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On the expansion of the giant component in percolated (n, d, λ) graphs

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Abstract

Let $d \geq d_0$ be a sufficiently large constant. A $(n, d, c\sqrt{d})$ graph G is a d-regular graph over n vertices whose second largest (in absolute value) eigenvalue is at most $c\sqrt{d}$. For any $0 < p < 1$, G_p is the graph induced by retaining each edge of G with probability p. It is known that for $p > \frac{1}{d}$ the graph G_p almost surely contains a unique giant component (a connected component with linear number vertices). We show that for $p \geq \frac{5c}{\sqrt{d}}$ the giant component of G_p almost surely has an edge expansion of at least $\frac{1}{\log_2 n}$.

1 Introduction

This paper deals with the affect of percolation on the edge expansion property of *algebraic expander* graphs. These are d-regular graphs in which the second largest eigenvalue (in absolute value) λ of their adjacency matrix is smaller than $d/5$. We call such a graph a (n, d, λ) algebraic expander. A more intuitive (combinatorial) notion of expansion for a finite graph G is the *edge expansion*, defined as:

$$c_E(G) = \inf_{S \subset V_G, |S| < |G|/2} \frac{|\partial_E S|}{|S|},$$

where $\partial_E S$ denotes the set of edges with exactly one vertex in S. It is known (due to Tanner, Alon and Milman [5], [22]) that algebraic expansion implies also a lower bound on the edge expansion: for a (n, d, λ) algebraic expander it holds that $c_E(G) \geq \frac{d-\lambda}{2}$. (There is also an inequality in the opposite direction: $c_E(G) \leq \sqrt{2d(d-\lambda)}$, see [1] for details).

Expander graphs received a considerable amount of attention in the literature in recent years, mostly because these graphs have numerous applications in theoretical computer science; see, for example, [6, 14, 21, 19]. It is well known

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that for any fixed $d \geq 3$, random d-regular graphs of size n are asymptotically almost surely expanders, as n grows. The problem of constructing infinite families of bounded degree expanders is more difficult, and there are several known constructions of this type [17, 16, 20, 8]. The result in this paper applies to the constructions of [16, 20, 8].

Various applications of expanders rely on their fault-tolerance as networks. For example, after deleting an appropriate constant fraction of the edges (arbitrarily), the remaining graph still contains some linear size connected components or some linear size paths; see [3, 23]. We show that for algebraic expanders if the deletions are random and independent then with high probability (with probability that tends to 1 as n increases) the giant component has an edge expansion proportional to $\frac{c\sqrt{d}}{\log_2 n}$. Up to constants, this bound is tight since with probability bounded away from 0, the giant component will contain a $\frac{\log_2 n}{2c\sqrt{d}}$ long "chain" of vertices each of them, except the first and the last ones, has degree of exactly 2 in the giant component. The edge expansion of such a "chain" is $\frac{4c\sqrt{d}}{\log_2 n}$.

Given a graph G, we use G_p to denote the subgraph of G obtained by retaining each edge of G independently with probability p. The graph G_p is the *percolated* version of G. For any graph property of G one can ask if this property is almost surely retained in G_p. A well studied example is the existence and the uniqueness of a giant component. Roughly speaking, a giant component is a connected component of G_p that contains linear fraction of vertices. A question of the same flavour can be asked also for an infinite graph G: for which values of p, G_p is likely to contain an infinite cluster (connected component) ? is the infinite cluster likely to be unique ? For several types of graphs, e.g. the d dimensional grid, the finite/infinite versions turned out to be related. For many interesting graphs the probability of containing a giant component (or infinite cluster in the infinite case) exhibits a sharp threshold around some value called the *critical probability* (this is due to 0/1 laws). The critical probability is denoted by p_c. For values of p slightly smaller than p_c the probability for giant component is close to 0 and for p slightly larger than p_c the probability for giant component is close to 1. Benjamini and Schramm [7] showed that if G is an infinite graph with a positive vertex Cheeger constant $c_V(G) > 0$ (the Cheeger constant can be defined with respect to the vertex boundary), then the critical probability for the existence of an infinite cluster in G_p is $< \frac{1}{1+c_V(G)} < 1$. They also observed that their proof can be applied to the finite case. Their technique can be easily applied also to the edge Cheeger constant as shown in [18].

A family of expanders is a sequence of d-regular graphs $G(n)$, where $G(n)$ has n vertices and edge expansion of least $b > 0$ (independent of n). Alon, Benjamini and Stacey [2] studied the existence and uniqueness of a giant component when percolation is applied to families of edge expander graphs. One of their results is about expander families with increasing girth (the girth of a graph G is the length of minimum size cycle in it). They show that for an expander family $G(n)$, with girth$(G(n))$ that goes to infinity as n increases, the critical probability p_c for the existence (and uniqueness) of a giant component is exactly $\frac{1}{d-1}$.

Specifically, for any fixed ϵ, and $p \geq \frac{1+\epsilon}{d-1}$ w.h.p. (with high probability, i.e. with probability that goes to 1 as n, the size of graph, goes to infinity) G_p contains a connected component with a linear number of vertices. The fraction of vertices in the giant component depends on ϵ. The girth, the edge expansion, d and ϵ influence the speed in which the probability for a g.c. (giant component) goes to 1. For $p \leq \frac{1-\epsilon}{d-1}$, w.h.p. G_p breaks into connected components of sub-linear size. It is further shown in [2] that if $G(n)$ is an infinite family of d-regular graphs, each one with edge expansion of at least $b > 0$, then for p sufficiently close to 1 (which depends on b) $G(n)_p$ is w.h.p. a $\frac{1}{\log_2 n}$ expander. They leave as an open problem the values of p which are close (from above) to the critical probability p_c. Notice that p_c can be as small as $\frac{1}{d-1}$ as in the case of an infinite family of expanders with girth that goes to infinity.

Percolation of (n, d, λ) graphs has been previously studied by Frieze, Krivelevich and Martin [12]. They gave tight results about the existence and the uniqueness of the giant component when $\lambda = o(d)$. Specifically, for $p < \frac{1}{d}$ the graph G_p almost surely contains only connected components of size $O(\log n)$. For $p > \frac{1}{d}$ the graph G_p has almost surely a unique giant component and all other components are of size at most $O(\log n)$.

1.1 Our result

Theorem 1.1. *Let $d \geq d_0$ be a fixed constant, let G be a $(n, d, c\sqrt{d})$ algebraic-expander and let $p \geq \frac{5c}{\sqrt{d}}$ (assuming $c < \frac{\sqrt{d}}{5}$). W.h.p. the edge expansion of the giant component in G_p is at least $\frac{c\sqrt{d}}{61 \log n}$.*

Theorem 1.1 implies that in the case of algebraic expanders even when $p << 1$ the giant component has edge expansion $\geq \frac{1}{\log_2 n}$. In contrast, the result in [2] is based on a weaker assumption (edge expansion greater than ϵ) but it implies that the giant component has edge expansion $\geq \frac{1}{\log_2 n}$ only for values of p close to 1. While Theorem 1.1 requires a somewhat stronger assumption (spectral gap) from G, it implies that the giant component in G_p has expansion $\geq \frac{1}{\log_2 n}$ also for values of p close to 0 (depending on the degree d and $\lambda = c\sqrt{d}$).

The main idea in the proof of Theorem 1.1 is to iteratively remove from G_p vertices of low degree until we are left with an induced subgraph G_p^k that has minimal degree $\geq \frac{3pd}{5}$. Using known techniques it can be shown that for large enough d this process removes only small fraction of the vertices. Moreover, the obtained subgraph G_p^k has edge expansion bounded away from 0. To show that the giant component of G_p has expansion $\geq \frac{1}{\log_2 n}$ (which is best possible up to constants) we need to handle sets of the giant component that contain vertices from $OUT \triangleq V \setminus G_p^k$. To do this it is enough to show that in the graph induced by G_p on OUT, the connected components are smaller than $\log_2 n$. Following the work of Alon and Kahale [4] on coloring random 3-colorable graphs, several papers [9, 11, 13, 10] dealt with similar versions of this problem: proving that a set OUT which is the outcome of some procedure applied to a random graph has no large connected components. Yet, in all the above cases the graph model was a simple variant of the $G_{n,p}$ model. Our result can be though of as a "derandomization" of the previous results as we deal with predetermined constant degree "pseudo-random" graphs for which there is less randomness in the induced model.

Remarks:

1. Possibly, Theorem 1.1 can be extended also for values of $p > \frac{c}{\sqrt{d}}$, using the same proof technique. To keep the proof simple, we did not try to optimize this constant.

2. Theorem 1.1 holds also for d which is a function of n if one of the following holds: $c\sqrt{d} = \lambda = \omega(\log d)$ or $d = o(n)$.

1.2 Notation

For a set $U \subset V$, $G[U]$ denotes the subgraph induced by the edges of G on the vertices of U. We use $e(S,S)$ to denote twice the number of edges having only vertices in S. The graph induced by retaining each edge of G independently with probability of p is denoted by G_p. The degree of a vertex v inside a graph G is denoted by $\deg_v^G$. The second largest eigenvalue (in absolute value) of G is denoted by λ. We use the term with high probability (w.h.p) to denote a sequence of probabilities that tends to 1 as n, the size of G, goes to infinity.

1.3 Spectral gap and pseudo-randomness

In the following proofs we will use the fact that a graph G with a noticeable spectral gap is pseudo-random. This is formulated by the following Lemma also known as the expander mixing lemma (see [6] for proof).

Lemma 1.2. *Let G be a d-regular graph with second largest (in absolute value) eigenvalue λ. Then, for any $S, T \subseteq V$:*

$$|e(S,T) - \frac{d}{n}|S||T|\,| < \lambda\sqrt{|S||T|},$$

where $e(S,T)$ is the number of directed edges from S to T in the adjacency matrix of G.

In terms of undirected edges (when G is undirected), $e(S,T)$ equals the number of edges between $S \setminus T$ to T plus twice the number of edges that contain only vertices of $S \cap T$.

Corollary 1.3. *Let G be a $(n, d, c\sqrt{d})$ algebraic expander. For any set U of size $\leq \frac{cn}{k\sqrt{d}}$ the average degree in $G[U]$ is at most $c\sqrt{d}(1 + 1/k)$.*

Proof. The number of edges inside $G[U]$ is $e(U,U)/2$ since every edge whose both endpoints are in U is in fact two directed edges from U to U. It follows that the average degree in $G[U]$ is $\frac{e(U,U)}{|U|}$. By the expander mixing lemma:

$$e(U,U) \leq \frac{d|U|^2}{n} + c\sqrt{d}|U| \leq c\sqrt{d}|U|\left(1 + \frac{\sqrt{d}|U|}{cn}\right) \leq c\sqrt{d}|U|(1 + 1/k)$$

□

We will frequently use the fact that small enough sets in G are rather sparse, as stated in Corollary 1.3. When c close to its smallest possible value for constant degree graphs (i.e. $\lambda = c\sqrt{d} \approx 2\sqrt{d-1}$, see [1] for details) there is a slightly stronger bound on the density of small sets given by [15]. We do not use this stronger bound as it gives asymptotically the same result for large values of c.

2 Proof of Theorem 1.1

We use a process similar to [23] which is aimed to reveal a large edge expanding subgraph of G_p. Let $S_0 = \{v \in G_p : \deg_v^{G_p} \notin [\frac{4pd}{5}, \frac{6pd}{5}]\}$. We begin by removing from G_p the vertices of S_0. The new induced graph, which we denote by G_p^0, may contain vertices whose degree in G_p^0 is $< \frac{4}{5}pd$, because removing S_0 affect the degrees of the remaining vertices. We then obtain a sequence of subgraphs G_p^i by iteratively removing from G_p^{i-1} any vertex whose degree in G_p^{i-1} is $< \frac{3}{5}pd$. The process stops once the minimal degree in the remaining graph G_p^k is at least $\frac{3}{5}pd$. We denote the set $V \setminus V(G_p^k)$ by OUT.

Remark: Since d is fixed and $n \to \infty$ it holds that the constant c (from $\lambda = c\sqrt{d}$) is at least 1 (in fact $c \geq 1 - o(1)$ even if d grows with n but it is $o(n)$). We will use this fact occasionally.

2.1 Proof overview

The main idea in the proof of Theorem 1.1 is as follows. We remove from G_p low degree vertices until the induced graph G_p^k has a large enough minimal degree. We first show that G_p^k itself has edge expansion of at least $\frac{pd}{13}$ and contains almost all the vertices of G (thus it must be contained in a giant component); this part of the proof uses standard techniques. We then show that in $G_p[OUT]$ the largest connected component is of size at most $\log_2 n$ (this implies the uniqueness of the giant component). The above two facts imply that any set that belongs to the giant component and is entirely in $V(G_p^k)$ or entirely in OUT, has expansion $\geq \frac{1}{\log_2 n}$. Using also the property that any vertex of $V(G_p^k)$ has at most $\frac{3pd}{5}$ neighbors in OUT we then prove that any set of the giant component with size $\leq n/2$ has edge expansion $\geq \frac{1}{\log_2 n}$.

The expected degree in G_p, which is pd, is large enough so that only few vertices are removed in the process of extracting G_p^k (namely OUT is small).

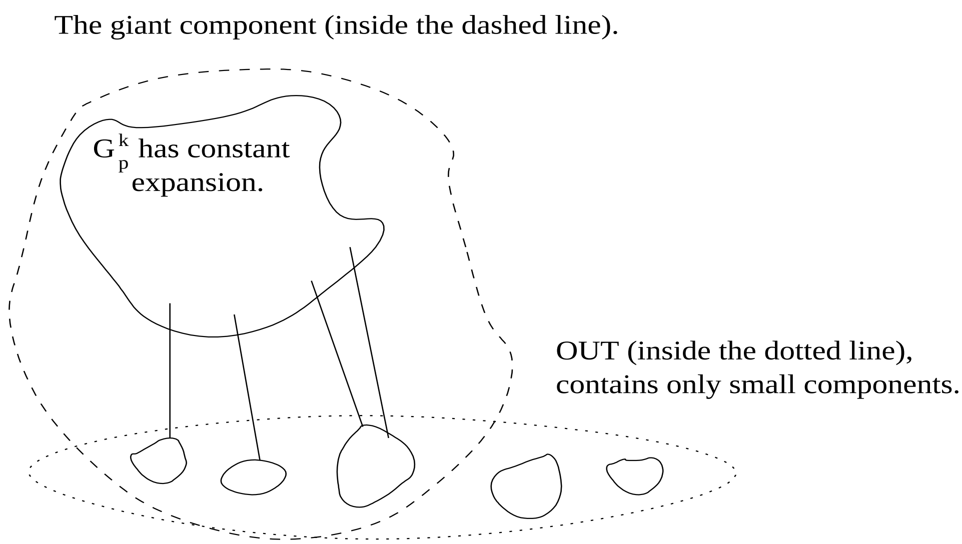

Figure 1: The structure of G_p.

To show it we use the following idea used in [4, 23]. Initially, the set S_0 is small (roughly $e^{-\Omega(c\sqrt{d})}n$). Every vertex which is removed in the iteration process has at least $\frac{pd}{5}$ edges to vertices that were previously removed. Thus if the iteration process is too long, the set $V \setminus V(G_p^i)$ becomes too dense with contradiction to Corollary 1.3.

Lemma 2.1. *W.h.p. the number of vertices in* G_p^k *is at least* $(1 - e^{-\frac{1}{12}c\sqrt{d}})n$.

The proof of Lemma 2.1 is deferred to Section 2.2.

The minimal degree in G_p^k is $\frac{3pd}{5} \geq 3c\sqrt{d}$. A set S smaller than $\frac{cn}{\sqrt{d}}$ contains at most $c|S|\sqrt{d}$ internal edges (by Corollary 1.3), thus for such set the expansion is at least $\frac{3pd}{5} - 2c\sqrt{d} \geq \frac{pd}{5}$. To establish the edge expansion of larger sets, a standard argument using the Chernoff and union bounds suffices. In Section 2.2 we give the full proof of the following Lemma.

Lemma 2.2. *W.h.p. the graph* G_p^k *has an edge expansion of at least* $\frac{pd}{13}$.

We next show that the connected components in $G_p[OUT]$ are of size at most $\log_2 n$. A direct "brute force" approach using the union bound over all possible trees of size $\log_2 n$ does not seem to work here because we don't have a good enough upper bound on the probability that a fixed tree T is in OUT. Notice that we can not simply claim that every vertex in OUT has a low degree in G_p (if this were true then probably a simple argument would have sufficed). It may be the case that a vertex in OUT has high degree in G_p but it is connected (directly or via other vertices) to vertices of low degree in G_p.

Following the work of Alon and Kahale [4] on coloring random 3-colorable graphs, several papers [9, 11, 13, 10] dealt with similar versions of this problem (proving that a set OUT which is the outcome of some procedure applied to a random graph has no large connected components). Yet, the analysis they gave is rather complicated. The reason for the difficulty is that OUT is not a random set independent of G_p; its vertices are in fact correlated and depend on the edges of G_p. A new aspect of the current paper is simplifying the proof that OUT has only small connected components. This is done using a reduction which will be described shortly. The outcome of the reduction is that instead of having to prove that w.h.p. $G_p[OUT]$ has no large trees, we need only to prove that w.h.p. $G_p[OUT]$ has no large *balanced* trees. A balanced tree is a tree in which at least $\frac{1}{3}$ of its vertices are low degree vertices in G_p. Proving that $G_p[OUT]$ contains no balanced trees can be done directly by applying the union bound over all possible sets that may form large trees in G_p.

We will now explain the reduction. We show that w.h.p. any maximal connected component U in $G_p[OUT]$ is $\frac{1}{3}$-*balanced* i.e. at least $1/3$ of its vertices are from S_0. The argument is as follows: every vertex removed in iterations $1, 2, .., k$ has at least $\frac{pd}{5}$ edges to previously removed vertices. Thus, if at least $2/3$ of the vertices of U are added during the iterations then the average degree in $G[U]$ is at least $\frac{2}{3}2c\sqrt{d}$. On the other hand, since $|U| < e^{-\frac{1}{12}c\sqrt{d}}n$ by Corollary 1.3 the average degree in $G[U]$ is at most $c\sqrt{d}(1 + e^{-c\sqrt{d}/2})$, which yields a contradiction. Having established that w.h.p. every maximal connected component in $G[OUT]$ is balanced, our problem (showing that w.h.p. there are no large connected components in $G[OUT]$) is reduced to showing that w.h.p. any maximal balanced connected component of $G[OUT]$ is of size $< \log_2 n$. Since proving the claim on balanced trees is simpler (it is a private case) we translate the claim into a claim on trees. This is done by showing that any $\frac{1}{3}$-balanced connected component of $G[U]$ of size $\geq \log_2 n$ contains a $\frac{1}{3}$-balanced tree whose size is in $[\log_2 n, 2\log_2 n]$; this follows from following Lemma whose proof is deferred to Section 2.2.

Lemma 2.3. *Let G be a connected graph whose vertices are partitioned into two sets: S and I. Let $\frac{1}{k}$ be a lower bound on the fraction of S vertices, where k is an integer. For any $1 \leq t \leq |V(G)|/2$ there exists a tree whose size is in $[t, 2t-1]$ and at least $\frac{1}{k}$ fraction of its vertices are from S.*

To summarize, in order to show that $G_p[OUT]$ has no connected components of size $\log_2 n$ we need only to prove the following Claim.

Claim 2.4. *W.h.p. $G_p[OUT]$ has no balanced trees of size in $[\frac{1}{2}\log_2 n, \log_2 n]$.*

Proof. We want to bound the probability that $G_p[OUT]$ contains any $\frac{1}{3}$-balanced tree of size in $[t/2, t]$ (we fix the parameter t later). Such a tree is called "bad".

$$\sum_{T \text{ is a tree}} \Pr[T \subset OUT \ \wedge \ \text{T is } \tfrac{1}{3}\text{-balanced}]$$

The number of trees of size t in a d-regular graph G is at most nd^{2t}, since each tree can be uniquely mapped into a closed path of length $2t$. For each tree of size t there are at most 2^t ways of choosing a subset of size $\geq t/3$. Any fixed set of size $\geq t/3$ is in S_0 with probability of at most $e^{-\frac{1}{2}\frac{1}{25}pdt/3}$. Thus the probability that there is a balanced bad tree in OUT is at most:

$$tnd^{2t}2^te^{-\frac{1}{300}pdt} \leq \exp\left(\log t + \log n + 2t\log d + t - \tfrac{1}{300}pdt\right) \overbrace{\leq}^{pd \geq 5c\sqrt{d}} \exp\left(\log n + 3t\log d - \tfrac{c\sqrt{d}t}{60}\right) = o(1), \quad (1)$$

for $t \geq \frac{61\log n}{c\sqrt{d}}$ (for fixed d and large enough n it holds that $c > 1$, see [1]).

Remark: Notice that in the last inequality we used $\frac{c\sqrt{d}}{60} > 3\log d$. This holds also when d is a function of n, if n is large enough and $d = o(n)$ (it is known that $c\sqrt{d} = \lambda \geq \sqrt{d - \frac{d^2}{n-1}}$). □

Since all connected components of $G[OUT]$ are of size at most $\frac{61\log_2 n}{c\sqrt{d}}$, sets from OUT that belong to the giant component have expansion of at least $\frac{c\sqrt{d}}{61\log_2 n}$. It remains to handle sets of the giant component that intersects both $V(G_p^k)$ and OUT.

Lemma 2.5. *W.h.p. any set S that belongs to the giant component and whose size is at most $n/2$ has edge expansion of at least $\frac{c\sqrt{d}}{61\log_2 n}$.*

Proof. We already handled sets which are completely in OUT or completely in $V(G_p^k)$. Let S be a set of the g.c. (giant component) that intersects both OUT and $V(G_p^k)$. Denote by $\bar{S}$ the complement of S in the giant component. Denote by S_1, S_2 the intersection of S with $OUT, V(G_p^k)$ respectively. We further partition S_1 into S_{11}, S_{12} as follows: S_{11} contains all the connected components of $G_p[S_1]$ that have at least one edge into $\bar{S}$ and S_{12} contains all the connected components of $G_p[OUT]$ that have only edges to S_2. It is enough to show that:

$$|E(S_{11}, \bar{S})| \geq \frac{|S_{11}|c\sqrt{d}}{61\log_2 n}, \qquad |E(S_{12}\cup S_2, \bar{S})| \geq \frac{|S_{12}\cup S_2|}{18}\ .$$

The first inequality follows immediately from the definition of S_{11}. The second inequality is derived as follows: $|E(S_{12}\cup S_2, \bar{S})| \geq |E(S_2, \bar{S})| \geq |S_2|pd/13$. Thus

$$\frac{|E(S_{12}\cup S_2, \bar{S})|}{|S_{12}| + |S_2|} \geq \frac{|S_2|pd/13}{|S_{12}| + |S_2|} \geq \frac{1}{\frac{13}{pd}(|S_{12}|/|S_2| + 1)} \geq \frac{1}{18},$$

where the last inequality holds because every vertex of G_p^k has at most $\frac{6pd}{5}$ neighbors in OUT. □

2.2 Proofs of lemmas 2.1, 2.2, 2.3

The proofs of Lemmas 2.2 and 2.1 are rather standard and are based on the fact that every small enough set S ($<< \frac{cn}{\sqrt{d}}$) contains at most $|S|c\sqrt{d}(1 + o(1))/2$ internal edges.

Proof of Lemma 2.1. A fixed vertex v belongs to S_0 with probability $< e^{-\frac{1}{2}(\frac{1}{5})^2 pd} \leq e^{-\frac{c\sqrt{d}}{10}}$. Thus, the expected size of S_0 is $e^{-\frac{1}{10}c\sqrt{d}}n$. With probability of $1-o(1)$ the cardinality of S_0 is at most $e^{-\frac{1}{12}c\sqrt{d}}n$. We briefly sketch the proof. We use the edge exposure martingale to prove that S_0 is concentrated around its expectation. We fix some order on the $m = nd/2$ edges of G. Let $X_0, X_1, ..., X_m$ be the martingale sequence, where X_i is the expectation of S_0 after exposing the first i edges of G_p. Notice that $X_0 = \mathbb{E}_{G_p}[S_0]$. The value of X_m is the value of the random variable S_0 where the probability measure is induced by G_p. To use the Azuma inequality we need to upper bound the martingale difference $|X_{i+1} - X_i|$ (for $i = 0, .., m-1$). It is known that if S_0 satisfies the edge Lipschitz condition with a constant Δ, then also the martingale difference is bounded by Δ (see [6]). It is clear that for a fixed graph G', adding/removing a single edge can change the value of S_0 by at most 2. By Azuma's inequality:

$$\Pr[X_m > X_0 + \lambda] \leq e^{-\lambda^2/(2m\Delta^2)}.$$

Substituting $\lambda = e^{-\frac{1}{10}c\sqrt{d}}n, \Delta = 2, m = nd/2$ we derive that w.h.p. $|S_0|$ is at most $e^{-\frac{1}{12}c\sqrt{d}}n$.

We next show that the number of vertices removed after removing S_0 (that is k) is at most $|2S_0|$. Every vertex that is removed in the iterative process has at least $\frac{pd}{5} \geq c\sqrt{d}$ edges which goto previously removed vertices, because its degree drops from at least $\frac{4pd}{5}$ (as it does not belong to S_0) down to at most $\frac{3pd}{5}$ (at the point it was removed). By contradiction, assume that $k \geq 2|S_0|$. Consider the situation immediately after iteration $i = 2|S_0|$. Denote by U the set of vertices not in G_p^i. The average degree in $G_p[U]$ is at least $\frac{2}{3}2c\sqrt{d}$. At this point $|U| \leq 3e^{-\frac{1}{12}c\sqrt{d}}n$. We derive a contradiction as by Corollary 1.3 the average degree in $G[U]$ is at most $c\sqrt{d}(1 + e^{-\frac{c\sqrt{d}}{15}})$. □

Proof of Lemma 2.2. The proof is divided into two parts. First consider sets of cardinality $\leq \frac{cn}{\sqrt{d}}$. Fix a set $S \subset V$. The edge expansion of S (in G_p^k) is at least:

$$\sum_{v \in S} \deg(v) - e(S, S),$$

(remember that $e(S, S)$ is twice the number of edges inside S). Every vertex v of G_p^k has degree of at least $\frac{3pd}{5} \geq 3c\sqrt{d}$ in G_p^k. By Corollary 1.3 $e(S, S)$ in G is at most $|S|c\sqrt{d}(1+1) = 2|S|c\sqrt{d}$. It follows that the edge expansion of S is at least $\frac{3pd}{5} - 2c\sqrt{d} \geq \frac{pd}{5}$.

Consider now a set $S \subset V$ of size αn such that $\frac{c}{\sqrt{d}} < \alpha \leq \frac{1}{2}$. By the expander mixing lemma, the number of edges between S and $V \setminus S$ in G is at least:

$$\begin{aligned} \alpha(1-\alpha)dn - c\sqrt{d}\sqrt{\alpha(1-\alpha)}n &= \alpha(1-\alpha)dn\left(1 - \frac{c}{\sqrt{d\alpha(1-\alpha)}}\right) \\ &\geq \alpha(1-\alpha)dn/3. \end{aligned}$$

The last inequality follows from $\frac{c}{\sqrt{d}} \leq \alpha \leq \frac{1}{2}$ and $c \leq \frac{\sqrt{d}}{5}$. The number of edges between S and $V \setminus S$ in G_p is at least $p\alpha(1-\alpha)dn/6$ with probability of $e^{-\frac{1}{8}p\alpha(1-\alpha)dn/3}$ (follows from the Chernoff bound). The number of subsets of size αn is at most $\left(\frac{ne}{\alpha n}\right)^{\alpha n} \leq e^{\alpha n(1+\log \frac{1}{\alpha})}$. Thus the probability for a "bad" set of size αn is at most:

$$\exp\left(\alpha n(1+\log\frac{1}{\alpha}) - \frac{1}{8}p\alpha(1-\alpha)dn/3\right) \leq \exp\left(\alpha n(1+\log\frac{1}{\alpha} - \frac{5c\sqrt{d}(1-\alpha)}{24})\right)$$
$$\leq \exp\left(-\alpha cn\sqrt{d}/10\right).$$

The last inequality holds for large enough d (and $\frac{1}{\alpha} < \sqrt{d}$). Summing over all values of αn gives that w.h.p. there is no bad set. In other words, every set of size in $[\frac{cn}{\sqrt{d}}, \frac{n}{2}]$ has an edges expansion of at least $\frac{pd}{12}$ in G_p. Since the number of edges that contain at least one vertex from OUT is bounded by $de^{-\frac{c\sqrt{d}}{12}}n$, we conclude that any subset U of $V(G_p^k)$ with size $\geq \frac{cn}{\sqrt{d}}$ has edge expansion of at least

$$\left(\frac{pd}{12}|U| - de^{-c\sqrt{d}/12}n\right)/|U| \geq \frac{pd}{13}.$$

□

Proof of lemma 2.3. We use the following well know fact: any tree T contains a *center* vertex v such that each subtree hanged on v contains strictly less than half of the vertices of T.

Let T be an arbitrary spanning tree of G, with center v. We proceed by induction on the size of T. Consider the subtrees $T_1, ..., T_k$ hanged on v. If there exists a subtree T_j with at least t vertices then also $T \setminus T_j$ has at least t vertices. In at least one of $T_j, T \setminus T_j$ the fraction of S vertices is at least $\frac{1}{k}$ and the lemma follows by induction on it. Consider now the case in which all the trees have less than t vertices. If in some subtree T_j the fraction of S vertices is at most $\frac{1}{k}$, then we remove it and apply induction to $T \setminus T_j$. The remaining case is that in all the subtrees the fraction of S vertices is strictly more than $\frac{1}{k}$. In this case we start adding subtrees to the root v until for the first time the number of vertices is at least t. At this point we have a tree with at most $2t-1$ vertices and the fraction of S vertices is at least $\frac{1}{k}$. To see that the fraction of S vertices is at least $\frac{1}{k}$, we only need to prove that the tree formed by v and the first subtree has $\frac{1}{k}$ fraction of S vertices. Let r be the number of S vertices in the first subtree and let b be the number of vertices in it. Since k is integer we have: $\frac{r}{b} > \frac{1}{k} \Longrightarrow \frac{r}{b+1} \geq \frac{1}{k}$. □

3 Open problems

We were able to show that a percolation applied to a family of d-regular expander graphs with eigenvalue gap retains some expansion properties of the original graphs, even when p is close to 0. There are still many open problems, we list here two of them:

1. Find other classes of expander families that retain expansion properties after percolated with values of p close to 0. For example, a family of expanders with girth that goes to infinity (for such a family some result is given at [2] for p close to 1).

2. Is Theorem 1.1 is tight ? If we drop the requirement that d is a constant and allow it to be a function of n, then the current proof of Claim 2.4 breaks done (when d is proportional to n). However it is plausible that a different counting argument may work; one example where a modified argument works is K_n (we have a proof for this case). If this is the case then Theorem 1.1 is tight (up to constant factors) because for the complete graph K_n it holds that $d = n-1, c = \frac{1}{\sqrt{n-1}}$ and for $p << \frac{c}{\sqrt{d}}$ the percolated graph is not likely to contain a giant component. Anyway, for constant d the question is interesting: there is a gap between the critical probability $\frac{1}{d}$ for which there is a giant component and the probability $\frac{5c}{\sqrt{d}}$ for which there is $\frac{1}{\log_2 n}$ edge expansion.

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