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3 – NASH is PPAD – Complete

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Abstract

In this paper, we improve a recent result of Daskalakis, Goldberg and Papadimitriou on **PPAD**-completeness of 4-NASH, showing that 3-NASH is **PPAD**-complete.

1 Introduction

Nash equilibrium has traditionally been one of the most influential tools in the study of many disciplines involved with strategies, such as Political Science and Economics. The rise of the Internet and the study of its anarchical environment have made Nash equilibrium an indispensable part of computer science. Great interests have shown in algorithmic issues of related problems. Research effort into them has been the central topic in the emerging discipline, Algorithmic Game Theory.

Naturally, the study of Nash Equilibrium's algorithmic complexity and related topics become one of theoreticians' favorite topics. It has attracted talents, derived new methodologies, and demanded hard works. Innovative ideas have been invented and applied on solving different versions of this problem.

Several interesting results are known for the computational complexity of finding a Nash equilibrium. An early result of Megiddo, [9] states that if 2-player Nash Equilibrium is NP-hard, then NP equals to co-NP. On the other hand, it is known NP-complete to find various specific kinds of Nash equilibria even in the 2-player case, developed by Gilboa and Zemel [5] as well as Conitzer and Sandholm [3]. The exact complexity characterization for Nash equilibrium remained beyond our comprehension until recently.

A recent result of Daskalakis, Goldberg and Papadimitriou [4] proved that 4-player NASH (4-NASH) is **PPAD**-complete. **PPAD** is a sub-class of **TFNP** [10], which was first defined by Papadimitriou [11, 12] to characterize total NP search problems whose totalities are implied by the parity argument proofs. Their results built on two approaches: the graphical game introduced by Kearns, Littman and Singh [8], and the reducibility among equilibrium problems

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by Goldberg and Papadimitriou [6]. They conjectured that 3-Nash is still **PPAD**-complete and 2-Nash could possibly be polynomial-time solvable.

In this work, we reduce 4-Nash to 3-Nash, and hence confirming the first of the conjectures, that is, 3-Nash is **PPAD**-complete.

Our proof relies on a new idea in the combined reductions [6] from r-Nash to degree 3 graphical Nash to 4-Nash. The main difference from the reduction of Goldberg and Papadimitriou is that theirs transfers an exact Nash equilibrium of the degree 3 graphical game to an exact one of the 4-player game but our reduction is approximate. Of course, their reduction also preserves the approximation relation of Nash equilibriums in different games. Ours does not preserve the exactness. The flexibility gained allows us to design a new structure in the thread of successive reductions to reduce the number of players by one.

The paper is organized as follows: We review the necessary definitions in Section 2. We present a threesome version of the matching pennies game in Section 3. The reduction from problem 4-Nash to 3-Nash is divided into two steps. In Section 4, we give a new coloring algorithm for the degree 3 graphical game necessary for the reduction. We finish our proof in Section 5 by showing that while embedding the graphical game into a 3-player game, the approximation of Nash equilibrium is preserved. We conclude in Section 6 with remarks and discussion.

2 Preliminaries

2.1 Games, Graphical Games and Nash Equilibriums

A game $\mathcal{G}$ between $r \geq 2$ players is composed of two parts. First, every player $p \in [r]$ where $[r] = \{0, 1, ... r\}$ has a set S_p of pure strategies. Second, for each $p \in [r]$ and $s \in S$ where

$$S = S_1 \times S_2 \times ... \times S_r,$$

we have u^p_s as the payoff or utility of player p. Here s is called a pure strategy profile of the game. For any p, we use S_{-p} to denote the set of all strategy profiles of players other than p. For any $j \in S_p$ and $s \in S_{-p}$, we use js to denote the pure strategy profile in S, which is combined by j and s. A mixed strategy x^p of player $p \in [r]$ is a probability distribution over S_p, that is, real numbers $x^p_j \geq 0$ for any $j \in S_p$ and $\sum_{j \in S_p} x^p_j = 1$. A profile of mixed strategies x of game $\mathcal{G}$ consists of r mixed strategies x^p, $p = 1, 2, ... r$. For any $p \in [r]$, x^p is a mixed strategy of player p. For any $p \in [r]$ and $s \in S_{-p}$, we define x_s as

$$x_s = \prod_{p' \in [r],\ p' \neq p} x^{p'}_{s_{p'}}$$

Now we give the definition of both accurate and approximate Nash equilibriums of a game. Intuitively, a Nash equilibrium is a profile of mixed strategies x such that no player can gain by unilaterally choosing a different mixed strategy, where the other strategies in the profile are kept fixed. The concept of approximate Nash equilibrium here was first proposed by [12].

Definition 1. *A Nash equilibrium of game $\mathcal{G}$ is a profile of mixed strategies x such that*

$$\sum_{s \in S_{-p}} u^p_{is} x_s > \sum_{s \in S_{-p}} u^p_{js} x_s \implies x^p_j = 0$$

for any $p \in [r]$ and $i, j \in S_p$.

Definition 2. *An ϵ-Nash equilibrium of game $\mathcal{G}$ is a profile of mixed strategies x such that*

$$\sum_{s \in S_{-p}} u^p_{is} x_s > \sum_{s \in S_{-p}} u^p_{js} x_s + \epsilon \implies x^p_j = 0$$

for any $p \in [r]$ and $i, j \in S_p$.

A useful class of games are graphical games, which was first defined in [8] and then generalized by [6]. Players in a graphical game $\mathcal{GG}$ are nodes of an underlying directed graph G. A player u can affect the payoffs to player v only if $uv \in G$. While general games require exponential data for their descriptions, graphical games have succinct representations. More exactly, when the in-degree of the underlying graph G is bounded, the representation of the game $\mathcal{GG}$ is polynomial in the number of players and strategies.

2.2 TFNP, PPAD and r-Nash

Let $R \subset \Sigma^* \times \Sigma^*$ be a polynomial-time computable, polynomially balanced relation (that is, there exists a polynomial p such that for any x and y satisfy $(x, y) \in R$, $|y| \leq p(|x|)$). The NP search problem Q_R specified by R is this: given input $x \in \Sigma^*$, return a $y \in \Sigma^*$ such that $(x, y) \in R$, if such a y exists, and return the string "no" otherwise. An NP search problem is said to be total if for every x, there exists a y such that $(x, y) \in R$. We use **TFNP** [10] to denote the class of total NP search problems.

Definition 3. *Given two problems $Q_{R_1}, Q_{R_2} \in$ **TFNP**, we say that Q_{R_1} is reducible to Q_{R_2} if there exists a pair of polynomial-time computable functions (f, g) such that, for every input x of R_1, if y satisfies $(f(x), y) \in R_2$, then $(x, g(y)) \in R_1$.*

One of the most interesting sub-classes of **TFNP** is **PPAD** which is the directed version of class **PPA**. The totality of problems in **PPAD** is guaranteed by the following trivial fact: in a directed graph, where the in-degree and out-degree of every vertex are no more than one, if there exists a source, there must be another source or sink. Many important problems were identified to be in **PPAD** [12], e.g. the search versions of Brouwer's fixed point theorem, Kakutani's fixed point theorem, Smith's theorem and Borsuk-Ulam theorem. r-Nash, that is, the problem of finding an approximate Nash equilibrium, also belongs to **PPAD** [12].

Definition 4. *The input of problem r-Nash is a pair $(\mathcal{G}, 0^k)$ where $\mathcal{G}$ is an r-player game in normal form, and the output is a $(1/2^k)$-Nash equilibrium of $\mathcal{G}$.*

It was shown in [4] that problem 4-Nash is **PPAD**-complete. In this work, we construct a reduction (definition 3) from 4-Nash to 3-Nash, and thus prove the latter is also complete.

3 3-Player Matching Pennies

In this section, we define a game called 3-player Matching Pennies which is a generalization of the 2-player Matching Pennies described in [6].

Definition 5 (3-Player Matching Pennies). *Let's call the three players* P_1*,* P_2 *and* P_3*. Each of them has* N *pure strategies* $[N] = \{1, 2, \dots N\}$*. Let* (i_1, i_2, i_3) *be any pure strategy profile of the game. For player* P_1*, it receives a payoff of* $u > 0$ *if* $i_1 = i_2$ *or* $i_1 = i_3$*, and* 0 *otherwise. For player* P_2*, it receives a payoff of* $-u$ *if* $i_2 = i_1$ *or* $i_2 = i_3$*, and* 0 *otherwise. For player* P_3*, it receives a payoff of* u *if* $i_3 = i_2$*, and* 0 *otherwise.*

The following lemma is easy to prove.

Lemma 1. 3*-Player Matching Pennies has a unique (accurate) Nash equilibrium* x *in which* x^i *is the uniform distribution over* $[N]$ *for any* $i \in [3]$*.*

Proof. First, we prove that x^2 is uniform. Otherwise, we define two non-empty sets

$$L = \left\{ i \in [N] \;\middle|\; x_i^2 > \frac{1}{N} \right\} \quad \text{and} \quad S = \left\{ i \in [N] \;\middle|\; x_i^2 \leq \frac{1}{N} \right\}.$$

As x is a Nash equilibrium, we have $x_s^3 = 0$ for any $s \in S$, and similarly $x_s^1 = 0$ for any $s \in S$. Pick any $l \in L$ such that $x_l^1 \neq 0$. Obviously, player P_2 prefers strategy s where $s \in S$ to strategy l. This contradicts with our assumption that $l \in L$ and $x_l^2 > 1/N > 0$.

Second, we prove that x^3 is uniform. Otherwise, we define two non-empty sets

$$L = \left\{ i \in [N] \;\middle|\; x_i^3 > \frac{1}{N} \right\} \quad \text{and} \quad S = \left\{ i \in [N] \;\middle|\; x_i^3 \leq \frac{1}{N} \right\}.$$

As x is a Nash equilibrium, we have $x_s^1 = 0$ for any $s \in S$. Obviously, for any strategy $l \in L$ and $s \in S$, P_2 prefers s to l, which contradicts with the fact that x^2 is uniform. Finally, it's easy to check that x^1 is also uniform, and the lemma is proven. □

4 Reduction from 4-Nash to 3-Nash : Step I

Let $(\mathcal{G}, 0^k)$ be any input pair of problem 4-Nash, in which every player $p \in [4]$ has M pure strategies $[M]$. We first construct a graphical game $\mathcal{GG}$ in the same way as the section 3 of [6]. Every player (vertex) in game $\mathcal{GG}$ has two strategies $\{0, 1\}$ and the underlying directed graph $G = (V \cup W, E)$ is bipartite with maximum in-degree 3. W and V are disjoint and each edge in E goes between V and W.

For any $j \in [M]$ and player $p \in [4]$, $\mathcal{GG}$ contains a vertex $v(x_j^p) \in V$. The construction of $\mathcal{GG}$ guarantees that, given any Nash equilibrium of $\mathcal{GG}$, if the probability of player $v(x_j^p)$ using strategy 1 (which is denoted by $\mathbf{p}[v(x_j^p)]$) is interpreted as the value of x_j^p, then the profile of mixed strategies x obtained is exactly a Nash equilibrium of the original game $\mathcal{G}$.

The main idea of the construction is described informally as follows. First, gadgets are designed to implement arithmetic operations of addition, multiplication and maximization. Here vertices in W are used to mediate between vertices of V, so that the latter ones obey the intended arithmetic relationship. For example, given two vertices v_1 and v_2, we can setup the payoffs of w and v_3 appropriately such that in any Nash equilibrium, $\mathbf{p}[v_3] = \mathbf{p}[v_1]\mathbf{p}[v_2]$. Second, $\mathcal{GG}$ contains two vertices $v(U_j^p)$ and $v(U_{\le j}^p)$ for any $j \in [M]$ and $p \in [4]$. Gadgets are assembled properly such that $\mathbf{p}[v(U_j^p)]$ is the payoff to player p if it chooses strategy j, and

$$\mathbf{p}[v(U_{\le j}^p)] = \max_{1 \le i \le j} \text{ expected payoff of player } p \text{ if it chooses strategy } i.$$

Finally, $\mathbf{p}[v(U_j^p)]$ and $\mathbf{p}[v(U_{\le j+1}^p)]$ are compared, and results feed back to $v(x_j^p)$ to make sure x satisfies the constraint in definition 1. Because of the importance of the three arithmetic gadgets in the reduction, we list all the related propositions in [6] below.

Proposition 1. *Let α be a non-negative real number. Let v_1, v_2 and w be vertices in a graphical game $\mathcal{GG}$, and suppose that the payoffs to v_2 and w are as follows.*

Payoffs to v_2 :

	$w:0$	$w:1$
$v_2:0$	0	1
$v_2:1$	1	0

Payoffs to w : $w:0$

	$v_2:0$	$v_2:1$
$v_1:0$	0	0
$v_1:1$	α	α

$w:1$

	$v_2:0$	$v_2:1$
$v_1:0$	0	1
$v_1:1$	0	1

Then, in any Nash equilibrium of $\mathcal{GG}$, $\boldsymbol{p}[v_2] = \min(\alpha \boldsymbol{p}[v_1], 1)$.

Proposition 2. *Let α, β and γ be non-negative real numbers. Let v_1, v_2, v_3 and w be vertices in a graphical game $\mathcal{GG}$, and suppose that the payoffs to v_3 and w are as follows.*

Payoffs to v_3 :

	$w:0$	$w:1$
$v_3:0$	0	1
$v_3:1$	1	0

Payoffs to w : $w:0$

	$v_2:0$	$v_2:1$
$v_1:0$	0	β
$v_1:1$	α	$\alpha+\beta+\gamma$

$w:1$

$v_3:0$	0
$v_3:1$	1

Then, in any Nash equilibrium of $\mathcal{GG}$, $\boldsymbol{p}[v_3] = \min(\alpha \boldsymbol{p}[v_1] + \beta \boldsymbol{p}[v_2] + \gamma \boldsymbol{p}[v_1]\boldsymbol{p}[v_2], 1)$.

Proposition 3. *Let $v_1, v_2, v_3, v_4, v_5, v_6, w_1, w_2, w_3$ and w_4 be vertices in a graphical game $\mathcal{GG}$, and suppose that the payoffs to vertices other than v_1 and v_2 are as follows.*

Payoffs to w_1 : $w_1:0$

	$v_2:0$	$v_2:1$
$v_1:0$	0	0
$v_1:1$	1	1

$w:1$

	$v_2:0$	$v_2:1$
$v_1:0$	0	1
$v_1:1$	0	1

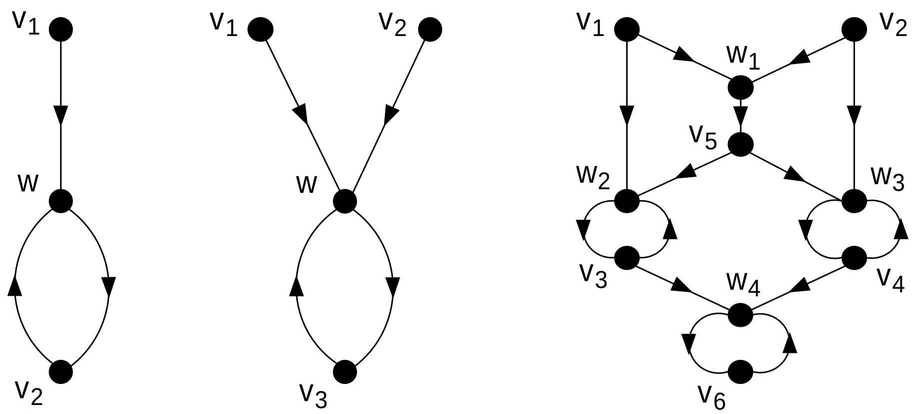

Figure 1: Underlying Graphs of Proposition $1-3$

Payoffs to v_5 *:*

	$w_1 : 0$	$w_1 : 1$
$v_5 : 0$	1	0
$v_5 : 1$	0	1

Payoffs to w_2 *and* v_3 *are chosen using Proposition 2 to ensure* $\boldsymbol{p}[v_3] = \boldsymbol{p}[v_1](1 - \boldsymbol{p}[v_5])$.
Payoffs to w_3 *and* v_4 *are chosen using Proposition 2 to ensure* $\boldsymbol{p}[v_4] = \boldsymbol{p}[v_2]\boldsymbol{p}[v_5]$.
Payoffs to w_4 *and* v_6 *are chosen using Proposition 2 to ensure* $\boldsymbol{p}[v_6] = \min(1, \boldsymbol{p}[v_3] + \boldsymbol{p}[v_4])$.
Then, in any Nash equilibrium of graphical game $\mathcal{GG}$, $\boldsymbol{p}[v_6] = \max(\boldsymbol{p}[v_1], \boldsymbol{p}[v_2])$.

The underlying graphs of the three propositions are shown in figure 1. In this work, we consider approximate Nash equilibriums. While the equalities in Proposition 1-3 don't hold anymore, the gadgets designed give good approximations of all the three arithmetic relations.

Proposition 4. *In any* ϵ*-Nash equilibrium of* $\mathcal{GG}$*, we have*

$$\boldsymbol{p}[v_2] = \min(\alpha \boldsymbol{p}[v_1], 1) \pm \epsilon \qquad \textit{in } \text{Proposition 1;}$$

$$\boldsymbol{p}[v_3] = \min(\alpha \boldsymbol{p}[v_1] + \beta \boldsymbol{p}[v_2] + \gamma \boldsymbol{p}[v_1]\boldsymbol{p}[v_2], 1) \pm \epsilon \qquad \textit{in } \text{Proposition 2;}$$

$$\boldsymbol{p}[v_6] = \max(\boldsymbol{p}[v_1], \boldsymbol{p}[v_2]) \pm 3\epsilon \qquad \textit{in } \text{Proposition 3.}$$

By $x = y \pm \epsilon$*, we mean that* $y - \epsilon \le x \le y + \epsilon$.

Using Proposition 4, it's easy to check that, given any approximate Nash equilibrium of $\mathcal{GG}$, we can compute an approximate Nash equilibrium of $\mathcal{G}$ very efficiently.

Property 1. *There exists a polynomial* $p_1(n)$ *such that, given any* ϵ'*-Nash equilibrium of the graphical game* $\mathcal{GG}$ *where* $\epsilon' = \epsilon 2^{-p_1(|\mathcal{G}|)}$*, an* ϵ*-Nash equilibrium of the original game* $\mathcal{G}$ *can be computed in polynomial time.*

The following three properties of graphical game $\mathcal{GG}$ can be easily found in [6].

Property 2. *The size of graphical game* $\mathcal{GG}$ *is polynomial of the size of game* $\mathcal{G}$.

We use $|\mathcal{GG}|$ and $|\mathcal{G}|$ to denote the size of game $\mathcal{GG}$ and $\mathcal{G}$ respectively. The size of a game is the number of bits necessary to describe it.

Property 3. *Every vertex in W has ≤ 3 incoming edges and ≤ 1 outgoing edge.*

Property 4. *Every vertex in V has ≤ 1 incoming edge and ≤ 2 outgoing edges.*

Further observation on the three gadgets would give us the following property.

Property 5. *Let w be any vertex in W with 3 incoming edges, then when player w chooses strategy 1, its payoff only depends on one player in V (which is named v_w and $v_w w \in E$); when player w chooses strategy 0, its payoff depends on two players in V which are different from v_w (they are named v_w^1 and v_w^2, $v_w^1 w, v_w^2 w \in E$). Furthermore, we have $wv_w^1, wv_w^2 \notin E$.*

Property 5 is the key which gives us an embedding of $\mathcal{GG}$ into a 3-player game. Before that, we normalize game $\mathcal{GG}$ and color all of its vertices in three colors $\{0, 1, 2\}$. After the normalization, $\mathcal{GG}$ and the 3-coloring c should satisfy the following four properties:

A. The underlying graph $G = (V \cup W, E)$ is still bipartite, where V and W are disjoint.

B. All sets V_i (the set of vertices with color i, where $i \in [3]$) have the same cardinality.

C. Every vertex u in graph G has either 0 or 4 incoming edges. For the latter case, we use $u_1u, u_2u, u_1'u, u_2'u \in E$ to denote the 4 incoming edges. The color of u is different from all the four vertices which satisfy that $c(u_1) = c(u_2) \neq c(u_1') = c(u_2')$. When u plays strategy 0 (or 1), its payoff only depends on two players with different colors.

D. Utilities (or payoffs) of graphical game $\mathcal{GG}$ lie in the range $[0, 1]$.

The normalization is described by the algorithm in figure 2 and all the four properties above are satisfied. Property 6 and 7 are simple corollaries of Property 2 and 1 respectively.

Property 6. *The size of graphical game $\mathcal{GG}$ after normalization is still polynomial of $|\mathcal{G}|$.*

Property 7. *There exists a polynomial $p_2(n)$ such that, given any ϵ'-Nash equilibrium of the graphical game $\mathcal{GG}$ after normalization where $\epsilon' = \epsilon 2^{-p_2(|\mathcal{G}|)}$, an ϵ-Nash equilibrium of the original game $\mathcal{G}$ can be computed in polynomial time.*

Now we know that, given any input pair $(\mathcal{G}, 0^k)$ of 4-Nash, we can construct a graphical game $\mathcal{GG}$ and a 3-coloring c in polynomial time. Let integer $n = k + p_2(|\mathcal{G}|)$, then given any $1/2^n$-Nash equilibrium of $\mathcal{GG}$, a $1/2^k$-Nash equilibrium of $\mathcal{G}$ can be computed very efficiently.

Property 8. *There exists a polynomial p_3 such that for any input pair $(\mathcal{G}, 0^k)$ of 4-Nash,*

$$\Big|(\mathcal{GG}, 0^n)\Big| \leq p_3\Big(\Big|(\mathcal{G}, 0^k)\Big|\Big).$$

In the next section, we show how to embed the graphical game $\mathcal{GG}$ into a 3-player game $\mathcal{G}^*$ in polynomial time (of $|(\mathcal{GG}, 0^n)|$). Given any $1/2^{k^*}$-Nash equilibrium of game $\mathcal{G}^*$, we can extract a $1/2^n$-Nash equilibrium of game $\mathcal{GG}$ very efficiently.

Normalization of Graphical Game $\mathcal{GG}$

set $c(w) = 1$ for all vertices w in W, and $c(v) = 2$ for all vertices v in V.
for any vertex $w \in W$ with 2 incoming edges $v^1w, v^2w \in E$ **do**
    if $c(v^1) = c(v^2)$ **then**
        according to Property 3, without loss of generality, we assume that $wv^1 \notin E$
        add vertex v' to V and w' to W
        replace edge v^1w with edges $v^1w', w'v', v'w'$ and $v'w$
        set $c(w') = 1$, choose $c(v') \in \{2, 3\} \neq (c(v^1) = c(v^2))$
        payoffs to w' and v' are chosen using Proposition 1 with $\alpha = 1$ ($\mathbf{p}[v'] = \mathbf{p}[v^1]$)
for any vertex $w \in W$ with 3 incoming edges **do**
    according to Property 5, name the three vertices in V as v_w, v_w^1 and v_w^2
    if $c(v_w^1) = c(v_w^2)$ **then**
        according to Property 5, we have edge $wv_w^1 \notin E$
        add vertex v' to V and w' to W
        replace edge v_w^1w with edges $v_w^1w', w'v', v'w'$ and $v'w$
        set $c(w') = 1$, choose $c(v') \in \{2, 3\} \neq (c(v_w^1) = c(v_w^2))$
        payoffs to w' and v' are chosen using Proposition 1 with $\alpha = 1$ ($\mathbf{p}[v'] = \mathbf{p}[v_w^1]$)
for any vertex u in the graph with ≥ 1 incoming edges **do**
    add extra vertices and incoming edges, color them properly to satisfy Property **C**
    [although we add incoming edges, the vertices added don't affect u's payoffs]
add idle isolated vertices, color them properly to satisfy Property **B**
re-scale the utilities of $\mathcal{GG}$ so that they lie in the range $[0, 1]$, and satisfy Property **D**

Figure 2: Normalization of Graphical Game $\mathcal{GG}$

5 Reduction From 4-Nash to 3-Nash : Step II

Game $\mathcal{G}^*$ has three players P_1,P_2,P_3 and every player has $2N$ strategies where $N = |V_i|$. For any $i \in [3]$, we pick an arbitrary one-to-one correspondence C_i from V_i to $[N]$ and define the set of pure strategies S_i of player P_i as

$$S_i = \left\{ \, (C_i(u), 0), \; (C_i(u), 1) \, \middle| \, u \in V_i \, \right\} = \left\{ \, (i_1, i_2), \; i_1 \in [N], \; i_2 \in \{0, 1\} \, \right\}.$$

For convenience, we will always neglect C_i and recognize vertex u itself as an integer in $[N]$. Utilities of $\mathcal{G}^*$ are described by the algorithm in figure 3, where constant $M = 2N^6L$ and $L = 2^{2n}$. Let k^* be the smallest integer satisfying $2^{k^*} > N^2L$. Although M is exponentially large, we still have the following property.

Property 9. *$|(\mathcal{G}^*, 0^{k^*})|$ is polynomial of $|(\mathcal{GG}, 0^n)|$, and thus polynomial of $|(\mathcal{G}, 0^k)|$.*

The correctness of our reduction from $\mathcal{GG}$ to $\mathcal{G}^*$ is guaranteed by the following theorem.

Theorem 1. *Let y be any $1/2^{k^*}$-Nash equilibrium of $\mathcal{G}^*$, then x is a $1/2^n$-Nash equilibrium of game $\mathcal{GG}$, which is obtained as follows: for any player u in $\mathcal{GG}$, let $c(u) = i \in [3]$, then*

$$x_0^u = \frac{y^i_{(u,0)}}{y^i_{(u,0)} + y^i_{(u,1)}} \qquad\qquad x_1^u = \frac{y^i_{(u,1)}}{y^i_{(u,0)} + y^i_{(u,1)}} \ .$$

The proof of Theorem 1 relies on Lemma 2 which states that, the distribution y^i of player P_i over the N vertices is very close to the uniform distribution. Here for any $j \in [N]$ and $i \in [3]$, we use y^i_j to denote the probability $y^i_{(j,0)} + y^i_{(j,1)}$.

Lemma 2. *Let y be any $1/2^{k^*}$-Nash equilibrium of $\mathcal{G}^*$, then for any $j \in [N]$ and $i \in [3]$,*

$$\frac{1}{N} - \frac{1}{NL} \le y^i_j \le \frac{1}{N} + \frac{1}{NL} \ .$$

Proof. We prove the following stronger result:

$$\forall\ j \in [N],\ y^1_j \le \frac{1}{N} + \frac{1}{N^2L} \quad \Big(\text{and thus, } \forall\ j \in [N],\ \frac{1}{N} - \frac{1}{NL} \le y^1_j \le \frac{1}{N} + \frac{1}{N^2L}\Big) \qquad (1)$$

$$\forall\ j \in [N],\ y^2_j \le \frac{1}{N} + \frac{1}{N^6L} \quad \Big(\text{and thus, } \forall\ j \in [N],\ \frac{1}{N} - \frac{1}{N^5L} \le y^2_j \le \frac{1}{N} + \frac{1}{N^6L}\Big) \qquad (2)$$

$$\forall\ j \in [N],\ y^3_j \le \frac{1}{N} + \frac{1}{N^4L} \quad \Big(\text{and thus, } \forall\ j \in [N],\ \frac{1}{N} - \frac{1}{N^3L} \le y^3_j \le \frac{1}{N} + \frac{1}{N^4L}\Big) \qquad (3)$$

Step 1: If (2) is not true, without loss of generality, we assume $y^2_1 > 1/N + 1/(N^6L)$ and set $S = \{\ s \in [N] \mid y^2_s \le 1/N\ \}$ which is non-empty. If P_3 chooses strategy $(1, 0)$ or $(1, 1)$, its expected payoff $p \ge y^2_1 M$. On the other hand, if P_3 plays strategy $(s, 0)$ or $(s, 1)$ where $s \in S$, then its expected payoff $p' \le 1 + y^2_s M$ (using Property **D**). As a result, we have $p - p' \ge 1 > 1/2^{k^*}$, and $y^3_s = 0$ for any $s \in S$. Similarly, we can prove $y^1_s = 0$ for any $s \in S$. To get a contradiction, we observe player P_2. Pick any $i \notin S$ satisfying $y^1_i \ge 1/N$. Obviously, player P_2 prefers strategies $(s, 1), (s, 0)$ where $s \in S$ to $(i, 0), (i, 1)$, and thus $y^2_i = 0$, which contradicts with our assumption that $y^2_i > 1/N$.

Step 2: If (3) is not true, without loss of generality, we assume $y^3_1 > 1/N + 1/(N^4L)$ and set $S = \{\ s \in [N] \mid y^3_s \le 1/N\ \}$ which is non-empty. Using the result of **Step 1**, we have

$$y^2_1 \ge \frac{1}{N} - \frac{1}{N^5L} \qquad \text{and} \qquad y^2_s \le \frac{1}{N} + \frac{1}{N^6L},\ \ \forall\ s \in S\ .$$

If P_1 plays strategy $(1, 0)$ or $(1, 1)$, its expected payoff $p \ge M(y^2_1 + y^3_1 - y^2_1 y^3_1)$. If P_1 plays strategy $(s, 0)$ or $(s, 1)$ where $s \in S$, its expected payoff $p' \le 1 + M(y^2_s + y^3_s - y^2_s y^3_s)$. As

$$p - p' \ge 2\left(N^2 - 2N + \frac{1}{N^3L} + \frac{1}{N}\right) - 1 > \frac{1}{2^{k^*}}\ ,$$

we have $y^1_s = 0$ for any $s \in S$. Similarly as **Step 1**, we pick any $i \notin S$ satisfying $y^1_i \ge 1/N$ and observe P_2. It's easy to check that, compared with $(i, 0), (i, 1)$, strategies $(s, 0), (s, 1)$ are preferred by player P_2 where $s \in S$, which contradicts with (2).

Step 3: If (1) is not true, without loss of generality, we assume $y_1^1 > 1/N + 1/(N^2L)$ and $s \in [N]$ satisfies that $y_s^1 \leq 1/N$. Using the result of **Step 2**, we have

$$y_1^3 \geq \frac{1}{N} - \frac{1}{N^3L} \qquad \text{and} \qquad y_s^3 \leq \frac{1}{N} + \frac{1}{N^4L}\ .$$

If P_2 plays strategy $(1,0)$ or $(1,1)$, its payoff $p \leq 1 - M(y_1^1 + y_1^3 - y_1^1y_1^3)$. On the other hand, if P_2 plays strategy $(s,0)$ or $(s,1)$, its payoff $p' \geq -M(y_s^1 + y_s^3 - y_s^1y_s^3)$. As

$$p' - p \geq 2\left(N^4 - 2N^3 + N + \frac{N}{L}\right) - 1 > \frac{1}{2^{k^*}}\,,$$

we have $y_1^2 = 0$, which contradicts with (2). □

Now we are ready to prove Theorem 1, and finally finish the reduction.

Proof of Theorem 1. Let y be any $1/2^{k^*}$-Nash equilibrium of game $\mathcal{G}^*$ and x be the profile of mixed strategies constructed in Theorem 1. For any u in $\mathcal{GG}$, we use p_0^u (p_1^u) to denote the expected payoff to u in $\mathcal{GG}$ when it plays strategy 0 (1). To prove that x is a $1/2^n$-Nash equilibrium of $\mathcal{GG}$, it's only necessary to prove the following inequalities for any u in $\mathcal{GG}$.

$$\begin{aligned}
p_0^u - p_1^u > \frac{1}{2^n} \quad &\Longrightarrow \quad p^c_{(u,0)} - p^c_{(u,1)} > \frac{1}{2^{k^*}} \qquad \left(\ \Longrightarrow \quad y^c_{(u,1)} = 0 \quad \Longrightarrow \quad x_1^u = 0\ \right)\\
p_1^u - p_0^u > \frac{1}{2^n} \quad &\Longrightarrow \quad p^c_{(u,1)} - p^c_{(u,0)} > \frac{1}{2^{k^*}} \qquad \left(\ \Longrightarrow \quad y^c_{(u,0)} = 0 \quad \Longrightarrow \quad x_0^u = 0\ \right)
\end{aligned}$$

where $c = c(u)$ and $p^c_{(u,0)}$ ($p^c_{(u,1)}$) is the payoff to P_c in $\mathcal{G}^*$ if it plays $(u,0)$ ($(u,1)$).

Using Lemma 2, we have

$$\begin{aligned}
p^c_{(u,0)} - p^c_{(u,1)} \quad &\geq \quad \left(\frac{1}{N} - \frac{1}{NL}\right)^2 p_0^u - \left(\frac{1}{N} + \frac{1}{NL}\right)^2 p_1^u\\
&> \quad \frac{1}{N^2}\left(1 - \frac{1}{L}\right)^2 \frac{1}{2^n} - \frac{4}{N^2L} \qquad (\text{ Property } \mathbf{D} \Rightarrow p_0^u, p_1^u \in [0,1]\ )\\
&> \quad \frac{1}{4N^2 2^n} - \frac{4}{N^2 2^{2n}} > \frac{1}{N^2 2^{2n}} > \frac{1}{2^{k^*}}
\end{aligned}$$

for any u in $\mathcal{GG}$. The other inequality can be proven similarly. □

6 Concluding Remarks

Two major open problems, 3-Nash and 2-Nash, were proposed in [4] after their **PPAD**-complete proof of 4-Nash. They conjectured that 3-Nash remains **PPAD**-complete, and suspected that 2-Nash may not be **PPAD**-complete. Our result confirms their first conjecture. Their second conjecture remains the most prominent open problem in Algorithmic Game

Utilities of the 3-Player Game $\mathcal{G}^*$

for any $s = ((i_1, j_1), (i_2, j_2), (i_3, j_3))$ where $i_1, i_2, i_3 \in [N]$ and $j_1, j_2, j_3 \in \{0, 1\}$ **do**
    if $i_1 = i_2$ or $i_1 = i_3$ **then**
        set (the payoff to player P_1) $u_s^1 = M$
    else
        set (the payoff to player P_1) $u_s^1 = 0$
    if $i_2 = i_1$ or $i_2 = i_3$ **then**
        set (the payoff to player P_2) $u_s^2 = -M$
    else
        set (the payoff to player P_2) $u_s^2 = 0$
    if $i_3 = i_2$ **then**
        set (the payoff to player P_3) $u_s^3 = M$
    else
        set (the payoff to player P_3) $u_s^3 = 0$
for any vertex u in $\mathcal{GG}$ with 4 incoming edges **do**
    assume $c(u) = 1$, other cases can be handled similarly
    let u_2 and u_3 be the two vertices which can affect the payoff to u when u plays 0
    let u_2' and u_3' be the two vertices which can affect the payoff to u when u plays 1
    assume $c(u_2) = c(u_2') = 2$ and $c(u_3) = c(u_3') = 3$
    for any $i, j \in \{0, 1\}$ **do**
        let a be the payoff to u in game $\mathcal{GG}$ when u plays 0, u_2 plays i and u_3 plays j
        set $u_s^1 = u_s^1 + a$ where pure strategy profile $s = ((u, 0), (u_2, i), (u_3, j))$
        let b be the payoff to u in game $\mathcal{GG}$ when u plays 1, u_2' plays i and u_3' plays j
        set $u_s^1 = u_s^1 + b$ where pure strategy profile $s = ((u, 1), (u_2', i), (u_3', j))$

Figure 3: Utilities of the 3-Player Game $\mathcal{G}^*$

Theory. It would require new insight and new techniques for developing a conclusive answer to this problem.

For two similar and very closely related problems, the oracle model of the Brouwer fixed point problem [1, 7], and the polynomial-time Turing machine model of the Sperner's problem [2], their algorithmic complexities for the 2-dimensional cases exhibit similar structures to the high dimensional cases. If 2-Nash indeed turns against such parallelism, it is possible that the impact could expand beyond what has already been the most amazing development in the interplays of Computer Science and Economics, started from fifteen years ago with the introduction of the **PPAD** class [11, 12] in a daring effort to characterize the parity argument in mathematical proofs, gradually made its way into our understanding of many important problems till the recent exalted inclusion of the r-Nash into the class of **PPAD**-complete problems. The challenge now is to settle the ultimate open problem of 2-Nash.

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