F} \) (not necessarily a single polynomial of degree at most \( d \) over \( s \)). Our semantic even permits the distribution to produce a null function with some probability. We interpret a null function as one that does not satisfy any property of the form “the function evaluates to...” (and hence satisfies every property of the form “the function does not evaluate to...”).

Lemma 6.3 (from list-decoding to decoding). Fix a distribution \( D \) over affine subspaces that samples well, i.e., there exists \( \Delta : [0, 1] \to [0, 1] \), such that for every set \( A \subseteq \mathbb{F}^m \), for every \( 0 < \varepsilon < 1 \),

\[
\Pr_{s \sim D} \left[ \left| \frac{|s \cap A|}{|s|} - \frac{|A|}{|\mathbb{F}^m|} \right| \geq \varepsilon \right] \leq \Delta(\varepsilon)
\]

Let \( A \) denote an oracle, and let \( \tilde{A} \) denote a distributional oracle. Assume

1. (the oracles are significantly consistent)

\[
\mathbb{E}_{\tilde{A}} \left[ \mathbb{E}_{s \sim D} \left[ \Pr_{\tilde{x} \sim s} \left[ \tilde{A}(s)(\tilde{x}) = A(\tilde{x}) \right] \right] \right] \geq \gamma
\]
There exist \( f_1, \ldots, f_t : \mathbb{F}^m \to \mathbb{F} \) such that,
\[
\mathbb{E}_A \left[ \mathbb{E}_{s \sim \mathcal{D}} \left[ \Pr_{\vec{x} \sim s} \left[ \vec{A}(s)(\vec{x}) \neq A(\vec{x}) \lor \exists i \left( f_i \equiv \vec{A}(s) \right) \right] \right] \right] \geq 1 - \delta
\]

For any \( 0 < \varepsilon < 1 \) such that \( \varepsilon \geq t \cdot \Delta(\varepsilon) \), there exists \( 1 \leq i \leq t \), such that
\[
\Pr_{\vec{x} \in \mathbb{F}^m} [f_i(\vec{x}) = A(\vec{x})] \geq \gamma - \delta - 2\varepsilon
\]

**Proof.** Assume, on way of contradiction, that for every \( 1 \leq i \leq t \), \( \Pr_{\vec{x} \in \mathbb{F}^m} [f_i(\vec{x}) = A(\vec{x})] < \gamma - \delta - 2\varepsilon \). Let us bound the consistency towards a contradiction to the first item of the premise. For every \( 1 \leq i \leq t \), define the set of points explained by \( f_i \),
\[
A_i = \left\{ \vec{x} \in \mathbb{F}^m \mid f_i(\vec{x}) = A(\vec{x}) \right\}
\]

For every \( 1 \leq i \leq t \), note that \( \mu_i = \frac{|A_i|}{|s|} < \gamma - \delta - 2\varepsilon \).

As \( \mathcal{D} \) samples well, for every \( 1 \leq i \leq t \), a random subspace \( s \sim \mathcal{D} \) is not likely to hit \( A_i \) much more than it is expected,
\[
\Pr_{\vec{s} \sim \mathcal{D}} \left[ \frac{|s \cap A_i|}{|s|} \geq \mu_i + \varepsilon \right] \leq \Delta(\varepsilon) \leq \frac{\varepsilon}{t}
\]

By the union bound,
\[
\Pr_{\vec{s} \sim \mathcal{D}} \left[ \exists i \in [t], \frac{|s \cap A_i|}{|s|} \geq \gamma - \delta - \varepsilon \right] \leq \varepsilon
\]

For a random oracle assignment \( \vec{A} \), a subspace \( s \sim \mathcal{D} \) and a uniformly distributed point \( \vec{x} \in s \) chosen independently at random, define the following events:

1. \( B : \exists i \in [t], |s \cap A_i| \geq (\gamma - \delta - \varepsilon) \cdot |s| \) (bad subspace).
2. \( C : \vec{A}(s)(\vec{x}) = A(\vec{x}) \) (consistent).
3. \( E : \exists i \in [t], (f_i \equiv \vec{A})(s) \) (explained).

In these notations, we have established that
\[
\Pr_{\vec{A}, s, \vec{x}} [C \land E] = \Pr_{\vec{A}, s, \vec{x}} [C \land E \land \neg B] + \Pr_{\vec{A}, s, \vec{x}} [C \land E \land B]
\]
\[
\leq \Pr_{\vec{A}, s, \vec{x}} [C \land E \land \neg B] + \Pr_{s} [B]
\]
\[
< (\gamma - \delta - \varepsilon) + \varepsilon = \gamma - \delta
\]

The second item of the premise implies
\[
\Pr_{\vec{A}, s, \vec{x}} [C] = \Pr_{\vec{A}, s, \vec{x}} [C \land \neg E] + \Pr_{\vec{A}, s, \vec{x}} [C \land E]
\]
\[
< \delta + (\gamma - \delta) = \gamma
\]

This contradicts the first item of the premise.
6.3 Degree Consolidation

Degree consolidation shows that if one reconstructs a polynomial of not too large degree that identifies with the oracle on many of our subspaces then the polynomial’s true degree is, in fact, low. The reason is that the polynomial’s degree does not decrease much when restricted to almost all our subspaces.

First we prove a lemma allowing us to deduce degree $d$ if one of the directions of our subspaces is distributed over $\mathbb{F}^m$ (rather than $\mathbb{H}^m$). This is used only in the analysis of the Randomness-Efficient Subspace vs. Point tester.

**Lemma 6.4 (degree $d$ consolidation).** Fix dimensions $k$ and $m$, $0 \leq k < m$. Fix an oracle $A$ assigning polynomials of degree at most $d$ to all affine subspaces. Suppose that a polynomial $Q : \mathbb{F}^m \rightarrow \mathbb{F}$ satisfies the following for some $0 \leq \delta \leq 1$:

1. $\deg Q \leq \delta |\mathbb{F}|$.

2. When picking independently at random $\bar{z} \in \mathbb{F}^m$ and $\bar{y}_1, \ldots, \bar{y}_k \in \mathbb{H}^m$ such that $\bar{z}, \bar{y}_1, \ldots, \bar{y}_k$ are linearly independent,

\[
\Pr_{\bar{z}, \bar{y}_1, \ldots, \bar{y}_k} \left[ \left( Q \equiv A \right)(\text{affine}(\bar{0}; \bar{z}, \bar{y}_1, \ldots, \bar{y}_k)) \right] > \delta + \frac{1}{|\mathbb{F}|}
\]

Then, $\deg Q \leq d$.

**Proof.** Assume on way of contradiction that $\deg Q > d$. Consider linearly independent $\bar{z} \in \mathbb{F}^m$ and $\bar{y}_1, \ldots, \bar{y}_k \in \mathbb{H}^m$. Denote $s = \text{affine}(\bar{0}; \bar{z}, \bar{y}_1, \ldots, \bar{y}_k)$, and observe the polynomial

\[Q|_s(\alpha_0, \alpha_1, \ldots, \alpha_k)\]

Note that each of the coefficients of this polynomial can be viewed as a polynomial in $z_1, \ldots, z_m$ and $y_{1,1}, \ldots, y_{1,m}, \ldots, y_{k,1}, \ldots, y_{k,m}$ of total degree at most $\deg Q$. In particular, observe the coefficient of the degree $\deg Q$ monomial $\alpha_0^{\deg Q}$ in $Q|_s$. Note that it depends solely on $z_1, \ldots, z_m$ (and not on $y_{1,1}, \ldots, y_{1,m}, \ldots, y_{k,1}, \ldots, y_{k,m}$). Hence, let us denote it by $P(z_1, \ldots, z_m)$.

To analyze $P$ we will need several notations. Denote $Q(x_1, \ldots, x_m) = \sum_{i_1 \ldots i_m} a_{i_1 \ldots i_m} x_1^{i_1} \cdots x_m^{i_m}$. Define $I = \{(i_1, \ldots, i_m) | \sum_j i_j = \deg Q\}$. Now, $P(z_1, \ldots, z_m) = \sum_{(i_1 \ldots i_m) \in I} a_{i_1 \ldots i_m} z_1^{i_1} \cdots z_m^{i_m}$. Thus, by definition, $\deg P = \deg Q$ and $P$ is not identically zero.

Clearly,

\[
\Pr_{\bar{z}, \bar{y}_1, \ldots, \bar{y}_k} \left[ \deg Q|_{\text{affine}(\bar{0}; \bar{z}, \bar{y}_1, \ldots, \bar{y}_k)} > d \right] \geq \Pr_{\bar{z}, \bar{y}_1, \ldots, \bar{y}_k} [P(\bar{z}) \neq 0]
\]

By the Schwartz-Zippel lemma, for a uniformly distributed $\bar{z} \in \mathbb{F}^m$ we have

\[
\Pr_{\bar{z} \in \mathbb{F}^m} [P(\bar{z}) \neq 0] \geq 1 - \frac{\deg Q}{|\mathbb{F}|} \geq 1 - \delta
\]

For any linearly independent $\bar{y}_1, \ldots, \bar{y}_k$, the probability that a uniformly distributed $\bar{z} \in \mathbb{F}^m$ satisfies: $\bar{z}, \bar{y}_1, \ldots, \bar{y}_k$ are linearly dependent, is at most $\frac{1}{|\mathbb{F}|}$. Therefore,

\[
\Pr_{\bar{z}, \bar{y}_1, \ldots, \bar{y}_k} \left[ \deg Q|_{\text{affine}(\bar{0}; \bar{z}, \bar{y}_1, \ldots, \bar{y}_k)} > d \right] \geq 1 - \delta - \frac{1}{|\mathbb{F}|}
\]

However, $\Pr_{\bar{z}, \bar{y}_1, \ldots, \bar{y}_k} \left[ \deg Q|_{\text{affine}(\bar{0}; \bar{z}, \bar{y}_1, \ldots, \bar{y}_k)} \leq d \right] > \delta + \frac{1}{|\mathbb{F}|}$.
Next we prove a lemma allowing us to deduce degree \( md \) (rather than \( d \)), even if we only observe affine subspaces in \( S_k^m \). This lemma will be used in the analysis of the Randomness-Efficient Plane vs. Point tester.

**Lemma 6.5 (degree \( md \) consolidation).** Fix dimensions \( k \) and \( m \), \( 1 \leq k \leq m \). Fix an oracle \( A \) assigning polynomials of degree at most \( d \) to all affine subspaces. Suppose that for some \( 0 \leq \delta \leq 1 \), there exists a polynomial \( Q : \mathbb{F}^m \rightarrow \mathbb{F} \), such that \( \deg Q \leq \delta |\mathbb{F}| \) and

\[
\Pr_{s \in S_k^m} [(Q \equiv A)(s)] > \delta + \frac{1}{|\mathbb{H}|}
\]

Then, \( \deg Q \leq md \).

**Proof.** By the premise and uniformity, when picking independently at random linearly independent directions \( \bar{y}_1, \ldots, \bar{y}_m \in \mathbb{H}^m \) and a point \( \bar{z} \in \mathbb{F}^m \),

\[
\Pr_{\bar{y}_1, \ldots, \bar{y}_m} \left[ \Pr_{\bar{z}} [(Q \equiv A) (\text{affine}(\bar{z}, \bar{y}_1, \ldots, \bar{y}_m))] > \delta \right] > \frac{1}{|\mathbb{H}|}
\]

Thus,

\[
\Pr_{\bar{y}_1, \ldots, \bar{y}_m} \left[ \Pr_{\bar{z}} \left[ \deg Q_{\text{affine}(\bar{z}; \bar{y})} \leq d \right] > \delta \right] > \frac{1}{|\mathbb{H}|}
\]

By proposition 3.3, there exist linearly independent \( \bar{y}_1, \ldots, \bar{y}_m \in \mathbb{H}^m \), such that for every \( 1 \leq i \leq m \),

\[
\Pr_{\bar{z} \in \mathbb{F}^m} \left[ \deg Q_{\text{affine}(\bar{z}; \bar{y}_i)} \leq d \right] > \delta
\]

\( \bar{y}_1, \ldots, \bar{y}_m \) is a basis for \( \mathbb{F}^m \). Thus, every point \( \bar{x} \in \mathbb{F}^m \) can be represented as \( \bar{x} = \sum_{i=1}^{m} \alpha_i \bar{y}_i \) for some \( \alpha_1, \ldots, \alpha_m \in \mathbb{F} \). Hence, view \( Q \) as a polynomial in variables \( \alpha_1, \ldots, \alpha_m \). Assume on way of contradiction that \( \deg Q > md \). Hence, there exists \( 1 \leq i \leq m \) such that the degree of \( Q \) in the variable \( \alpha_i \), which we will denote by \( D \), is larger than \( d \). The coefficient of \( \alpha_i^D \) in the polynomial \( Q_{\text{affine}(\bar{z}; \bar{y}_i)} \) is a non-zero polynomial \( P(z_1, \ldots, z_m) \) of degree at most \( D \leq \deg Q \). Hence, by the Schwartz-Zippel lemma,

\[
\Pr_{\bar{z} \in \mathbb{F}^m} \left[ P(z_1, \ldots, z_m) = 0 \right] \leq \frac{\deg Q}{|\mathbb{F}|} \leq \delta
\]

Thus, \( \Pr_{\bar{z} \in \mathbb{F}^m} \left[ \deg Q_{\text{affine}(\bar{z}; \bar{y}_i)} \leq d \right] \leq \delta \), which is a contradiction. 

7 Consistency Graph

Fix a dimension \( k \geq 3 \). In this section we define and analyze a graph that captures the consistency among hyperplanes in \( \mathbb{F}^k \), i.e., affine subspaces of dimension that is smaller by 1 than \( k \). Using the graph we prove a list decoding lemma (lemma 7.4). This lemma is used in the analysis of the Randomness-Efficient Plane vs. Point tester to go up one dimension (see section 8). Lemma 7.4 is also the only lemma in this section that is is used outside it.

The idea is a variation of the analysis of Raz and Safra for the non-randomness-efficient setting [17]. Our crucial observation is that we can essentially still apply their analysis when considering only directions with coordinates in a subfield \( \mathbb{H} \subseteq \mathbb{F} \), instead of the entire field \( \mathbb{F} \).
7.1 Graph Construction

Given an oracle \( \mathcal{A} \) assigning affine subspaces polynomials of degree at most \( d \), define a simple undirected graph \( G_\mathcal{A} = (V, E_\mathcal{A}) \) that captures the consistency among affine subspaces in \( \mathcal{S}_{k-1}^k \) as follows. Let the vertices be all those subspaces, \( V = \mathcal{S}_{k-1}^k \). Let the edges indicate whether two affine subspaces are assigned polynomials that are consistent on the intersection of the subspaces,

\[
E_\mathcal{A} = \{ (s_1, s_2) \mid \forall \bar{x} \in s_1 \cap s_2, \mathcal{A}(s_1)(\bar{x}) = \mathcal{A}(s_2)(\bar{x}) \}
\]

Note that every two subspaces in \( \mathcal{S}_{k-1}^k \) are either parallel (i.e., identify or do not intersect) or intersect by an affine subspace from \( \mathcal{S}_{k-2}^k \) (see closedness under intersection; proposition 4.6).

7.2 Graph is Almost-Transitive

We first wish to establish that the graph is almost-transitive in the sense that every two vertices that are not neighbors do not have too many common neighbors (whereas, if the graph had been transitive, they would not have had common neighbors at all):

**Lemma 7.1 (almost transitivity).** If \( (s_1, s_2) \notin E_\mathcal{A} \), then

\[
\Pr_{s_3 \in V} [(s_1, s_3) \in E_\mathcal{A} \land (s_3, s_2) \in E_\mathcal{A}] \leq \frac{1}{|R|} + \frac{d}{|\bar{F}|}
\]

**Proof.** Assume \( (s_1, s_2) \notin E_\mathcal{A} \). By definition, there exists \( \bar{x} \in s_1 \cap s_2 \), for which \( \mathcal{A}(s_1)(\bar{x}) \neq \mathcal{A}(s_2)(\bar{x}) \). Hence, \( a \equiv s_1 \cap s_2 \in \mathcal{S}_{k-2}^k \) and \( \mathcal{A}(s_1) \) and \( \mathcal{A}(s_2) \) induce two different polynomials of degree at most \( d \) on \( a \). Let us denote these polynomials by \( P_1 \) and \( P_2 \). Fix a representation in \( \mathcal{R}_{k-2}^k \) for \( a \). We say that a vertex \( s_3 \in V \) spots inconsistency, if there exists \( \bar{x} \in s_3 \cap a \), such that \( P_1(\bar{x}) \neq P_2(\bar{x}) \). We wish to argue that a random vertex \( s_3 \in V \) is likely to spot inconsistency.

Pick uniformly \( r = (\bar{z}; \bar{y}_1, \ldots, \bar{y}_{k-1}) \in \mathcal{R}_{k-1}^k \). Let us say that \( s_3 = \text{affine}(r) \) is bad, if \( s_3 \) either contains \( a \) or does not intersect it. Since \( (k - 2) + (k - 1) \geq k \), for \( s_3 \) to be bad, \( a \)'s directions must be linearly dependent in \( \bar{y}_1, \ldots, \bar{y}_{k-1} \). Hence, by uniformity and by proposition 3.2,

\[
\Pr_{s_3 \in V} [s_3 \text{ is bad}] \leq \frac{1}{|R|}
\]

By the Schwartz-Zippel lemma, \( \Pr_{\bar{x} \in a} [P_1(\bar{x}) \neq P_2(\bar{x})] \geq 1 - \frac{d}{|\bar{F}|} \). For all the hyperplanes \( s \) that do not contain \( a \) but do intersect it, the dimension of their intersection with \( a \) is \( (k - 1) + (k - 2) - k = k - 3 \). Let \( I = \{ s \cap a \mid s \in V; a \notin s \land s \cap a \neq \phi \} \). By closedness under intersection and uniformity, \( \mathbb{E}_{a' \in I} [\Pr_{\bar{x} \in a'} |P_1(\bar{x}) \neq P_2(\bar{x})|] = \Pr_{\bar{x} \in a} [P_1(\bar{x}) \neq P_2(\bar{x})] \geq 1 - \frac{d}{|\bar{F}|} \). By uniformity,

\[
\Pr_{s_3 \in V} [s_3 \text{ spots inconsistency}|s_3 \text{ is not bad}] \geq 1 - \frac{d}{|\bar{F}|}
\]

Combining inequalities 1 and 2, we get

\[
\Pr_{s_3} [s_3 \text{ spots inconsistency}] \geq 1 - \frac{1}{|R|} - \frac{d}{|\bar{F}|}
\]

If \( s_3 \) spots inconsistency then either \( (s_1, s_3) \notin E_\mathcal{A} \) or \( (s_3, s_2) \notin E_\mathcal{A} \). Thus, \( (s_1, s_3) \in E_\mathcal{A} \) and \( (s_3, s_2) \in E_\mathcal{A} \) with probability at most \( \frac{1}{|R|} + \frac{d}{|\bar{F}|} \). \( \blacksquare \)
7.3 Graph-Based List Decoding

The almost-transitivity of the graph $G_A$ can be used to prove that, other than relatively few edges, the graph is truly transitive, i.e., composed of disjoint cliques. Moreover, these cliques are relatively large. This was shown by Raz and Safra [17],

**Lemma 7.2 (graph partition).** Fix $\epsilon = \frac{1}{|H|} + \frac{d}{|V|}$. There exists a partition of the vertices of $G_A$ into cliques, $V = \bigcup_{i=1}^t V_i$, such that

1. (all non-trivial cliques are large) For every $1 \leq i \leq t$, either $|V_i| = 1$, or $|V_i| > 2\sqrt{\epsilon} |V|$.

2. (almost all edges are within cliques)

$$\Pr_{s_1, s_2 \in V} [(s_1, s_2) \notin E_A \lor \exists i \ s_1, s_2 \in V_i] \geq 1 - 5\sqrt{\epsilon}$$

*Proof.* By lemma 7.1 and the combinatorial lemma of Raz and Safra [17] (for completeness we include a proof for this lemma; see lemma A.1 in the appendix).

A large clique in $G_A$ corresponds to a single relatively-low degree polynomial identifying with the oracle $A$ on all affine subspaces associated with the vertices in the clique,

**Lemma 7.3 (from large clique to polynomial).** For every large clique $U \subseteq V$, $|U| > \left(\frac{2d}{|H|} + \frac{1}{|V|}\right) |V|$, there exists a polynomial $Q : \mathbb{F}^k \to \mathbb{F}$ with $\deg Q \leq 2d$, such that for every $s \in U$, $(Q \equiv A)(s)$.

*Proof.* For linearly independent $\vec{y}_1, \ldots, \vec{y}_{k-1} \in \mathbb{F}^k$, there are exactly $|\mathbb{F}|$ different hyperplanes of the form $\vec{z} + \text{span}\{\vec{y}_1, \ldots, \vec{y}_{k-1}\}$ for some $\vec{z} \in \mathbb{F}^k$. Pick uniformly at random linearly independent $\vec{y}_1, \ldots, \vec{y}_{k-1} \in \mathbb{H}^k$ and consider the random variable $X$ denoting the fraction of hyperplanes in $U$ among the $|\mathbb{F}|$ induced hyperplanes. By linearity of expectations,

$$\mathbb{E}_{\vec{y}_1, \ldots, \vec{y}_{k-1}} [X] = \frac{|U|}{|V|} > \frac{2d}{|H|} + \frac{1}{|V|}$$

Hence, by Markov's inequality, since $0 \leq X \leq 1$,

$$\Pr_{\vec{y}_1, \ldots, \vec{y}_{k-1}} \left[ X > \frac{2d}{|H|} \right] > \frac{1}{|V|}$$

By proposition 3.2, there necessarily exists a basis $\vec{y}_1, \ldots, \vec{y}_k \in \mathbb{H}^k$ for $\mathbb{F}^k$ as well as $2 \cdot (2d + 1)$ scalars $c_0, \ldots, c_{2d}, c'_0, \ldots, c'_{2d} \in \mathbb{F}$ such that

$$s_0 = \text{affine}(c_0 \vec{y}_k; \vec{y}_1, \ldots, \vec{y}_{k-1}) \in U$$

$$s_2d = \text{affine}(c_{2d} \vec{y}_k; \vec{y}_1, \ldots, \vec{y}_{k-1}) \in U$$

$$s'_0 = \text{affine}(c'_0 \vec{y}_1; \vec{y}_2, \ldots, \vec{y}_k) \in U$$

$$s'_{2d} = \text{affine}(c'_{2d} \vec{y}_1; \vec{y}_2, \ldots, \vec{y}_k) \in U$$
Let us define a polynomial \( Q : \mathbb{F}^k \rightarrow \mathbb{F} \) such that for every \( 0 \leq i \leq d \), \( (Q \equiv A)(s_i) \). This is done using Lagrange’s interpolation formula:

\[
Q \left( \sum_{i=1}^{k} \alpha_i \tilde{y}_i \right) = \sum_{i=0}^{d} \prod_{j \in \{0, \ldots, d\} - \{i\}} (\alpha_k - c_j) \cdot A(s_i) \left( c_i \tilde{y}_k + \sum_{j=1}^{k-1} \alpha_j \tilde{y}_j \right)
\]

The degree of \( Q \) in \( \alpha_k \) is at most \( d \) and its total degree is \( \deg Q \leq 2d \).

We would like to argue that for every \( s \in U \), \( (Q \equiv A)(s) \). For every line of the form \( l = \text{affine} \left( \sum_{i=1}^{k-1} a_i \tilde{y}_i; \tilde{y}_k \right) \subset s_j \), the polynomial \( Q_l \) has degree at most \( d \). Moreover, for every \( 0 \leq i \leq d \), \( Q_{l_i} \) and \( A(s_j') \) identify on \( l \cap s_i \). By the Schwartz-Zippel lemma, \( Q_{l_i} \) and \( A(s_j') \) identify on the entire line \( l \). Thus, for every \( 0 \leq j \leq 2d \), \( Q \) and \( A \) identify on \( s_j \).

Let \( s \in U \). Necessarily, \( s \) intersects the \( s_j \)’s or the \( s_j' \)’s (or both). Hence, \( Q_{|s} \) and \( A(s) \) identify on more than \( 2d|s| \) of the points on \( s \). \( Q_{|s} \) is of degree at most \( 2d \). Thus, by the Schwartz-Zippel lemma, \( Q \) and \( A \) identify on \( s \).

The partition of \( G_A \) into cliques yields list decoding,

**Lemma 7.4 (hyperplane vs. hyperplane).** Assume \( A \) assigns polynomials of degree at most \( d \) to affine subspaces. For any \( \delta \geq 8\sqrt{\frac{d}{|V|} + \frac{1}{|V|^2}} \), there exists a list of polynomials \( Q_1, \ldots, Q_t : \mathbb{F}^k \rightarrow \mathbb{F}, t \leq \frac{4}{\delta} \), with \( \deg Q_i \leq 2d \), such that

\[
\Pr_{s_1, s_2 \in V} \left[ (s_1, s_2) \notin E_A \lor \exists i, (Q_i \equiv A)(s_1) \land (Q_i \equiv A)(s_2) \right] > 1 - \delta
\]

**Proof.** Consider the partition of lemma 7.2. Let \( S_1, \ldots, S_l \) denote the small cliques in this partition, i.e., cliques whose size is \( |S_i| < \frac{\delta}{4} |V| \). Clearly,

\[
\sum_{i=1}^{l} |S_i|^2 < \frac{\delta^2}{4} |V|^2 \cdot \sum_{i=1}^{l} |S_i| \leq \frac{\delta}{4} |V|^2
\]

Hence, \( \Pr_{s_1, s_2 \in V} \left[ \exists i, s_1, s_2 \in S_i \right] < \frac{\delta}{4} \). Let \( L_1, \ldots, L_t \) be the set of all large cliques \( |L_i| \geq \frac{\delta}{4} |V| \). We have \( t \leq \frac{4}{\delta} \). Moreover,

\[
\Pr_{s_1, s_2 \in V} \left[ (s_1, s_2) \notin E_A \lor \exists i, s_1, s_2 \in L_i \right] > 1 - \frac{5}{8} \delta - \frac{1}{4} \delta > 1 - \delta
\]

For every \( 1 \leq i \leq t \), let \( Q_i : \mathbb{F}^k \rightarrow \mathbb{F} \) be the polynomial associated with \( L_i \) according to lemma 7.3. We have \( \deg Q_i \leq 2d \), and

\[
\Pr_{s_1, s_2 \in V} \left[ (s_1, s_2) \notin E_A \lor \exists i, (Q_i \equiv A)(s_1) \land (Q_i \equiv A)(s_2) \right] > 1 - \delta
\]

Note that the lemma is meaningful only when the density of the graph, \( |E_A|/|V|^2 \), is large enough with respect to \( \delta \), otherwise, the list might be empty. This corresponds to the fact that the oracle must assign the affine subspaces somewhat consistent polynomials if we wish to (list) decode.
8 Going Up One Dimension

Fix dimension $k \geq 3$. Let $A$ be an oracle assigning polynomials of degree at most $d$ to affine subspaces. In this section we prove that if there is $\gamma$ consistency between affine subspaces of dimension $(k - 1)$ in $\mathbb{F}^k$ and points within them, then there exists a polynomial $Q : \mathbb{F}^k \to \mathbb{F}$ of degree at most $2d$ that identifies with the oracle on almost $\gamma$ of the points. This is done in several steps:

1. We use an argument of counting in several manners to transform our setting to one that resembles that of the consistency graph of section 7.

2. We use the analysis of the consistency graph to prove the claim we want but with not as good consistency parameter.

3. We fix the consistency parameter via the consistency consolidation of section 6.

The final result of this section is given in lemma 8.3. This is also the only lemma in this section used outside it. Note that the degree parameter grows from $d$ to $2d$, and we indeed need to take care of that when we use this lemma.

8.1 From Hyperplane vs. Point to Hyperplane vs. Hyperplane

We start by showing that $\gamma$ consistency between hyperplanes and points within them implies that for an average pair $(s_1, s_2)$ of intersecting hyperplanes, $A(s_1)$ and $A(s_2)$ identify (with each other and with $A$) on at least $\gamma^2$ of the points in the intersection of $s_1$ and $s_2$.

The proof uses repeatedly the trick of counting in several manners, which is made possible due to uniformity considerations (see section 4).

For an affine subspace $a \in \mathcal{S}_{k-2}^k$, denote the set of hyperplane pairs that intersect on $a$ by $\mathcal{S}_a \equiv \{(s_1, s_2) \mid s_1, s_2 \in \mathcal{S}_{k-1}^k, s_1 \cap s_2 = a\}$.

Lemma 8.1 (counting in several manners). If for an oracle $A$,

$$
\mathbb{E}_{s \in \mathcal{S}_{k-1}^k} \left[ \mathbb{E}_{\vec{x} \in s} \left[ \Pr [A(s)(\vec{x}) = A(\vec{x})] \right] \right] \geq \gamma
$$

Then,

$$
\mathbb{E}_{a \in \mathcal{S}_{k-2}^k} \left[ \mathbb{E}_{(s_1, s_2) \in \mathcal{S}_a} \left[ \mathbb{E}_{\vec{x} \in a} \left[ \Pr [A(s_1)(\vec{x}) = A(\vec{x}) = A(s_2)(\vec{x})] \right] \right] \right] \geq \gamma^2 - \frac{1}{|H|}
$$

Proof. For a space $s \in \mathcal{S}_{k-1}^k$, a sub-space of it $a \subset s$, $a \in \mathcal{S}_{k-2}^k$, and a point $\vec{x} \in a$, let $I_{s,a,\vec{x}}$ be an indicator variable for the event $A(s)(\vec{x}) = A(\vec{x})$.

By the premise and uniformity,

$$
\mathbb{E}_{s} \left[ \mathbb{E}_{a \subset s} \left[ \mathbb{E}_{\vec{x} \in a} [I_{s,a,\vec{x}}] \right] \right] \geq \gamma
$$

By uniformity, we can also count in a different manner and obtain:

$$
\mathbb{E}_{a} \left[ \mathbb{E}_{\vec{x} \in a} \left[ \mathbb{E}_{s \supset a} [I_{s,a,\vec{x}}] \right] \right] \geq \gamma
$$
By convexity considerations,
\[
\mathbb{E}_a \left[ \mathbb{E}_{\bar{x} \in a} \left( \mathbb{E}_{s \supseteq a} \left[ I_{s,a,\bar{x}} \right]^2 \right) \right] \geq \gamma^2
\]

Or, in other words,
\[
\mathbb{E}_a \left[ \mathbb{E}_{\bar{x} \in a} \left[ \mathbb{E}_{s_1,s_2 \supseteq a} \left[ I_{s_1,a,\bar{x}} I_{s_2,a,\bar{x}} \right] \right] \right] \geq \gamma^2
\]

By uniformity, we can change the order of summation once again, and get:
\[
\mathbb{E}_a \left[ \mathbb{E}_{s_1,s_2 \supseteq a} \left[ \mathbb{E}_{\bar{x} \in a} \left[ I_{s_1,a,\bar{x}} I_{s_2,a,\bar{x}} \right] \right] \right] \geq \gamma^2
\]

The lemma follows noticing that the probability that \( s_1 = s_2 \) given that \( s_1, s_2 \supseteq a \) is at most \( \frac{1}{|a|} \).

\section{8.2 Hyperplane vs. Point Lemma}

Next, we show that considerable consistency between \((k-1)\)-dimensional affine subspaces and points implies a significant correspondence of the values assigned to points with a relatively low degree polynomial over \( \mathbb{F}^k \). The heart of the proof is the analysis of the consistency graph (lemma 7.4).

\begin{lemma}[hyperplane vs. point] Assume \( \mathcal{A} \) assigns polynomials of degree at most \( d \) to affine subspaces. Fix \( \delta = 16 \max \left\{ \sqrt{\frac{d}{|a|}}, \sqrt{\frac{1}{|a|}} \right\} \). Assume that

\[
\mathbb{E}_{a \in \mathcal{S}_{k-1}} \left[ \Pr_{\bar{x} \in a} \left[ \mathcal{A}(s)(\bar{x}) = \mathcal{A}(\bar{x}) \right] \right] \geq \gamma
\]

Then there exists a polynomial \( Q : \mathbb{F}^k \to \mathbb{F} \), with \( \deg Q \leq 2d \), such that

\[
\Pr_{\bar{x} \in \mathbb{F}^k} \left[ Q(\bar{x}) = \mathcal{A}(\bar{x}) \right] \geq \gamma^2 - 3\delta
\]

\end{lemma}

\begin{proof}
Lemma 8.1 allows us to translate the consistency given in this lemma to consistency between pairs of hyperplanes on points,

\[
\mathbb{E}_{a \in \mathcal{S}_{k-2}} \left[ \mathbb{E}_{(s_1,s_2) \in \mathcal{S}_a} \left[ \Pr_{\bar{x} \in a} \left[ \mathcal{A}(s_1)(\bar{x}) = \mathcal{A}(\bar{x}) = \mathcal{A}(s_2)(\bar{x}) \right] \right] \right] \geq \gamma^2 - \frac{1}{|a|^2}
\]

Lemma 7.4 gives list decoding \( Q_1, \ldots, Q_t : \mathbb{F}^k \to \mathbb{F}, \deg Q_i \leq 2d, t \leq \frac{d^2}{\delta} \), for consistency among pairs of hyperplanes. We wish to argue that at least one of these polynomials also identifies with the oracle on many of the points.

Let us define appropriate notations. Choose independently and uniformly at random a subspace \( a \in \mathcal{S}_{k-2} \), hyperplanes that intersect on \( a \), \((s_1,s_2) \in \mathcal{S}_a \), and a point \( \bar{x} \in a \). Define the following events:

1. \( X : \mathcal{A}(s_1)(\bar{x}) = \mathcal{A}(\bar{x}) = \mathcal{A}(s_2)(\bar{x}) \) (hyperplanes consistent on (and with) point).

2. \( C : (s_1,s_2) \in E_A \) (hyperplanes consistent).
3. $E : \exists i \ (Q_i \equiv \mathcal{A})(s_1) \land (Q_i \equiv \mathcal{A})(s_2)$ (hyperplanes explained).

In these notations we have $\Pr_{a,s_1,s_2,\bar{x}}[X] \geq \gamma^2 - \frac{1}{|F|}$. By uniformity, $s_1, s_2$ are uniformly distributed over the set of all pairs with $s_1 \cap s_2 \in S_{k-2}^d$. Since for a uniformly distributed pair $s_1, s_2 \in V$, the probability that $s_1 \cap s_2 \notin S_{k-2}^d$ is bounded by $\frac{1}{|F|}$ (see proposition 3.2), the list decoding translates into

$$\Pr_{s_1,s_2}[\neg C \lor E] \geq 1 - \delta - \frac{1}{|H|}$$

$\bar{x}$ is uniformly distributed within $s_1 \cap s_2$. Hence, by the Schwartz-Zippel lemma, $\Pr_{a,s_1,s_2,\bar{x}}[X|\neg C] \leq \frac{d}{|F|}$. Therefore, the probability that $s_1, s_2$ are consistent on $\bar{x}$ but not explained is small,

$$\Pr_{a,s_1,s_2,\bar{x}}[X \land \neg E] = \Pr_{a,s_1,s_2,\bar{x}}[C \land X \land \neg E] + \Pr_{a,s_1,s_2,\bar{x}}[\neg C \land X \land \neg E]$$

$$\leq \Pr_{s_1,s_2}[C \land \neg E] + \Pr_{a,s_1,s_2,\bar{x}}[\neg C \land X]$$

$$\leq 1 - \Pr_{s_1,s_2}[\neg C \lor E] + \Pr_{a,s_1,s_2,\bar{x}}[X|\neg C]$$

$$\leq \delta + \frac{1}{|H|} + \frac{d}{|F|}$$

Thus, the probability that $s_1, s_2$ are consistent on $\bar{x}$ and are explained is large

$$\Pr_{a,s_1,s_2,\bar{x}}[X \land E] \geq \Pr_{a,s_1,s_2,\bar{x}}[X] - \Pr_{a,s_1,s_2,\bar{x}}[X \land \neg E]$$

$$\geq \gamma^2 - \frac{1}{|H|} - \delta - \frac{1}{|H|} - \frac{d}{|F|}$$

$$\geq \gamma^2 - 2\delta \quad (3)$$

Let us define a distributional oracle $\tilde{\mathcal{A}}$, assigning each affine subspace $a \in S_{k-2}^d$, a distribution over polynomials of degree at most $d$ over $a$ (for clarification of our notion of distributional oracles, see the discussion before lemma 6.3). To define the distribution $\tilde{\mathcal{A}}(a)$, we indicate how to sample a polynomial accordingly:

- Pick uniformly at random hyperplanes that intersect on $a$, $(s_1, s_2) \in S_a$.
- If there is $i$ such that $(Q_i \equiv \mathcal{A})(s_1)$ and $(Q_i \equiv \mathcal{A})(s_2)$, output the restriction of $Q_i$ to $a$ (note that if there are two (or more) such polynomials, they must identify on $a$).
- Otherwise, output a null polynomial.

If $\tilde{\mathcal{A}}(a)$ is not null, then there exists $i$ such that $(Q_i \equiv \tilde{\mathcal{A}})(a)$, while if $\tilde{\mathcal{A}}(a)$ is not null, $\tilde{\mathcal{A}}(a)(\bar{x}) \neq \mathcal{A}(\bar{x})$ for every $\bar{x} \in a$. Thus,

$$E_{\tilde{\mathcal{A}}} \left[ E_{\bar{x} \in a} \left[ \Pr_{\bar{x} \in a} \left[ (\tilde{\mathcal{A}}(a)(\bar{x}) \neq \mathcal{A}(\bar{x}) \lor \exists i \ (Q_i \equiv \tilde{\mathcal{A}})(a) \right) \right] \right] = 1$$

By the construction of $\tilde{\mathcal{A}}$ and inequality $3$, $\tilde{\mathcal{A}}$ has large consistency with $\mathcal{A}$,

$$E_{\tilde{\mathcal{A}}} \left[ E_{\bar{x} \in a} \left[ \Pr_{\bar{x} \in a} \left[ (\tilde{\mathcal{A}}(a)(\bar{x}) = \mathcal{A}(\bar{x})) \right] \right] \right] \geq \gamma^2 - 2\delta$$

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By corollary 5.6, the uniform distribution on $S^{k-2}_{k}$ samples well: for every set $A \subseteq \mathbb{F}^{k}$, for every $0 < \varepsilon < 1$,

$$\Pr_{a \in S^{k-2}_{k}} \left[ \left| \frac{|a \cap A|}{|a|} - \frac{|A|}{|\mathbb{F}^{k}|} \right| \geq \varepsilon \right] \leq \frac{1}{\varepsilon^2 |\mathbb{H}|}$$

Thus, by lemma 6.3, since $\frac{\delta}{2} \geq t \cdot \frac{4}{|\mathbb{H}|}$, there exists $1 \leq i \leq t$ such that

$$\Pr_{\bar{x} \in \mathbb{F}^{k}} [Q_i(\bar{x}) = A(\bar{x})] \geq \gamma^2 - 3\delta$$

\[ \blacksquare \]

8.3 Consolidating

We can apply consistency consolidation to improve the result of the last subsection. The following summarizes what we establish in this section:

**Lemma 8.3 (consistency consolidated).** Denote $\theta_0 = \frac{1}{2} \cdot \left( \sqrt{\frac{1}{|\mathbb{V}|}} + \frac{\sqrt{d}}{|\mathbb{F}|} \right)$. Fix $k \geq 3$. Fix an oracle $A$ assigning polynomials of degree at most $d$ to all affine subspaces. Assume that

$$\mathbb{E}_{s \in S^{k}_{k-1}} \left[ \Pr_{\bar{x} \in \mathbb{F}^{k}} [A(s)(\bar{x}) = A(\bar{x})] \right] \geq \gamma$$

Then there exists a polynomial $Q : \mathbb{F}^{k} \rightarrow \mathbb{F}$, with $\deg Q \leq 2d$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^{k}} [Q(\bar{x}) = A(\bar{x})] \geq \gamma - 2\theta_0$$

**Proof.** Assume $\theta_0 \leq 1$ (otherwise, the claim trivially holds). Denote $\epsilon_0 = \sqrt{\frac{2d}{|\mathbb{F}|}}$ and $\delta_0 = 16 \max \left\{ \sqrt{\frac{d}{|\mathbb{V}|}}, \frac{1}{|\mathbb{H}|} \right\}$. Define $f(\gamma) = \gamma^2 - 3\delta_0$. It holds that

$$f(\theta_0 - \epsilon_0) - \epsilon_0 = (\theta_0 - \epsilon_0)^2 - 3\delta_0 - \epsilon_0 \geq \frac{\theta_0^2}{2}$$

where $\theta_0^2/2 \geq 2\epsilon_0$. Apply lemma 6.1 on lemma 8.2 to deduce the existence of $t \leq 4/\theta_0^2$ polynomials $Q_1, \ldots, Q_t : \mathbb{F}^{k} \rightarrow \mathbb{F}$, with $\deg Q_i \leq 2d$, such that

$$\mathbb{E}_{s \in S^{k}_{k-1}} \left[ \Pr_{\bar{x} \in \mathbb{F}^{k}} [A(s)(\bar{x}) \neq A(\bar{x}) \vee \exists i(Q_i \equiv A)(s)] \right] \geq 1 - \theta_0$$

By lemma 6.3 (using sampling corollary 5.6), there exists $1 \leq i \leq t$, such that

$$\Pr_{\bar{x} \in \mathbb{F}^{k}} [Q_i(\bar{x}) = A(\bar{x})] \geq \gamma - 2\theta_0$$

\[ \blacksquare \]

9 The Randomness-Efficient Plane vs. Point Tester is Sound

We wish to show that if the average consistency between planes and points is large then the oracle assigns points values that are close to a low degree polynomial. Theorem 1 will follow.

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Lemma 9.1 (from dimension 2 to dimension k). Denote \( \theta_k = 2^4 \left( \sqrt{\frac{kd}{|F|}} + \sqrt{\frac{1}{|F^2|}} \right) \). For every dimension \( k \geq 2 \), for every \( 0 < \gamma \leq 1 \) and oracle \( A \), if

\[
\mathbb{E}_{s \in S^2_k} \left[ \Pr_{\bar{x} \in s} \left[ A(s)(\bar{x}) = A(\bar{x}) \right] \right] \geq \gamma
\]

then there exists a polynomial \( Q : \mathbb{F}^k \to \mathbb{F} \) with \( \deg Q \leq kd \) such that

\[
\Pr_{\bar{x} \in \mathbb{F}^k} \left[ Q(\bar{x}) = A(\bar{x}) \right] \geq \gamma - (8k - 10)\theta_k
\]

Proof. We prove the lemma by induction on \( k \). Let us formulate two inductive claims. The first argues slightly better consistency, but worse degree:

**Claim 1** [\( k \)]: For every \( 0 < \gamma \leq 1 \) and oracle \( A \), if

\[
\mathbb{E}_{s \in S^2_k} \left[ \Pr_{\bar{x} \in s} \left[ A(s)(\bar{x}) = A(\bar{x}) \right] \right] \geq \gamma
\]

then there exists a polynomial \( Q : \mathbb{F}^k \to \mathbb{F} \) with \( \deg Q \leq 2(k-1)d \) such that

\[
\Pr_{\bar{x} \in \mathbb{F}^k} \left[ Q(\bar{x}) = A(\bar{x}) \right] \geq \gamma - (8k - 16)\theta_k
\]

**Claim 2** [\( k \)]: For every \( 0 < \gamma \leq 1 \) and oracle \( A \), if

\[
\mathbb{E}_{s \in S^2_k} \left[ \Pr_{\bar{x} \in s} \left[ A(s)(\bar{x}) = A(\bar{x}) \right] \right] \geq \gamma
\]

then there exists a polynomial \( Q : \mathbb{F}^k \to \mathbb{F} \) with \( \deg Q \leq kd \) such that

\[
\Pr_{\bar{x} \in \mathbb{F}^k} \left[ Q(\bar{x}) = A(\bar{x}) \right] \geq \gamma - (8k - 10)\theta_k
\]

**Claim 1** [\( 2 \)] holds by taking \( Q \) to be \( A(s) \) for the only plane \( s \). Hence, the lemma will follow if we prove that for every \( k \geq 2 \),

**Claim 1** [\( k \)] \( \Rightarrow \) **Claim 2** [\( k \)] \( \Rightarrow \) **Claim 1** [\( k + 1 \)]

**Claim 9.1.1.** **Claim 1** [\( k \)] \( \Rightarrow \) **Claim 2** [\( k \)]

Proof. Fix \( 0 < \gamma \leq 1 \) and oracle \( A \) such that

\[
\mathbb{E}_{s \in S^2_k} \left[ \Pr_{\bar{x} \in s} \left[ A(s)(\bar{x}) = A(\bar{x}) \right] \right] \geq \gamma
\]

Assume that \( (8k - 10)\theta_k \leq 1 \) (otherwise, we are done). Denote \( \epsilon_0 = \sqrt{2(k-1)d/|F|} \). Define \( f(\gamma) = \gamma - (8k - 16)\theta_k \). Let \( \delta = (8k - 14)\theta_k \). It holds that

\[
f(\delta - \epsilon_0) - \epsilon_0 = (8k - 14)\theta_k - \epsilon_0 - (8k - 16)\theta_k - \epsilon_0 \geq \theta_k
\]

where \( \theta_k \geq 2\epsilon_0 \). By lemmata 6.1 and 6.2 applied on **Claim 1** [\( k \)], there exist \( t \leq 2/\theta_k \) polynomials \( Q_1, \ldots, Q_t : \mathbb{F}^k \to \mathbb{F} \), \( \deg Q_i \leq 2(k-1)d \), such that
1. (each identifies with many planes) for every $1 \leq i \leq t$,

$$\Pr_{s \in S_k} [(Q_i \equiv A)(s)] \geq \frac{\theta_k}{t} > \frac{2(k - 1)d}{|F|} + \frac{1}{|W|}$$

2. (all explain almost all the consistency)

$$\mathbb{E}_{s \in S_k} \left[ \Pr_{\bar{x} \in s} [A(s)(\bar{x}) \neq A(\bar{x}) \lor \exists i (Q_i \equiv A)(s)] \right] \geq 1 - \delta - \theta_k$$

By lemma 6.5, for every $1 \leq i \leq t$, $\deg Q_i \leq kd$. By lemma 6.3 (using sampling corollary 5.6), there exists $1 \leq i \leq t$ such that

$$\Pr_{\bar{x} \in \mathbb{F}^k} [Q_i(\bar{x}) = A(\bar{x})] \geq \gamma - (8k - 10)\theta_k$$

Claim 9.1.2. $\text{Claim}_2[k] \Rightarrow \text{Claim}_1[k+1]$

Proof. Fix $0 < \gamma \leq 1$ and oracle $A$ such that

$$\mathbb{E}_{s \in S_k} \left[ \Pr_{\bar{x} \in s} [A(s)(\bar{x}) = A(\bar{x})] \right] \geq \gamma$$

Let $s \in S_{k+1}$. Define an oracle relative to $s$, $A_s$, as follows: for every affine subspace $s' = \text{affine}(r)$ in $\mathbb{F}^k$, let $A_s(s') = A(\text{affine}_s(r))$ (the notation $\text{affine}_s$ was introduced in section 4). Let the consistency within $s$ be

$$\gamma_s = \mathbb{E}_{s' \in S_k} \left[ \Pr_{\bar{x} \in s'} [A_s(s')(\bar{x}) = A_s(\bar{x})] \right]$$

By uniformity, the average consistency within $s \in S_k$ is large,

$$\mathbb{E}_{s \in S_k} [\gamma_s] = \mathbb{E}_{s' \in S_k} \left[ \Pr_{\bar{x} \in s'} [A(s')(\bar{x}) = A(\bar{x})] \right] \geq \gamma$$

Claim $2[k]$ implies the existence of a new oracle $A'$ that assigns each hyperplane $s \in S_k$ a polynomial of degree at most $kd$ that identifies with $A$ on at least $\gamma_s - (8k - 10)\theta_k$ of its points. It holds that

$$\mathbb{E}_{s \in S_k} \left[ \Pr_{\bar{x} \in s'} [A'(s)(\bar{x}) = A(\bar{x})] \right] \geq \mathbb{E}_{s \in S_k} [\gamma_s - (8k - 10)\theta_k] \geq \gamma - (8k - 10)\theta_k$$

By lemma 8.3, there exists a polynomial $Q : \mathbb{F}^{k+1} \rightarrow \mathbb{F}$ with $\deg Q \leq 2kd$ such that

$$\Pr_{\bar{x} \in \mathbb{F}^{k+1}} [Q(\bar{x}) = A(\bar{x})] \geq \gamma - (8(k + 1) - 16)\theta_{k+1}$$

Lemma 9.1 follows by induction.

The soundness of the Randomness-Efficient Plane vs. Point tester easily follows.
Proof. (of theorem 1) Assume that
\[ \Pr_{\mathbf{z} \in \mathbb{F}^m, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{F}^m} \left[ \text{PlanePoint}^A(\mathbf{z}, \mathbf{y}_1, \mathbf{y}_2) \right] = \gamma \]

The probability that \( \mathbf{y}_1, \mathbf{y}_2 \) are linearly dependent is at most \( \frac{1}{|\mathbb{F}|^m} + \frac{1}{|\mathbb{H}|^m} \leq \frac{2}{|\mathbb{H}|} \). Thus,
\[ \mathbb{E}_{s \in \mathbb{S}_2^m} \left[ \Pr_{\mathbf{x} \in s} \left[ A(s)(\mathbf{x}) = A(\mathbf{x}) \right] \right] \geq \gamma - \frac{2}{|\mathbb{H}|} \]

By lemma 9.1, we have decoding: there exists a polynomial \( Q : \mathbb{F}^m \to \mathbb{F} \) with \( \deg Q \leq md \), such that
\[ \Pr_{\mathbf{x} \in \mathbb{F}^m} \left[ Q(\mathbf{x}) = A(\mathbf{x}) \right] \geq \gamma - \varepsilon \]

By lemma 6.1, we have list-decoding: for every \( \delta, \delta > 2\varepsilon \), there exist \( t \cdot 2^{2\varepsilon} \) polynomials \( Q_1, \ldots, Q_t : \mathbb{F}^m \to \mathbb{F} \) with \( \deg Q_i \leq md \), such that
\[ \mathbb{E}_{s \in \mathbb{S}_2^m} \left[ \Pr_{\mathbf{x} \in s} \left[ A(s)(\mathbf{x}) \neq A(\mathbf{x}) \lor \exists i (Q_i = A)(s) \right] \right] \geq 1 - \delta - 2\varepsilon + \frac{2}{|\mathbb{H}|} \]

Therefore,
\[ \Pr_{\mathbf{z} \in \mathbb{F}^m, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{H}^m} \left[ \neg \text{PlanePoint}^A(\mathbf{z}, \mathbf{y}_1, \mathbf{y}_2) \lor \exists i (Q_i = A)(\text{affine}(\mathbf{z}; \mathbf{y}_1, \mathbf{y}_2)) \right] \geq 1 - \delta - 2\varepsilon \]

### 10 The Randomness-Efficient Subspace vs. Point Tester is Sound

In this section we use the result from the previous section, namely, the soundness of the Randomness-Efficient Plane vs. Point tester, to prove the soundness of the Subspace vs. Point tester.

Consider the distribution \( \mathcal{D} \) over three-dimensional affine subspaces induced by the tester: pick uniformly \( \mathbf{z} \in \mathbb{F}^m, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{H}^m \), such that \( \mathbf{z}, \mathbf{y}_1, \mathbf{y}_2 \) are linearly independent, and output \( \text{affine}(\mathbf{0}; \mathbf{z}, \mathbf{y}_1, \mathbf{y}_2) \).

**Lemma 10.1 (from Plane vs. Point to Subspace vs. Point).** Fix dimension \( m \geq 3 \). Fix \( \varepsilon \triangleq 2^7 m \left( \sqrt{\frac{\deg}{|\mathbb{F}|}} + \sqrt{\frac{1}{|\mathbb{H}|}} \right) \). If an oracle \( A \) assigning polynomials of degree at most \( d \) to affine subspaces satisfies
\[ \mathbb{E}_{s \sim \mathcal{D}} \left[ \Pr_{\mathbf{x} \in s} \left[ A(s)(\mathbf{x}) = A(\mathbf{x}) \right] \right] \geq \gamma \]
then there exists a polynomial \( Q : \mathbb{F}^m \to \mathbb{F} \) with \( \deg Q \leq md \) such that
\[ \Pr_{\mathbf{x} \in \mathbb{F}^m} \left[ Q(\mathbf{x}) = A(\mathbf{x}) \right] \geq \gamma - \varepsilon \]

**Proof.** Let us construct a new oracle \( A' \). For every plane \( p \in \mathbb{S}_2^m \) that does not contain the origin, let \( A'(p) \) be the restriction of \( A(s) \) to \( p \), where \( s \) is the unique three-dimensional linear subspace that contains \( p \). Let \( A' \) identify with \( A \) on all other affine subspaces.

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For a subspace \( s \sim \mathcal{D}, \) \( s = \text{affine}(\overline{0}; \overline{z}, \overline{y}_1, \overline{y}_2), \) and a random scalar \( \alpha \in \mathbb{F}, \) let \( s_\alpha = \text{affine}(\alpha \overline{z}; \overline{y}_1, \overline{y}_2). \) Clearly, the premise implies that

\[
\mathbf{E}_{s,\alpha} \left[ \Pr_{\overline{x} \in s_\alpha} [A(s)(\overline{x}) = A(\overline{x})] \right] \geq \gamma
\]

The plane \( s_\alpha \) is distributed as follows: with probability \( \frac{1}{|\mathbb{F}|}, \) \( s_\alpha \) is uniformly distributed within the planes in \( S_2^m \) that contain the origin; with probability \( 1 - \frac{1}{|\mathbb{F}|}, \) \( s_\alpha \) is uniformly distributed within the planes in \( S_2^m \) that do not contain the origin.

Hence, noticing that a uniformly distributed plane in \( S_2^m \) contains the origin with probability \( \frac{1}{|\mathbb{F}|}, \)

\[
\mathbf{E}_{p \in S_2^m} \left[ \Pr_{\overline{x} \in p} [A'(p)(\overline{x}) = A(\overline{x})] \right] \geq \gamma - \frac{1}{|\mathbb{F}|}
\]

By lemma 9.1, there exists a polynomial \( Q : \mathbb{F}^m \rightarrow \mathbb{F} \) with \( \deg Q \leq md \) such that

\[
\Pr_{\overline{x} \in \mathbb{F}^m} [Q(\overline{x}) = A(\overline{x})] \geq \gamma - \varepsilon
\]

Now we can apply degree consolidation and get

**Lemma 10.2 (degree consolidated).** Fix dimension \( m \geq 3. \) Fix \( \varepsilon = 2^{-7} m \left( \sqrt{\frac{md}{|\mathbb{F}|}} + \frac{1}{|\mathbb{H}|} \right). \) If an oracle \( \mathcal{A} \) assigning polynomials of degree at most \( d \) to affine subspaces satisfies

\[
\mathbf{E}_{s \sim \mathcal{D}} \left[ \Pr_{\overline{x} \in s} [A(s)(\overline{x}) = A(\overline{x})] \right] \geq \gamma
\]

then there exists a polynomial \( Q : \mathbb{F}^m \rightarrow \mathbb{F} \) with \( \deg Q \leq d \) such that

\[
\Pr_{\overline{x} \in \mathbb{F}^m} [Q(\overline{x}) = A(\overline{x})] \geq \gamma - 2\varepsilon
\]

**Proof.** Assume \( \varepsilon \leq \frac{1}{2} \) (otherwise, we are done). Denote \( \epsilon_0 = \sqrt{\frac{md}{|\mathbb{F}|}}, \delta = 1.5\varepsilon - \epsilon_0. \)

Applying lemma 6.1 and lemma 6.2 on lemma 10.1, we know that there exist \( t \leq 8/\varepsilon \) polynomials \( Q_1, \ldots, Q_t : \mathbb{F}^m \rightarrow \mathbb{F} \) with \( \deg Q_i \leq md, \) such that

1. (each identifies with many planes) for every \( 1 \leq i \leq t, \)

\[
\Pr_{s \sim \mathcal{D}} [(Q_i \equiv \mathcal{A})(s)] > \frac{\epsilon_0}{t} \geq \frac{md}{|\mathbb{F}|} + \frac{1}{|\mathbb{F}|}
\]

2. (all explain almost all the consistency)

\[
\mathbf{E}_{s \sim \mathcal{D}} \left[ \Pr_{\overline{x} \in s} [A(s)(\overline{x}) \neq A(\overline{x}) \lor \exists i(Q_i \equiv \mathcal{A})(s)] \right] \geq 1 - \delta - \epsilon_0
\]

By lemma 6.4, for every \( 1 \leq i \leq t, \deg Q_i \leq d. \) By corollary 5.8, \( \mathcal{D} \) samples well: for every set \( A \subseteq \mathbb{F}^m, \) for every \( 0 < \epsilon < 1, \)

\[
\Pr_{s \sim \mathcal{D}} \left[ \left| \frac{|s \cap A|}{|s|} - \frac{|A|}{|\mathbb{F}^m|} \right| \geq \epsilon \right] \leq \frac{1}{\epsilon^2} \cdot \left( \frac{1}{|\mathbb{H}|} + \frac{1}{|\mathbb{F}|} \right)
\]

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Hence, by lemma 6.3, there exists $1 \leq i \leq t$, such that
\[
\Pr_{\bar{x} \in \mathbb{F}^m} [Q_i(\bar{x}) = A(\bar{x})] \geq \gamma - 2\varepsilon
\]

Our main theorem stating the soundness of the Randomness Efficient Subspace vs. Point tester follows:

**Proof.** (of theorem 2) Assume that
\[
\Pr_{\bar{z} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} [SpacePoint^A(\bar{z}, \bar{y}_1, \bar{y}_2)] = \gamma
\]
The probability that $\bar{z}, \bar{y}_1, \bar{y}_2$ are linearly dependent is very small,
\[
\Pr_{\bar{z} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} [-\text{ind}(\bar{z}, \bar{y}_1, \bar{y}_2)] \leq \frac{1}{|\mathbb{H}|^m} + \frac{1}{|\mathbb{H}|^{m-1}} + \frac{1}{|\mathbb{F}|^{m-2}} \leq \frac{2}{|\mathbb{H}|} + \frac{1}{|\mathbb{F}|}
\]
Hence,
\[
\mathbb{E}_{s \sim \mathcal{D}} \left[ \Pr_{\bar{x} \in \mathcal{S}} [A(s)(\bar{x}) = A(\bar{x})] \right] \geq \gamma - \frac{2}{|\mathbb{H}|} - \frac{2}{|\mathbb{F}|}
\]
By lemma 10.2 we have decoding: there exists a polynomial $Q : \mathbb{F}^m \rightarrow \mathbb{F}$ with $\deg Q \leq d$ such that
\[
\Pr_{\bar{x} \in \mathbb{F}^m} [Q(\bar{x}) = A(\bar{x})] \geq \gamma - 2.5\varepsilon
\]
Lemma 6.1 applied on lemma 10.2 gives list-decoding: there exist $t \leq 2/\delta$ polynomials $Q_1, \ldots, Q_t : \mathbb{F}^m \rightarrow \mathbb{F}$ with $\deg Q_i \leq d$ such that
\[
\mathbb{E}_{s \sim \mathcal{D}} \left[ \Pr_{\bar{x} \in \mathcal{S}} [A(s)(\bar{x}) \neq A(\bar{x}) \lor \exists i (Q_i \equiv A)(s)] \right] \geq 1 - \delta - 2.75\varepsilon
\]
Therefore,
\[
\Pr_{\bar{z} \in \mathbb{F}^m, \bar{y}_1, \bar{y}_2 \in \mathbb{H}^m} [-SpacePoint^A(\bar{z}, \bar{y}_1, \bar{y}_2) \lor \exists i (Q_i \equiv A)(\text{affine}(\bar{0}; \bar{z}, \bar{y}_1, \bar{y}_2))] \geq 1 - \delta - 3\varepsilon
\]

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**References**


A Combinatorial Lemma

For the sake of self-containment, let us prove the lemma of Raz and Safra [17] that we use. First, let us introduce several notations. Given a graph \( G = (V, E) \) and a vertex \( v \in V \), the neighbors of \( v \) are \( \mathcal{N}_G(v) \triangleq \{ u \in V \mid (u, v) \in E \} \). The degree of \( v \) is \( d_G(v) = |\mathcal{N}_G(v)| \). The connected component of \( v \) is \( C_G(v) \triangleq \{ u \in V \mid u \text{ is reachable from } v \} \). The non-neighbors of \( v \) within its connected component are denoted \( D_G(v) = C_G(v) \setminus \{ v \} \cap \mathcal{N}_G(v) \).

**Lemma A.1 (graph partition, [17]).** Let \( G = (V, E) \) be an undirected graph in which every two non-neighbors \( u, v \in V \), \( (u, v) \notin E \), have at most \( \epsilon |V| \) common neighbors. Then, there exists a partition of the vertices into cliques, \( V = \bigcup_{i=1}^{t} V_i \), such that

1. (all non-trivial cliques are large) For every \( 1 \leq i \leq t \), either \( |V_i| = 1 \), or \( |V_i| > 2\sqrt{\epsilon} |V| \).
2. (almost all edges are within cliques)\[ \Pr_{(u, v) \in E} \left[ (u, v) \notin E \lor \exists i, u, v \in V_i \right] \geq 1 - 5\sqrt{\epsilon} \]

**Proof.** Consider the following operation on graphs, meant to improve transitivity by removing some edges:

Pick a vertex \( v \in V \).

1. If \( d_G(v) \leq 2\sqrt{\epsilon} |V| \), remove all the edges that touch \( v \).
2. If \( d_G(v) > 2\sqrt{\epsilon} |V| \), remove all edges between neighbors of \( v \) and non-neighbors of \( v \) (these edges are necessarily within \( v \)'s connected component).

If there is no vertex for which this operation causes removal of edges, then the graph is necessarily transitive, and, moreover, all its non-trivial cliques are of size more than \( 2\sqrt{\epsilon} |V| \).

Hence, iteratively perform this operation, picking each time an arbitrary vertex for which edges would be removed, until this is no longer possible. Let \( v_1, v_2, \ldots, v_l \) denote the picked (not necessarily distinct) vertices. Let \( G_1, G_2, \ldots, G_l \) denote the subgraphs obtained in the \( l \) iterations. Let \( I_1 \) be the set of all indices \( 1 \leq i \leq l \) in which step 1 was performed. Let \( I_2 \) be the set of all indices \( 1 \leq i \leq l \) in which step 2 was performed.

We will bound the total number of edges removed. Observe that if step 1 is performed for a vertex \( v_i \), then its connected component becomes a singleton. Thus, \( |I_1| \leq |V| \), and we have

\[
\sum_{i \in I_1} |\mathcal{N}_{G_i}(v_i)| \leq \sum_{i \in I_1} 2\sqrt{\epsilon} |V| = |I_1| \cdot 2\sqrt{\epsilon} |V| \leq 2\sqrt{\epsilon} |V|^2
\]

Observe that if step 2 is performed for a vertex \( v_i \), then after the \( i \)'th operation, the vertices of \( \mathcal{N}_{G_i}(v_i) \) and the vertices of \( D_{G_i}(v_i) \) do not belong to the same connected component. Thus, \( \sum_{i \in I_2} |D_{G_i}(v_i)| \cdot |\mathcal{N}_{G_i}(v_i)| \leq |V|^2 \) (no pair of vertices appears twice in this sum). By the almost-transitivity, for every \( i \in I_2 \), every vertex \( u \in D_{G_i}(v_i) \) has at most \( \epsilon |V| \) neighbors in \( \mathcal{N}_{G_i}(v_i) \) (each is a common neighbor of \( u \) and \( v_i \)). Therefore, we can bound the total number of edges removed in step 2 by

\[
\sum_{i \in I_2} |D_{G_i}(v_i)| \cdot \epsilon |V| < \sum_{i \in I_2} |D_{G_i}(v_i)| \cdot \epsilon \cdot \frac{|\mathcal{N}_{G_i}(v_i)|}{2\sqrt{\epsilon}} \leq \frac{\sqrt{\epsilon}}{2} \sum_{i \in I_2} |D_{G_i}(v_i)| \cdot |\mathcal{N}_{G_i}(v_i)| \leq \frac{\sqrt{\epsilon}}{2} |V|^2
\]

Therefore, the total number of edges removed is at most \( 2.5\sqrt{\epsilon} |V|^2 \) and the total number of pairs \( u,v \in V \) for which \( (u,v) \in E \) but \( u \) and \( v \) are not in the same clique is at most \( 5\sqrt{\epsilon} |V|^2 \).