Electronic Colloquium on Computational Complexity, Report No. 103 (2005)

A proof of hyperbolic van der Waerden conjecture : the right generalization is the ultimate simplification

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August 15, 2005

Abstract

Consider a homogeneous polynomial $p(z_1, ..., z_n)$ of degree n in n complex variables . Assume that this polynomial satisfies the property :

$|p(z_1, ..., z_n)| \geq \prod_{1 \leq i \leq n} Re(z_i)$ on the domain $\{(z_1, ..., z_n) : Re(z_i) \geq 0, 1 \leq i \leq n\}$.

We prove that $|\frac{\partial^n}{\partial z_1 ... \partial z_n} p| \geq \frac{n!}{n^n}$.
Our proof is relatively short and self-contained (i.e. we only use basic properties of hyperbolic polynomials).
As the van der Waerden conjecture for permanents , proved by D.I. Falikman and G.P. Egorychev , as well Bapat's conjecture for mixed discriminants , proved by the author , are particular cases of this result.
We also prove so called "small rank" lower bound (in the permanents context it corresponds to sparse doubly-stochastic matrices , i.e. with small number of non-zero entries in each column). The later lower bound generalizes (with simpler proofs) recent results by A.Schrijver for k-regular bipartite graphs.
Some important algorithmic applications are presented in the last section .

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ISSN 1433-8092

1 Hyperbolic polynomials

The following concept of hyperbolic polynomials was originated in the theory of partial differential equations [14], [6] ,[7] .
A homogeneous polynomial $p(x), x \in R^m$ of degree n in m real varibles is called hyperbolic in the direction $e \in R^m$ (or e- hyperbolic) if for any $x \in R^m$ the polynomial $p(x - \lambda e)$ in the one variable λ has exactly n real roots counting their multiplicities. We assume in this paper that $p(e) > 0$. Denote an ordered vector of roots of $p(x-\lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq ...\lambda_n(x))$. It is well known that the product of roots is equal to $\frac{p(x)}{p(e)}$. Call $x \in R^m$ e-positive (e-nonnegative) if $\lambda_n(x) > 0$ ($\lambda_n(x) \geq 0$). The fundamental result [14] in the theory of hyperbolic polynomials states that the set of e-nonnegative vectors is a closed convex cone. A k-tuple of vectors $(x_1, ...x_k)$ is called e-positive (e-nonnegative) if $x_i, 1 \leq i \leq k$ are e-positive (e-nonnegative). We denote the closed convex cone of e-nonnegative vectors as $N_e(p)$, and the open convex cone of e-positive vectors as $C_e(p)$.

Recent interest in the hyperbolic polynomials got sparked by the discovery [9] ,[8] that $\log(p(x))$ *is a self-concordant barrier for the opened convex cone* $C_e(p)$ *and therefore the powerful mashinery of interior-point methods can be applied . It is an important open problem whether this cone* $C_e(p)$ *has a semi-definite representation .*

It has been shown in [14] (see also [22]) that an e-hyperbolic polynomial p is also d- hyperbolic for all e-positive vectors $d \in C_e(p)$; for all $d \in C_e(p)$ the set equalities $C_d(p) = C_e(p), N_d(p) = N_e(p)$.

Let us fix n real vectors $x_i \in N_e(p) \subset R^m, 1 \leq i \leq n$ such that $\sum_{1 \leq i \leq n} x_i \in C_e(p)$ and define the following homogeneous polynomial:

$$P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = p(\sum_{1 \leq i \leq n} \alpha_i x_i) \tag{1}$$

We will call such polynomials P-hyperbolic .
In other words ,a homogeneous polynomial $p(\alpha), \alpha \in R^n$ of degree n in n real variables is P-hyperbolic if it is $(1, 1, .., 1)$-hyperbolic ($e = (1, 1, .., 1)$) and its closed cone of e-nonnegative vectors contains the nonnegative orthant $R^n_+ = \{(x_1, ..., x_n) : x_i \geq 0, 1 \leq i \leq n\}$. It follows from [22] that the coefficients of$P$-hyperbolic polynomials are nonnegative real numbers .

Following [22] , we define the p-mixed form of an n-vector tuple $\mathbf{X} = (x_1, .., x_n)$ as

$$M_p(\mathbf{X}) =: M_p(x_1, .., x_n) = \frac{\partial^n}{\partial \alpha_1 ... \partial \alpha_n} p(\sum_{1 \leq i \leq n} \alpha_i x_i) \tag{2}$$

Equivalently, the p-mixed form $M_p(x_1, .., x_n)$ can be defined by the polarization (see [22]) :

$$M_p(x_1, .., x_n) = 2^{-n} \sum_{b_i \in \{-1,+1\}, 1 \leq i \leq n} p(\sum_{1 \leq i \leq n} b_i x_i) \prod_{1 \leq i \leq n} b_i \tag{3}$$

Associate with any vector $r = (r_1, ..., r_n) \in I_{n,n}$ an n-tuple of m-dimensional vectors $\mathbf{X}_r$ consisting of r_i copies of $x_i (1 \leq i \leq n)$. It follows from the Taylor's formula that

$$P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) = \sum_{r \in I_{n,n}} \prod_{1 \leq i \leq n} \alpha_i^{r_i} M_p(\mathbf{X}_r) \frac{1}{\prod_{1 \leq i \leq n} r_i!} \tag{4}$$

For an e-nonnegative tuple $\mathbf{X} = (x_1, .., x_n)$, define its capacity as:

$$Cap(\mathbf{X}) = \inf_{\alpha_i > 0, \prod_{1 \leq i \leq n} \alpha_i = 1} P_{x_1,..,x_n}(\alpha_1, ..., \alpha_n) \tag{5}$$

Probably the best known example of a hyperbolic polynomial comes from the hyperbolic geometry :

$$P(\alpha_0, ..., \alpha_k) = \alpha_0^2 - \sum_{1 \leq i \leq k} \alpha_i^2 \tag{6}$$

This polynomial is hyperbolic in the direction $(1, 0, 0, ..., 0)$. Another "popular" hyperbolic polynomial is $\det(X)$ restricted on a linear real space of hermitian $n \times n$ matrices . In this case mixed forms are just mixed discriminants , hyperbolic direction is the identity matrix I , the corresponding closed convex cone of I-nonnegative vectors coincides with a closed convex cone of positive semidefinite matrices .
Less known , but very interesting , hyperbolic polynomial is the Moore determinant $M\det(Y)$ restricted on a linear real space of hermitian quaternionic $n \times n$ matrices . The Moore determinant is , essentially , the Pfaffian (see the corresponding definitions and the theory in a very readable paper [36]) . The following definition is from [5].

Definition 1.1: A polynomial $P(z_1, ..., z_n)$ in n complex variables is said to have the "half-plane property" if $P(z_1, ..., z_n) \neq 0$ provided $Re(z_i) > 0$. ∎

In a control theory literature (see [32]) the same property is called *Wide sense stability* . And *Strict sense stability* means that
$P(z_1, ..., z_n) \neq 0$ provided $Re(z_i) \geq 0$.

The following simple fact shows that for homogeneous polynomials the "half-plane property" is , up to a single factor , the same as P-hyperbolicity .

Proposition 1.2: *A homogeneous polynomial $R(z_1, ..., z_n)$ has the "half-plane" property if and only if the exists real α such that the polynomial $e^{i\alpha} R(z_1, ..., z_n)$ is P-hyperbolic polynomial with real nonnegative coefficients .*

Proof:

1. Suppose that $R(z_1, ..., z_n) = e^{-i\alpha} Q(z_1, ..., z_n)$ where α is real and Q is P-hyperbolic. Then Q is $(1, 1, ...,)$-hyperbolic and all real vectors $(x_1, ..., x_n)$ with positive coordinates are $(1, 1, ...,)$-positive . Therefore Q is $(x_1, ..., x_n)$-hyperbolic for all real vectors $(x_1, ..., x_n) \in R^n_{++}$ with positive coordinates .It follows that $|R(x_1 + iy_1, ..., x_n + iy_n)| = |Q(x_1 + iy_1, ..., x_n + iy_n)| = |Q(x_1, ..., x_n) \prod_{1 \leq k \leq n} (1 + i\lambda_k)|$, where $(\lambda_1, ..., \lambda_n)$ are real

roots of the real vector $(y_1, ..., y_n)$ in the direction $(x_1, ..., x_n)$.
This gives the following inequality , which is equivalent to the "half-plane property" of R :

$$|R(x_1 + iy_1, ..., x_n + iy_n)| \geq |R(x_1, ..., x_n)| = \tag{7}$$
$$= |Q(x_1, ..., x_n)| > 0 :$$
$$(x_1, ..., x_n) \in R^n_{++}, (y_1, ..., y_n) \in R^n$$

2. Suppose that $R(z_1, ..., z_n)$ has the "half-plane property" and consider the roots of the following polynomial equation in one complex variable : $P(x_1 - z, x_2 - z, ..., x_n - z) = 0$, where $(x_1, ..., x_n) \in R^n$ is a real vector , $z = x + iy \in C$. If the imaginery part $Im(z) = y$ is not zero then , using the homogeniuty , $R(i\frac{x-x_1}{y} + 1, ..., i\frac{x-x_n}{y} + 1) = 0$, which is impossible as R has the "half-plane property". Therefore all roots of $R(X - te) = 0$ are real for all real vectors $X \in R^n$ (here $e = (1, 1, ..., 1)$). In the same way all roots of $R(X - te) = 0$ are real positive numbers if $X \in R^n_{++}$. It follows that if $X \in R^n$ then $R(X) = R(e) \prod 1 \leq k \leq n \lambda_k(X)$, where $(\lambda_1, ..., \lambda_n)$ are (real) roots of the equation $R(X - te) = 0$. Thus the polynomial $(\frac{1}{R(e)})R$ takes real values on R^n and therefore its coefficients are real . In other words , the polynomial $(\frac{1}{R(e)})R$ is P-hyperbolic . If $R(1, 1, ..., 1) = e^{-i\alpha}|R(1, 1, ..., 1)|$ then the polynomial $e^{i\alpha}R$ is also P-hyperbolic .
(Recall that the coefficients of any P-hyperbolic polynomial p are nonnegative for they are p-mixed forms of e-nonnegative tuples , and p-mixed forms of e-nonnegative tuples are nonnegative if $p(e) > 0$ [22].)

∎

Corollary 1.3: *Let $p(x_1, ..., x_n)$ be a homogeneous polynomial in n variables and of degree n . Assume that $p(1, 1, ..., 1) > 0$. Then the property "polynomial p is P-hyperbolic and its capacity $Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, .., x_n) = C > 0$ " is equivalent to the property "$\inf_{Re(z_i) > 0, \prod_{1 \leq i \leq n} Re(z_i) = 1} |p(z_1, ..., z_n)| = C > 0$".*

Proof: Suppose that
"polynomial p is P-hyperbolic and its capacity $Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, .., x_n) = C > 0$ " .
Then , as in (7) ,

$$|p(x_1 + iy_1, ..., x_n + iy_n)| \geq p(x_1, ..., x_n) \geq C \prod_{1 \leq i \leq n} x_i; x_i \geq 0, y_i \in R, 1 \leq i \leq n.$$

Therefore $\inf_{Re(z_i) > 0, \prod_{1 \leq i \leq n} Re(z_i) = 1} |p(z_1, ..., z_n)| = C$.

Assume that
"$\inf_{Re(z_i) > 0, \prod_{1 \leq i \leq n} Re(z_i) = 1} |p(z_1, ..., z_n)| = C > 0$" .

Since $p(1,1,...,1) > 0$, it follows from Proposition 1.2 that p is P-hyperbolic . The equality $Cap(p) = \inf_{Re(z_i)>0, \prod_{1 \le i \le n} Re(z_i)=1} |p(z_1,...,z_n)|$ follows . ∎

Remark 1.4: Corollary 1.3 essentially says that if $p(x_1,...,x_n)$ is a homogeneous polynomial in n variables and of degree n with real nonnegative coefficients and its complex capacity
$C - Cap(p) =: \inf_{Re(z_i)>0, \prod_{1 \le i \le n} Re(z_i)=1} |p(z_1,...,z_n)| = C > 0$
then its (real) capacity
$Cap(p) = \inf_{x_i>0, \prod_{1 \le i \le n} x_i=1} p(x_1,..,x_n) = C - Cap(p)$.
If $C - Cap(p) = 0$ then this statement can be wrong . I.e. consider $q(x_1,...,x_n) = \frac{\sum_{1 \le i \le n} x_i^n}{n}$.
Then $Cap(q) = 1$ and $C - Cap(q) = 0$. ∎

We use in this paper the following class of hyperbolic in the direction $(1,1,...,1)$ polynomials of degree k :
$Q(\alpha_1,...,\alpha_k) = M_p(\sum_{1 \le i \le k} \alpha_i x_i, ..., \sum_{1 \le i \le k} \alpha_i x_i, x_{k+1}, ..., x_n)$, where p is a e-hyperbolic polynomial of degree $n > k$, $(x_1,..,x_n)$ is e-nonnegative tuple , and the p-mixed form
$M_p(\sum_{1 \le i \le k} x_i, ..., \sum_{1 \le i \le k} x_i, x_{k+1}, ..., x_n) > 0$.

2 Main Theorem

Theorem 2.1:

1. *Let $q(x_1,x_2,...,x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n . Then*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} q(0,...,0) \ge \frac{n!}{n^n} Cap(q) \tag{8}$$

2. *This bound is attained only on the following class of polynomials :*

$$q_{a_1,...,a_n}(x_1,...,x_n) = (\frac{\sum_{1 \le i \le n} a_i x_i}{n})^n; a_i > 0, 1 \le i \le n.$$

(Notice that $Cap(q_{a_1,...,a_n}) = \prod_{1 \le i \le n} a_i$.)

2.1 Auxiliary Results

Proposition 2.2:

1. *Let $c_1,...,c_n$ be real numbers ; $0 \le c_i \le 1, 1 \le i \le n$ and $\sum_{1 \le i \le n} c_i = n-1$.*
   Define the following symmetric functions :

$$S_n = \prod_{1 \le i \le n} c_i, S_{n-1} = \sum_{1 \le i \le n} \prod_{j \ne i} c_j.$$

Then the following entropic inequality holds :

$$S_{n-1} - nS_n \ge e^{\sum_{1 \le i \le n} c_i \log(c_i)}.$$

2. *(Mini van der Waerden conjecture)*
   Consider a doubly-stochastic $n \times n$ matrix $A = [a|b|...|b]$. I.e. A has $n-1$ columns equal to the column vector b , and one column equal to the column vector a . Let $a = (a_1, ..., a_n)^T : a_i \geq 0, sum_{1 \leq i \leq n} a_i = 1$; $b = (b_1, ..., b_n)^T : b_i = \frac{1-a_i}{n-1}, 1 \leq i \leq n$. Then the permanent $Per(A) \geq \frac{n!}{n^n}$.

Proof:

1. Doing simple "algebra" we get that

$$S_{n-1} - nS_n = \prod_{1 \leq i \leq n} c_i (\sum_{1 \leq i \leq n} \frac{1-c_i}{c_i}).$$

   Notice that $0 \leq 1 - c_i \leq 1$ and $\sum_{1 \leq i \leq n} (1-c_i) = 1$. Using the concavity of the logarithm we get that

$$\log(S_{n-1} - nS_n) \geq \sum_{1 \leq i \leq n} \log(c_i) + \sum_{1 \leq i \leq n} (1-c_i) \log(\frac{1}{c_i}) = \sum_{1 \leq i \leq n} c_i \log(c_i).$$

2.

$$per(A) = \frac{(n-1)!}{(n-1)^{n-1}} \sum_{1 \leq i \leq n} a_i \prod_{j \neq i} (1 - a_j).$$

   Define $c_i = 1 - a_i$. Then $0 \leq 1 - c_i \leq 1$, $\sum_{1 \leq i \leq n} c_i = n - 1$ and the permanent

$$Per(A) = \frac{(n-1)!}{(n-1)^{n-1}} (S_{n-1} - nS_n).$$

   It is easy to prove and well known that

$$\min_{0 \leq 1-c_i \leq 1; \sum_{1 \leq i \leq n} c_i = n-1} \sum_{1 \leq i \leq n} c_i \log(c_i) = \sum_{1 \leq i \leq n} \frac{n-1}{n} \log(\frac{n-1}{n}) = \log((\frac{n-1}{n})^{n-1}).$$

   Using the entropic inequality from the first part we get the following equality

$$\min_{0 \leq 1-c_i \leq 1; \sum_{1 \leq i \leq n} c_i = n-1} S_{n-1} - nS_n = (\frac{n-1}{n})^{n-1}.$$

   Which gives the needed inequality

$$Per(A) \geq \frac{(n-1)!}{(n-1)^{n-1}} (\frac{n-1}{n})^{n-1} = \frac{n!}{n^n}.$$

   It is easy to see (strict concavity of $\sum_{1 \leq i \leq n} c_i \log(c_i)$) that the last inequality is strict unless $A(i,j) = \frac{1}{n}; 1 \leq i, j \leq n$.

∎

Corollary 2.3: *Define capacity of $n \times n$ matrix A with nonnegative entries as*

$$Cap(A) = \inf_{x_j > 0} \frac{\prod_{1 \le i \le n} \sum_{1 \le j \le n} A(i,j) x_j}{\prod_{1 \le j \le n} x_j}$$

If $A = [c|d|...|d]$ then $Per(A) \ge \frac{n!}{n^n} Cap(A)$.

Proof: Sinkhorn's diagonal scaling to doubly-stochastic matrices does the job. I.e. , if $A = [c|d|...|d]$ and all entries of A are positive then there exist two diagonal matrices with positive entries D_1, D_2 and a doubly-stochastic matrix $B = [a|b|...|b]$ such that $\det(D_1 D_2) = Cap(A)$ and $A = D_1 B D_2$. ∎

Corollary 2.4: *Consider an univariate polynomial* $R(t) = \sum_{0 \le i \le n} d_i t^i = \prod_{1 \le i \le n} (a_i t + b_i)$ *, where $a_i, b_i \ge 0$. If for some positive real number C the inequality $R(t) \ge Ct$ holds for all $t \ge 0$ then*

$$d_1 = \frac{\partial}{\partial t} R(0) \ge C\left(\left(\frac{n-1}{n}\right)^{n-1}\right) \tag{9}$$

The inequality (9) is attained on the polynomial $R(t) = n^{-n}(t + n - 1)^n$.

Proof: Associate with polynomial $R(t) = \prod_{1 \le i \le n} (a_i t + b_i)$ the following matrix $A = [a|c|...|c]$, where $a = (a_1, ..., a_n)^T, c = \frac{1}{n-1}(b_1, ..., b_n)^T$. The condition $R(t) \ge Ct, \forall t \ge 0$ is equivalent to the inequality $Cap(A) \ge C$. And $d_1 = \frac{(n-1)!}{(n-1)^{n-1}} Per(A)$. It follows from Corollary 2.3 that

$$d_1 = \left(\frac{(n-1)!}{(n-1)^{n-1}}\right)^{-1} Per(A) \ge \left(\frac{(n-1)!}{(n-1)^{n-1}}\right)^{-1} \left(\frac{n!}{n^n} C\right) = C\left(\left(\frac{n-1}{n}\right)^{n-1}\right).$$

∎

Proposition 2.5: *Let $p(X)$ be e-hyperbolic (homogeneous) polynomial of degree n , $p(e) > 0$. Consider two e-nonnegative vectors $Z, Y \in N_e(p)$ such that $Z + Y \in C_e(p)$, i.e. $Z + Y$ is e-positive . Then*

$$p(tZ + Y) = \prod_{1 \le i \le n} (a_i t + b_i); a_i, b_i \ge 0, a_i + b_i > 0, 1 \le i \le n. \tag{10}$$

Proof: As the vector $Z + Y = D$ is e-positive hence $p(Z + Y) > 0$,the polynomial p is $Z + Y$-hyperbolic and any e-positive (e-nonnegative) is also $Z + Y$-positive($Z + Y$-nonnegative) [22] . Doing simple algebra , we get that $p(tZ + Y) = p((t-1)Z + D)$.

Let $0 \le \lambda_1 \le \lambda_2 \le ... \le \lambda_n$ be nonnegative roots of the equation $p(Z - xD)) = 0$. Since $D - X = Y \in N_e(p) = N_D(p)$ hence $\lambda_n \le 1$. Therefore

$$p(tZ + Y) = p((t-1)Z + D) = p(D) \prod_{1 \le i \le n} (t\lambda_i + (1 - \lambda_i))$$

We can put $a_i = (p(Z + Y))\lambda_i \ge 0, b_i = (p(Z + Y))(1 - \lambda_i) \ge 0$. ∎

Proposition 2.6: *Let $q(x_1, x_2, ..., x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n . Define a new homogeneous polynomial of degree $n-1$ in $n-1$ variables :*

$$r(x_2, ..., x_n) = \frac{\partial}{\partial x_1} q(0, x_2, ..., x_n).$$

If $Cap(q) > 0$ then the polynomial r is also P-hyperbolic .

Proof: Proved in [35] , easy modification of the argument in [22] , essentially the Rolle's theorem . ▮

The next Lemma is the final auxiliary Result .

Lemma 2.7:

Define $F(n) = \frac{n!}{n^n}$. The following inequality holds :

$$Cap(r) \geq \frac{F(n)}{F(n-1)} Cap(q) = ((\frac{n-1}{n})^{n-1}) Cap(q) \tag{11}$$

Proof: Fix positive real numbers $(x_2, ..., x_n)$ such that $\prod_{2 \leq i \leq n} x_i = 1$. Define the following two real n-dimensionals vectors with nonnegative coordinates : $Z = (1, 0, 0, ..., 0), Y = (0, x_2, ..., x_n)$. The vector $Z + Y$ is e-positive . Consider the next univariate polynomial $R(t) = p(tZ + Y)$. It follows from Proposition 2.5 that

$$R(t) = \prod_{1 \leq i \leq n} (a_i t + b_i) = \sum_{0 \leq i \leq n} d_i t^i,$$

where $a_i, b_i \geq 0$ and $r(x_2, ..., x_n) = d_1$.
We get from the definition of $Cap(q)$ that

$$R(t) = \prod_{1 \leq i \leq n} (a_i t + b_i) = p(t, x_2, ..., x_n) \geq Cap(q) t \prod_{2 \leq i \leq n} x_i = tCap(q).$$

Using Corollary 2.4 , we get that

$$r(x_2, ..., x_n) = d_1 \geq ((\frac{n-1}{n})^{n-1}) Cap(q).$$

In other words , that $Cap(r) \geq \frac{F(n)}{F(n-1)} Cap(q) = (\frac{n-1}{n})^{n-1}) Cap(q)$.

2.2 Proof of the Main Theorem

(Only first part of Theorem 2.1 is proved in this draft . The uniqueness part will be presented in the final version .)
Proof: Our proof is by (simple and natural) induction in n . Theorem 2.1 is clearly true for $n = 1$. Suppose it is true for all $k \leq n-1$. Let $q(x_1, x_2, ..., x_n)$ be a P-hyperbolic

(homogeneous) polynomial of degree n and $Cap(q) = C > 0$. Then using Lemma 2.7 we get that

$$Cap(r) \geq ((\frac{n-1}{n})^{n-1})C = \frac{F(n)}{F(n-1)}C,$$

where $F(n) = \frac{n!}{n^n}$ and $r(x_2, ..., x_n) = \frac{\partial}{\partial x_1} q(0, x_2, ..., x_n)$ is a P-hyperbolic (homogeneous) polynomial of degree $n-1$. Using induction we get the needed inequality

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} q(x_1, ..., x_n) = \frac{\partial^{n-1}}{\partial x_2 ... \partial x_n} r(x_2, ..., x_n) \geq F(n-1)Cap(r) \geq F(n-1)\frac{F(n)}{F(n-1)}Cap(q) = \frac{n!}{n^n}Cap(q).$$

∎

Example 2.8: Consider a n-tuple of quaternionic hermitian $n \times n$ matrices $\mathbf{H} = (H_1, ..., H_n)$ and define the following homogeneous polynomial of degree n in n real variables :

$$Q_{\mathbf{H}}(x_1, ..., x_n) = M\det(\sum_{1 \leq i \leq n} x_i H_i),$$

where $M\det$ is the Moore determinant (consult the fantastic survey [36] on the subject of various quaternionic determinants). It is well known that right eigenvalues of quaternionic hermitian matrices are real (in this case the Moore's determinant is equal to the product of right eigenvalues) , quaternionic hermitian matrices with all right eigenvalues being nonnegative called quaternionic positive semidefinite (we write $H \succeq 0$ if the quaternionic hermitian matrix H is quaternionic positive semidefinite .) The $tr(H)$ is equal to the sum of all (real) right eigenvalues of H .
A n-tuple of quaternionic hermitian $n \times n$ matrices $\mathbf{H} = (H_1, ..., H_n)$ is called doubly stochastic if :

$$H_i \succeq 0, tr(H_i) = 1, 1 \leq i \leq n; \sum_{1 \leq i \leq n} H_i = I.$$

It is straigthforward to prove that if the tuple $\mathbf{H} = (H_1, ..., H_n)$ is doubly stochastic then the polynomial $Q_{\mathbf{H}}(x_1, ..., x_n)$ is P-hyperbolic and $Cap(Q_{\mathbf{H}}) = 1$. It follows from Theorem 2.1 that if the tuple $\mathbf{H} = (H_1, ..., H_n)$ is doubly stochastic then the following inequlity holds :

$$HM(\mathbf{H}) =: \frac{\partial^n}{\partial x_1 ... \partial x_n} M\det(\sum_{1 \leq i \leq n} x_i H_i) \geq \frac{n!}{n^n} \qquad (12)$$

If the tuple $\mathbf{H} = (H_1, ..., H_n)$ consists of real diagonal positive semidefinite matrices then inequality (12) is the statement of the van der Waerden conjecture for permanents proved in [12] ; if the tuple $\mathbf{H} = (H_1, ..., H_n)$ consists of complex hermitian positive semidefinite matrices then inequality (12) is the statement of the Bapat's conjecture [3] for mixed discriminants proved by the author in [31] . Even this quaternionic case seems to be a new result . ∎

Remark 2.9: Notice that we did not use Falikman-Egorychev theorem ([12] , [11]) which proves the "first" van der Waerden Conjecture [2] , bur rather its particularly simple case (Proposition 2.2) . Theorem 2.2 generalizes all known variants of van der Waerden Conjecture ([3] , [31] and others ...). It also proves as Hall's theorem on perfect bipartite matchings , Rado's theorem and its hyperbolic analogue [35] , [15] . And we did not use the Alexandrov-Fenchel inequalities ... The main "spring" of our proof is that we work in a very large class of P-hyperbolic polynomials , this class is large enough to allow the easy induction. In fact , the clearest (in our opinion) proof of the Alexandrov-Fenchel inequalities for mixed discriminants is in A.G. Khovanskii' 1984 paper [22] . The Khovanskii' proof is based on the similar induction (via partial differentions) to the one used in our paper . In a way , the Alexandrov-Fenchel inequalities are "hidden" in our proof .

∎

2.3 Small Rank Lower Bound

Definition 2.10: Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction e. Denote an ordered vector of roots of $p(x - \lambda e)$ as $\lambda(x) = (\lambda_1(x) \geq \lambda_2(x) \geq ...\lambda_n(x))$. We define the p-rank of $x \in R^m$ in direction e as $Rank_p(x) = |\{i : \lambda_i(x) \neq 0\}|$. It follows from Theorem 1.5 that the p-rank of $x \in R^m$ in any direction $d \in C_e$ is equal to the p-rank of $x \in R^m$ in direction e , which we call the p-rank of $x \in R^m$. ∎

Consider the following polynomial in one variable $D(t) = p(td + x) = \sum_{0 \leq i \leq n} c_i t^i$. It follows from the identity (4) that

$$\begin{aligned} c_n &= M_p(d, .., d)(n!)^{-1} = p(d), \\ c_{n-1} &= M_p(x, d, .., d)(1!(n-1)!)^{-1}, ..., \\ c_0 &= M_p(x, .., x)(n!)^{-1} = p(x). \end{aligned} \tag{13}$$

Let $(\lambda_1^{(d)}(x) \geq \lambda_2^{(d)}(x) \geq ... \geq \lambda_n^{(d)}(x))$ be the (real) roots of x in the e-positive direction d, i.e. the roots of the equation $p(td - x) = 0$. Define (canonical symmetric functions) :

$$S_{k,d}(x) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1}(x)\lambda_{i_2}(x)...\lambda_{i_k}(x).$$

Then $S_{k,d}(x) = \frac{c_{n-k}}{c_n}$. Clearly if x is e-nonnegative then for any e-positive vector d the p-rank $Rank_p(x) = \max\{k : S_{k,d}(x) > 0\}$. The following usefull result can be found in [22] (the proof is essentially the same induction via partial differentions).

Fact 2.11: Consider a homogeneous polynomial $p(x), x \in R^m$ of degree n in m real variables which is hyperbolic in the direction $e, p(e) > 0$. Then the following statements are true :

1. The p-mixed form $M_p(y_1, ..., y_n)$ is linear in each $y_i \in R^M$ when the rest is fixed .
2. If the vectors $y_i, 1 \leq i \leq n$ are e-positive (e-nonnegative) then $M_p(y_1, ..., y_n) > 0$ ($M_p(y_1, ..., y_n) \geq 0$).

3. If the vectors $y_i, z_i, y_i - z_i \in R^M$:; $1 \leq i \leq n$ are e-nonnegative then

$$M_p(y_1, ..., y_n) \geq M_p(z_1, ..., z_n).$$

One of the corollaries of this fact is that for e-nonnegative vectors x the number of positive roots of the univariate equation $p(td - x) = 0$ is the same for all e-positive vectors d.

Proposition 2.12: *Let $q(x_1, x_2, ..., x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n and $Cap(q) > 0$. Define a new homogeneous polynomial of degree $n-1$ in $n-1$ variables :*

$$r(x_2, ..., x_n) = \frac{\partial}{\partial x_1} q(0, x_2, ..., x_n).$$

Let $(e_1, e_2, ..., e_n)$ be a canonical basis in R^n . In other words , the vector $e_i \in R^n$ is the ith column of $n \times n$ identity matrix I . Then for all $2 \leq i \leq n$ the following inequality holds

$$Rank_r(e_i) \leq \min(Rank_q(e_i), n-1) \tag{14}$$

Proof: First we recall the following formula , expressing the polynomial $\frac{\partial}{\partial x_1} q(0, x_2, ..., x_n)$ in terms of q-mixed forms ([22],[35]) :

$$r(x_2, ..., x_n) = M_q(e_1, z, ..., z)((n-1)!)^{-1}, z = (0, x_2, ..., x_n)^T.$$

Clearly , $Rank_r(e_i) \leq n-1 \leq \min(Rank_q(e_i), n-1)$ if $Rank_q(e_i) \geq n-1$. Suppose that $Rank_q(e_i) = R_i \leq n-2$. Since the vectors $(e_1, e_2, ..., e_n)$ are e-nonnegative hence

$$M_q(e, ..., e, e_i, ..., e_i) = 0,$$

where the n-tuple $(e, ..., e, e_i, ..., e_i)$ contains $R_i + 1 = Rank_r(e_i) + 1$ copies of e_i and $n - 1 - R_i$ copies of $e = (1, 1, ..., 1)^T$. Define $d = \sum_{2 \leq i \leq n} e_i = e - e_1$. To prove that $Rank_r(e_i) \leq \min(Rank_q(e_i), n-1)$ we need to prove that $M_r(d, .., d, e_i, ..., e_i) = 0$, where the $n-1$-tuple $(d, .., d, e_i, ..., e_i)$ contains $R_i + 1 = Rank_r(e_i) + 1$ copies of e_i and $n - 2 - R_i$ copies of d. But

$$M_r(d, .., d, e_i, ..., e_i) = M_q(e_1, d, ..., d, e_i, ..., e_i).$$

We have now two n-tuples $\mathbf{T}_1 = (e, ..., e, e_i, ..., e_i)$ and $\mathbf{T}_2 = (e_1, d, ..., d, e_i, ..., e_i)$. The n-tuples $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_1 - \mathbf{T}_2$ consist of e-nonnegative vectors . Therefore , using the monotonicity result from [22] , we get that $M_r(d, .., d, e_i, ..., e_i) \leq M_q(e, ..., e, e_i, ..., e_i) = 0$. ∎

Lemma 2.13: *Let $q(x_1, x_2, ..., x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n and $Rank_q(e_1) = k$. Then following inequality holds :*

$$Cap(r) \geq ((\frac{k-1}{k})^{k-1}) Cap(q) \tag{15}$$

Proof: (Very similar to the proof of Lemma 2.7).

Fix positive real numbers $(x_2, ..., x_n)$ such that $\prod_{2 \leq i \leq n} x_i = 1$. Define the following two real n-dimensionals vectors with nonnegative coordinates : $Z = (1, 0, 0, ..., 0), Y = (0, x_2, ..., x_n)$. The vector $Z + Y$ is e-positive . Consider the next univariate polynomial $R(t) = p(tZ + Y)$. It follows from Proposition 2.5 that

$$R(t) = \prod_{1 \leq i \leq n} (a_i t + b_i) = \sum_{0 \leq i \leq n} d_i t^i,$$

where $a_i, b_i \geq 0$ and $r(x_2, ..., x_n) = d_1$ and the cardinality $|\{i : a_i > 0\}| = k$. In other words the degree $deg(R) = k$
We get from the definition of $Cap(q)$ that

$$R(t) = \prod_{1 \leq i \leq n} (a_i t + b_i) = p(t, x_2, ..., x_n) \geq Cap(q) t \prod_{2 \leq i \leq n} x_i = tCap(q).$$

Using Corollary 2.4 , we get that

$$r(x_2, ..., x_n) = d_1 \geq ((\frac{k-1}{k})^{k-1}) Cap(q).$$

In other words , that $Cap(r) \geq (\frac{k-1}{k})^{k-1}) Cap(q)$. ▮

Theorem 2.14:

1. *Let $q(x_1, x_2, ..., x_n)$ be a P-hyperbolic (homogeneous) polynomial of degree n ; $Rank_q(e_i) = R_i$. Define $G_i = \min(R_i, n + 1 - i)$ Then*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} q(0, ..., 0) \geq \prod_{1 \leq i \leq n} (\frac{G_i - 1}{G_i})^{G_i - 1} Cap(q) \tag{16}$$

2. *If $Rank_q(e_i) \leq k \leq n$ then*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} q(0, ..., 0) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k} Cap(q) \tag{17}$$

Proof: We use the same induction as in the proof of Theorem 2.1 together with Proposition 2.12 and Lemma 2.13 . ▮

The following result is a direct corollary Theorem 2.14 . Even the permanental inequality (18) seems to be new (compare (18) with the corresponding result from [33]). The easiness of our proof (compare again with [33]) suggests that the "method of hyperbolic polynomials" introduced in this paper is very powerful and natural .

Corollary 2.15:

1. *Consider a doubly-stochastic n-tuple $\mathbf{A} = (A_1, ..., A_n)$ of $n \times n$ hermitian positive semidefinite matrices ,*
   i.e. $A_i \succeq 0, tr(A_i = 1; 1 \leq i \leq n$ and $\sum_{1 \leq i \leq n} A_i = I$.
   If $Rank(A_i) \leq k \leq n$ then the mixed discriminant

$$M(A_1, ..., A_n) =: \frac{\partial^n}{\partial x_1 ... \partial x_n} \det(\sum_{1 \leq i \leq n} x_i A_i) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k}.$$

2. *Let $A = \{A(i,j) : 1 \leq i,j \leq n\}$ be a doubly-stochastic $n \times n$ matrix . Suppose that the cardinalities $|\{j : A(i,j) > 0\}| \leq k \leq n$ for $1 \leq i \leq n-k$. Then the following permanental inequality holds :*

$$Per(A) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \frac{k!}{k^k}. \qquad (18)$$

3 Applications

Suppose that a P-hyperbolic (aka *Strict sense stable* homogeneous polynomial)

$$p(x_1, ..., x_n) = \sum_{\sum_{1 \leq i \leq n} r_i = n} a_{(r_1, ..., r_n)} \prod_{1 \leq i \leq n} x_i^{r_i}$$

has nonnegative integer components coefficients and given as an oracle . I.e. we don't have a list coefficients , but can evaluate $p(x_1, ..., x_n)$ on rational inputs .
An algorithm is called deterministic polynomial-time oracle if it evaluates the given polynomial $p(.)$ at a number of rational vectors $(q_1, ..., q_n)$ which is polynomial in n and $\log(p(1,1,..,1))$; these rational vectors $(q_1, ..., q_n)$ are supposed to have bit-wise complexity which is polynomial in n and $\log(p(1,1,..,1))$; and the additional auxilary arithmetic computations also take a polynomial number of steps in n and $\log(p(1,1,..,1))$.

The following theorem combines the algorithm from [35] and Theorem 2.1 .

Theorem 3.1: *There exists a deterministic polynomial-time oracle algorithm which computes for given as an oracle P-hyperbolic polynomial $p(x_1, ..., x_n)$ a number $F(p)$ satisfying the inequality*

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0, ..., 0) \leq F(p) \leq e^n \frac{\partial^n}{\partial x_1 ... \partial x_n} p(x_1, ..., x_n).$$

Theorem 3.1 can be (slightly) improved . I.e. it can be applied to the polynomial

$$p_k(x_{k+1}, ..., x_n) = \frac{\partial^k}{\partial x_1 ... \partial x_k} p(0, .., 0, x_{k+1}, ..., x_n).$$

Notice that the polynomial p_k is a homogeneous polynomial of degree $n-k$ in $n-k$ variables . It is easy to prove that if $p = p_0$ is P-hyperbolic and $Cap(p) > 0$ then for all $0 \leq k \leq n$ the

polynomials p_k are also P-hyperbolic and $Cap(p_k) > 0$.
The trick is that if $k = m \log_2(n)$ then the polynomial p_k can be evaluated using $O(n^{m+1})$ oracle calls of the (original) polynomial p . This observations allows to decrease the multiplicative factor in Theorem 3.1 from e^n to $\frac{e^n}{n^m}$ for any fixed m . If the polynomial p can be explicitly evaluated in deterministic polynomial time , this observation results in deterministic polynomial time algorithms to approximate $\frac{\partial^n}{\partial x_1 ... \partial x_n} p(0,...,0)$ within multiplicative factor $\frac{e^n}{n^m}$ for any fixed m . Which is an improvement of results in [16] (permanents , p is a multilinear polynomial) and in [17] , [18] (mixed discriminants p is a determinantal polynomial) .

4 Open Problems and Acknowledgements

Problem 4.1: Is first part of Theorem 2.1 true for the volume polynomials $p(x_1, ..., x_n) = Volume(\sum_{1 \leq i \leq n} x_i C_i)$, where $C_i, 1 \leq i \leq n$ are convex compact subsets of R^n ?
Not all volume polynomials are P-hyperbolic (see the example in [22]) . ∎

Problem 4.2: What is a "good" model of a random P-hyperbolic polynomial ? By "good" we mean that with high probability the inequality (8) is much tighter . I.e. with high probability

$$\frac{\partial^n}{\partial x_1 ... \partial x_n} q(0,...,0) \geq (1 + O(n^{-1}))Cap(q)$$

∎

After the first draft had been posted Hugo Woerdeman found more direct proof of Corollary 2.4 .
I would like to thank Mihai Putinar , Sergey Fomin , George Soules , Alex Samorodnitsky for the interest to this paper .

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