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Three-Player Games Are Hard

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Abstract

We prove that computing a Nash equilibrium in a 3-player game is PPAD-complete, solving a problem left open in (2).

1 Introduction

In 1951 Nash proved that every game has a mixed Nash equilibrium (5); however, the complexity of the computational problem of finding such an equilibrium in polynomial time had remained open for a long time. This problem was one of those that motivated the definition of the class PPAD (6) — the class of all search problems whose solutions are guaranteed to exist by means of a directed graph argument; however, unlike the problem of finding a Brouwer fixed point, the problem of finding a Nash equilibrium was not known to be PPAD-complete (and thus, presumably, intractable). Recently this open problem was resolved: It was shown (2) that finding a Nash equilibrium in a 4-player game is indeed PPAD-complete. It was conjectured in (2) that the 3-player case is also PPAD-complete. In this note we prove this conjecture.

The proof in (2) is a reduction from a stylized version of Brouwer's problem, and utilizes certain "gadgets" invented in a previous paper (3): Games that behave like arithmetical and logical gates. The reduction then constructs a graphical game (a game whose players are nodes in a graph and who affect each other only if they are neighbors on the graph), combining the various gadgets, that ends up "computing" the value of a Brouwer function — in fact, averaged over a fine grid of points. Completeness of the 4-player case then follows from a result in (3) whereby any game can be simulated by a 4-player game. The number 4 comes up as the chromatic number of a particular undirected graph associated with a graphical game constructed in the proof of (2) (called here the *moralized* graph of the graphical game).

Our proof is based on a variant of the arithmetical gadget of (3), containing few extra nodes. The new gadget, when used in the reduction from Brouwer's problem in (2), has the effect of creating a graphical game whose moralized graph is 3-colorable, thus reducing the number of players needed to simulate the graphical game to three.

Independently, Chen and Deng (1) have also come up with a proof of the same result. Their proof is quite different from ours: Without changing the gadgets, they come up with a more sophisticated version of the simulation in (3) in which just three players are needed to simulate a general game (despite the fact that 4 colors are required to color the moralized graph). They do this by exploiting a subtle form of disconnectedness in the graphical game constructed in (3).

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2 Definitions

Games

An *n-player game in normal form*, $n \geq 2$, is a set $[n]$ of players and a finite set of *strategies* S_i for each player i. The set S of *pure strategy profiles* is the Cartesian products of the S_i's. We denote the set of all strategy profiles of players other than i by S_{-i}. Finally, for each $i \leq n$ and $s \in S$ we have an integer *payoff* or *utility* u^i_s.

A *mixed strategy* for player i is a distribution on S_i, that is, real numbers $x_j \geq 0$ for each strategy $j \in S_p$ such that $\sum_{j \in S_i} x_j = 1$. A set of n mixed strategies $x^i_j, i = 1, \ldots, n, j \in S_i$ are called a *(mixed) Nash equilibrium* if, for each i, the expected utility $\sum_{s \in S} u^i_s x^1_{s_1} \cdot x^2_{s_2} \cdots x^n_{s_n}$ is maximized over all mixed strategies of player i. The x^i_j's are an ϵ-approximate Nash equilibrium if the expected utility is maximum within an additive ϵ.

A normal-form game with n players and s strategies per player requires ns^n integers (utilities) for its representation. *Graphical games* (4) allow a much more succinct representation. A *graphical game* is defined in terms of a directed graph $G = ([n], E)$, where $[n]$ is the set of players. Let $G[i]$ denote the set $\{i\} \cup \{j : (j, i) \in E\}$. The utility of player i in a graphical game is a function $u_i : \prod_{j \in G[i]} S_j \to \mathbb{Z}$. In other words, only players with edges into i can affect i. (In (4) and the literature it is assumed that G is symmetric, hence the present formulation is, strictly speaking, a generalization.) If d is the highest in-degree of G, and each player has s strategies, the graphical game requires ns^{d+1} integers for its representation. Since graphical games are just a different representation of normal-form games, the definition of Nash equilibrium is carried over.

Complexity

We next define certain important search problems in NP (see (2) for a more extensive review of the complexity theory of search problems). k-NASH is the search problem in which the input is a k-player game in normal form together with a binary integer A (in binary, the *accuracy specification*), and the set of solutions is the set of $\frac{1}{A}$-approximate Nash equilibria of the game. Similarly, d-GRAPHICAL NASH is the search problem with inputs the set of all graphical games with degree at most d, plus an accuracy specification, and solutions the corresponding approximate Nash equilibria. A *reduction* from search problem A to search problem B consists of a polynomial algorithm that transforms any instance I of A to an instance I' of B, and any solution of I' to a solution of I. In (3) the following is shown:

Theorem 1 *(3) There are polynomial-time reductions from* k-NASH *and* d-GRAPHICAL NASH*, for any* $k, d \geq 2$*, to both* 4-NASH *and* 3-GRAPHICAL NASH.

A search problem is *total* if the set of solutions is guaranteed to be nonempthy. For example, Nash's 1951 theorem (5) implies that k-NASH is total. Obviously, the same is true of d-GRAPHICAL NASH. An important class of total search problems is PPAD, defined as the class of all total search problems reducible to the following:

END OF THE LINE: Given two circuits P and C with n input bits and n output bits, such that $P(0^n) = 0^n \neq C(0^n)$, find an input $x \in \{0, 1\}^n$ such that $P(C(x)) \neq x$ or $C(P(x)) \neq x \neq 0^n$.

Intuitively, END OF THE LINE creates a directed graph with vertex set $\{0, 1\}^n$ and an edge from x to y whenever $P(y = C(x)) = x$ (C and P stand for "possible child" and "possible parent"). This graph has indegree and outdegree at most one, and at least one source, namely 0^n, so it must have a sink. We are looking either for a sink, or for a source other than 0^n. Thus, PPAD ("polynomial parity argument, directed version") is the class of all total search problems whose totality is proven via the following simple combinatorial argument: "If a directed graph has an unbalanced node, then it muct have another."

A search problem S in PPAD is called *PPAD-complete* if all problems in PPAD reduce to it. Obviously, END OF THE LINE is PPAD-complete; the main result in (2) is the following.

Theorem 2 *(2)* 4-NASH *is PPAD-complete.*

This result is proved by a reduction from another problem proved PPAD-complete in (2), 3-DIMENSIONAL BROUWER. We are given a Brouwer function ϕ of a very special form on the 3-dimensional unit cube, defined in terms of its values at the centers of the 2^{3n} cubelets with side 2^{-n}; those values are in turn given in terms of a circuit with $3n$ binary inputs. At the center c of the cublet

$$K_{ijk} = \{(x, y, z) : i2^{-n} \le x \le (i+1)2^{-n}, j2^{-n} \le y \le (j+1)2^{-n}, k2^{-n} \le z \le (k+1)2^{-n}\},$$

where i, j, k are integers in $[2^n]$, the value of $\phi(c)$ is $c + \delta$, where δ is one of the following four displacement vectors:

- $\delta_1 = (\alpha, 0, 0)$
- $\delta_2 = (0, \alpha, 0)$
- $\delta_3 = (0, 0, \alpha)$
- $\delta_0 = (-\alpha, -\alpha, -\alpha)$

where $\alpha > 0$ is much smaller than the cubelet side, say 2^{-2n}. The correct index j of δ_j for the center of K_{ijk} is computed by the given circuit.

Call a vertex of a cubelet *panchromatic* if, among the eight cubelets adjacent to it, there are four that have all four increments $\delta_0, \delta_1, \delta_2, \delta_3$. 3-DIMENSIONAL BROUWER is then the following problem: Given a circuit C as described above, find a panchromatic vertex.

3 The Main Result

In this section we show how to modify the reduction in (2) to prove the following.

Theorem 3 *3*-NASH *is PPAD-complete.*

3.1 The Structure of the Reduction in (2)

The PPAD-completeness proof in (2) is a reduction from 3-DIMENSIONAL BROUWER. Given an instance C of 3-DIMENSIONAL BROUWER, we construct a graphical game $\mathcal{GR}$ and an $\epsilon > 0$ such that any ϵ-Nash equilibrium of $\mathcal{GR}$ identifies a panchromatic cubelet of the instance.

$\mathcal{GR}$ is a *binary* graphical game if each vertex has two strategies, called 0 and 1. Any mixed strategy profile in a binary game of r players can be seen as an r-tuple of numbers in $[0, 1]^r$, by interpreting the probability that vertex i plays 1 as the "value" of r, denoted $\mathbf{p}[r]$ (or $\mathbf{p}[r : 1]$).

The graphical game $\mathcal{GR}$ in the construction of (2) is composed of certain special binary games invented in (3). (By $\alpha = \beta \pm \gamma$ we mean $\beta - \gamma \le \alpha \le \beta + \gamma$.)

Lemma 1 *(2) There are binary graphical games $G_\zeta, G_{\times\zeta}$ where ζ is any real in $[0, 1]$, G_+, G_-, $G_<$,$G_\vee$,$G_\wedge$,$G_\neg$ with vertices a, b, c, d, such that:*

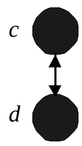

Figure 1: $\mathcal{G}_\zeta$

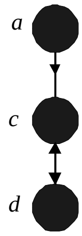

Figure 2: $\mathcal{G}_=, \mathcal{G}_{\times\zeta}$

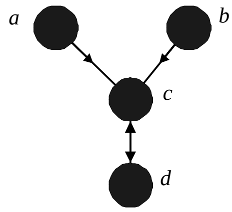

Figure 3: $\mathcal{G}_+$

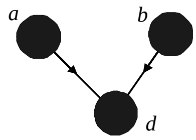

Figure 4: $\mathcal{G}_<$,$\mathcal{G}_\vee$,$\mathcal{G}_\wedge$

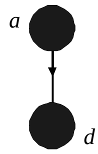

Figure 5: $\mathcal{G}_\neg$

1. *the underlying graphs are those shown in figures 1 through 5 (notice that certain games do not have all four nodes);*
2. *at any ϵ-Nash equilibrium in G_ζ we have $\mathbf{p}[d] = \zeta \pm \epsilon$;*
3. *at any ϵ-Nash equilibrium in $G_{\times\zeta}$ we have $\mathbf{p}[d] = \zeta\mathbf{p}[a] \pm \epsilon$;*
4. *at any ϵ-Nash equilibrium in $G_=$ we have $\mathbf{p}[d] = \mathbf{p}[a] \pm \epsilon$;*
5. *at any ϵ-Nash equilibrium in G_+ we have $\mathbf{p}[d] = \min\{1, \mathbf{p}[a] + \mathbf{p}[b]\} \pm \epsilon$;*
6. *at any ϵ-Nash equilibrium in G_- in which $\mathbf{p}[a] \geq \mathbf{p}[b]$, we have $\mathbf{p}[d] = \mathbf{p}[a] - \mathbf{p}[b] \pm \epsilon$;*
7. *at any ϵ-Nash equilibrium in $G_<$ we have $\mathbf{p}[d] = 1$ if $\mathbf{p}[a] < \mathbf{p}[b] - \epsilon$ and $\mathbf{p}[d] = 0$ if $\mathbf{p}[a] > \mathbf{p}[b] + \epsilon$.*
8. *at any ϵ-Nash equilibrium in $G_\vee$ in which $\mathbf{p}[a], \mathbf{p}[b] \in \{0, 1\}$, $\mathbf{p}[d] = 1$ if at least one of $\mathbf{p}[a], \mathbf{p}[b]$ is 1, and $\mathbf{p}[d] = 0$ otherwise;*
9. *and similarly for the other logical connectives $G_\wedge$ and $G_\neg$.*

Of these, $G_<$ is called the *brittle comparator* because of its unreliability when its inputs are close together.

Using these components, we construct a binary graphical game with three distinguished players x, y, z, whose values encode the three coordinates in the instance of 3-DIMENSIONAL BROUWER. Starting from the values of these three players, various components perform the following:

1. Extract the $3n$ bits of the integers i, j, k such that point (x, y, z) lies within cubelet K_{ijk} of the cube: $i = \lfloor x/2^{-n} \rfloor$ and similarly for j, k. This can be done by $3n$ sequences of a comparison, a subtraction, and a doubling. The only problem is that, because of the brittle comparators, at the cubelet boundaries the results will be essentially arbitrary; this is dealt with in item (3) below.

2. Starting from the bits of i, j, k, we use the logical gate gadgets to simulate the given circuit C that computes the displacement at (x, y, z).

3. To deal with the brittle comparator problem at the boundaries of the cubelets, we choose $\epsilon = \alpha^2 = 2^{-4n}$ -where α is the length of the standard displacement chosen in the definition of 3-DIMENSIONAL BROUWER, repeat this calculation with inputs on the fine grid $x + p \cdot \alpha, y + q \cdot \alpha, z + r \cdot \alpha$, $p, q, r = -20, -19, \ldots, 20$, and average the results.

4. Finally, we close the loop by "nudging" the values of x, y, z by the calculated displacement, thus making sure that at any Nash equilibrium the displacement is zero (more accurately, less than the required accuracy).

This completes our sketch of the parts of the proof in (2) that are relevant to our argument in this paper.

3.2 The Moralized Graph

Given a (graphical game with) directed graph $G = ([n], E)$, define the *moralized graph of* G to be the undirected graph $G^m = ([n], E')$, where $\{i, j\} \in E'$ if one of the following three conditions hold: (a) $(i, j) \in E$; (b) $(j, i) \in E$; (c) there is a $k \in [n]$ such that both $(i, k), (j, k) \in E$.

The importance of this graph lies in the following Lemma, implicit in (3):

Lemma 2 *Let $\mathcal{GR}$ be a graphical game on graph $G = ([n], E)$ and suppose that the moralized graph G^m is k-colorable for an integer $k > 1$. Then a k-player normal-form game $\mathcal{G}$ can be constructed in polynomial time such that from any ϵ-approximate Nash equilibrium of $\mathcal{G}$ we can recover an ϵ-approximate Nash equilibrium of $\mathcal{GR}$.*

Proof. Each player in $\mathcal{G}$ is a color class of G^m; the definition of G^m ensures that there are no "conflicts of interest" or "conspiracy opportunities" within a class. The strategies of a player in $\mathcal{G}$ have two components: a choice of a vertex in the color class, and a choice of a strategy for this vertex. The payoffs are the payoffs to the vertices under these choices of strategies. To ensure that the color classes spread their probabilities evenly on their players, we make them also play a generalized "rock-paper-scissors" game at high stakes. See the proof of theorem 1 in (3) for details. □

3.3 The New Gadget

Note that the game for addition $\mathcal{G}_+$ described above (see figure 3) does has a moralized graph K_4, which is not 3-colorable. We fix this ibelow.

Lemma 3 *There is a graphical game $\mathcal{G}_+$ with two input players a and b, one output player c and several intermediate players, with the following properties:*

- *the moralized graph of the game is* 3*-colorable*
- *at any ϵ-Nash equilibrium in $\mathcal{G}_+$ it holds that* $\mathbf{p}[c] = \min\{1, \mathbf{p}[a] + \mathbf{p}[b]\} \pm 25\epsilon$

Proof. The graph of the game and the labeling of the nodes is shown in figure 6. All players of $\mathcal{G}_+$ have strategy set $\{0, 1\}$ except for player b' who has three strategies $\{0, 1, *\}$. Below we give the payoff tables of all the players of the game. For ease of understanding we partition the game $\mathcal{G}_+$ into four subgames:

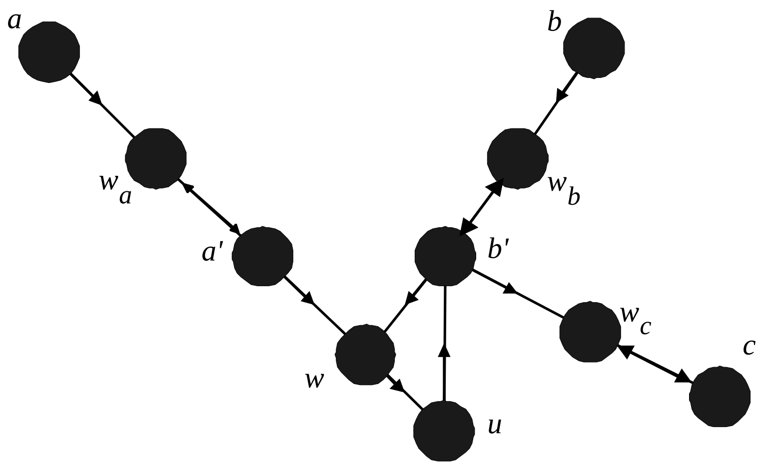

Figure 6: The new addition game.

1. Game played by players a, w_a, a':

Payoffs to a' :

	w_a plays 0	w_a plays 1
a' plays 0	0	1
a' plays 1	1	0

Payoffs to w_a:

w_a plays 0 :

	a' plays 0	a' plays 1
a plays 0	0	0
a plays 1	1/8	1/8

w_a plays 1 :

	a' plays 0	a' plays 1
a plays 0	0	1
a plays 1	0	1

2. Game played by players b', w_c, c:

Payoffs to c :

	w_c plays 0	w_c plays 1
c plays 0	0	1
c plays 1	1	0

Payoffs to w_c:

w_c plays 0 :

	c plays 0	c plays 1
b' plays 0	0	0
b' plays 1	0	0
b' plays $*$	8	8

w_c plays 1 :

	c plays 0	c plays 1
b' plays 0	0	1
b' plays 1	0	1
b' plays $*$	0	1

3. Game played by players b, w_b, b':

Payoffs to w_b:

w_b plays 0 :

	b plays 0	b plays 1
b' plays 0	0	1/8
b' plays 1	0	1/8
b' plays $*$	0	1/8

w_b plays 1 :

	b plays 0	b plays 1
b' plays 0	0	0
b' plays 1	1	1
b' plays $*$	0	0

Payoffs to b':

b' plays 0 :

	w_b plays 0	w_b plays 1
u plays 0	0	1
u plays 1	0	0

b' plays 1 :

	w_b plays 0	w_b plays 1
u plays 0	1	0
u plays 1	1	0

b' plays $*$:

	w_b plays 0	w_b plays 1
u plays 0	0	0
u plays 1	0	1

4. Game played by players a', b', w, u:

Payoffs to w:

w plays 0 :

	a' plays 0	a' plays 1
b' plays 0	0	1
b' plays 1	2	3
b' plays $*$	0	1

w plays 1 :

	a' plays 0	a' plays 1
b' plays 0	0	0
b' plays 1	1	1
b' plays $*$	1	1

Payoffs to u:

	w plays 0	w plays 1
u plays 0	0	1
u plays 1	1	0

Claim 1 *At any ϵ-Nash equilibrium of $\mathcal{G}_+$: $\mathbf{p}[a'] = \frac{1}{8}\mathbf{p}[a] \pm \epsilon$.*

Proof. If w_a plays 0, then the expected payoff to w_a is $\frac{1}{8}\mathbf{p}[a]$, whereas if w_a plays 1, the expected payoff to w_a is $\mathbf{p}[a']$. Therefore, in an ϵ-Nash equilibrium, if $\frac{1}{8}\mathbf{p}[a] > \mathbf{p}[a'] + \epsilon$, then $\mathbf{p}[w_a] = 0$. However, note also that if $\mathbf{p}[w_a] = 0$ then $\mathbf{p}[a'] = 1$, which is a contradiction to $\frac{1}{8}\mathbf{p}[a] > \mathbf{p}[a'] + \epsilon$. Consequently, $\frac{1}{8}\mathbf{p}[a]$ cannot be strictly larger than $\mathbf{p}[a'] + \epsilon$. On the other hand, if $\mathbf{p}[a'] > \frac{1}{8}\mathbf{p}[a] + \epsilon$, then $\mathbf{p}[w_a] = 1$ and consequently $\mathbf{p}[a'] = 0$, a contradiction. The claim follows from the above observations. $\square$

Claim 2 *At any ϵ-Nash equilibrium of $\mathcal{G}_+$: $\mathbf{p}[b' : 1] = \frac{1}{8}\mathbf{p}[b] \pm \epsilon$.*

Proof. If w_b plays 0, then the expected payoff to w_b is $\frac{1}{8}\mathbf{p}[b]$, whereas, if w_b plays 1, the expected payoff to w_b is $\mathbf{p}[b' : 1]$.

If, in a ϵ-Nash equilibrium, $\frac{1}{8}\mathbf{p}[b] > \mathbf{p}[b' : 1] + \epsilon$, then $\mathbf{p}[w_b] = 0$. In this regime, the payoff to player b' is 0 if b' plays 0, 1 if b' plays 1 and 0 if b' plays $*$. Therefore, $\mathbf{p}[b' : 1] = 1$ and this contradicts the hypothesis that $\frac{1}{8}\mathbf{p}[b] > \mathbf{p}[b' : 1] + \epsilon$.

On the other hand, if, in a ϵ-Nash equilibrium, $\mathbf{p}[b' : 1] > \frac{1}{8}\mathbf{p}[b] + \epsilon$, then $\mathbf{p}[w_b] = 1$. In this regime, the payoff to player b' is $\mathbf{p}[u : 0]$ if b' plays 0, 0 if b' plays 1 and $\mathbf{p}[u : 1]$ if b' plays $*$. Since $\mathbf{p}[u : 0] + \mathbf{p}[u : 1] = 1$, it follows that $\mathbf{p}[b' : 1] = 0$ [1]. This contradicts the hypothesis that $\mathbf{p}[b' : 1] > \frac{1}{8}\mathbf{p}[b] + \epsilon$. The claim follows from the above observations. $\square$

Claim 3 *At any ϵ-Nash equilibrium of $\mathcal{G}_+$: $\mathbf{p}[b' : *] = \frac{1}{8}\mathbf{p}[a] + \frac{1}{8}\mathbf{p}[b] \pm 3\epsilon$.*

[1]Recall that, in the reduction of (2), ϵ is chosen to be $\epsilon = \alpha^2$ -where $\alpha = 2^{-2n}$ is the parameter in the definition of the 3-DIMENSIONAL BROUWER problem- so ϵ is a very small value compared to 1/2; since $\mathbf{p}[u : 0] + \mathbf{p}[u : 1] = 1$, at least one of $\mathbf{p}[u : 0]$, $\mathbf{p}[u : 1]$ will be at least $1/2 >> \epsilon$.

Proof. If w plays 0, then the expected payoff to w is $\mathbf{p}[a'] + 2\mathbf{p}[b' : 1]$, whereas, if w plays 1, the expected payoff to w is $\mathbf{p}[b' : 1] + \mathbf{p}[b' : *]$.

If, in a ϵ-Nash equilibrium, $\mathbf{p}[a'] + 2\mathbf{p}[b' : 1] > \mathbf{p}[b' : 1] + \mathbf{p}[b' : *] + \epsilon$, then $\mathbf{p}[w] = 0$ and, consequently, $\mathbf{p}[u] = 1$. In this regime, the payoff to player b' is 0 if b' plays 0, $\mathbf{p}[w_b : 0]$ if b' plays 1 and $\mathbf{p}[w_b : 1]$ if b' plays $*$. Since $\mathbf{p}[w_b : 0]+\mathbf{p}[w_b : 1] = 1$, it follows that $\mathbf{p}[b' : 0] = 0$ or, equivalently, that $\mathbf{p}[b' : 1] + \mathbf{p}[b' : *] = 1$. So the hypothesis can be rewritten as $\mathbf{p}[a'] + 2\mathbf{p}[b' : 1] > 1 + \epsilon$. Using claims 1 and 2 this can be restated as $\frac{1}{8}\mathbf{p}[a] + \frac{1}{4}\mathbf{p}[b' : 1] \pm 2\epsilon > 1 + \epsilon$ which is a contradiction since $\epsilon = \alpha^2 = 2^{-4n}$.

On the other hand, if, in a ϵ-Nash equilibrium, $\mathbf{p}[b' : 1] + \mathbf{p}[b' : *] > \mathbf{p}[a'] + 2\mathbf{p}[b' : 1] + \epsilon$, then $\mathbf{p}[w] = 1$ and consequently $\mathbf{p}[u] = 0$. In this regime, the payoff to player b' is $\mathbf{p}[w_b : 1]$ if b' plays 0, $\mathbf{p}[w_b : 0]$ if b' plays 1 and 0 if b' plays $*$. Since $\mathbf{p}[w_b : 0] + \mathbf{p}[w_b : 1] = 1$, it follows that $\mathbf{p}[b' : *] = 0$. So the hypothesis can be rewritten as $0 > \mathbf{p}[a'] + \mathbf{p}[b' : 1] + \epsilon$ which is a contradiction.

Therefore, in any ϵ-Nash equilibrium, $\mathbf{p}[b' : 1] + \mathbf{p}[b' : *] = \mathbf{p}[a'] + 2\mathbf{p}[b' : 1] \pm \epsilon$, or, equivalently, $\mathbf{p}[b' : *] = \mathbf{p}[a'] + \mathbf{p}[b' : 1] \pm \epsilon$. Using claims 1 and 2 this can be restated as $\mathbf{p}[b' : *] = \frac{1}{8}\mathbf{p}[a] + \frac{1}{8}\mathbf{p}[b] \pm 3\epsilon$. □

Claim 4 *At any ϵ-Nash equilibrium of $\mathcal{G}_+$:* $\mathbf{p}[c] = \min\{1, \mathbf{p}[a] + \mathbf{p}[b]\} \pm 25\epsilon$.

Proof. If w_c plays 0, the expected payoff to w_c is $8\mathbf{p}[b' : *]$, whereas, if w_c plays 1, the expected payoff to w_c is $\mathbf{p}[c]$. Therefore, in a ϵ-Nash equilibrium, if $\mathbf{p}[c] > 8\mathbf{p}[b' : *] + \epsilon$, then $\mathbf{p}[w_c] = 1$ and, consequently, $\mathbf{p}[c] = 0$, which is a contradiction to $\mathbf{p}[c] > 8\mathbf{p}[b' : *] + \epsilon$.

On the other hand, if $8\mathbf{p}[b' : *] > \mathbf{p}[c] + \epsilon$, then $\mathbf{p}[w_c] = 0$ and consequently $\mathbf{p}[c] = 1$. Hence, $\mathbf{p}[c]$ cannot be less than $\min\{1, 8\mathbf{p}[b' : *] - \epsilon\}$.

From the above observations it follows that $\mathbf{p}[c] = \min\{1, 8\mathbf{p}[b' : *]\} \pm \epsilon$ and, using claim 3, $\mathbf{p}[c] = \min\{1, \mathbf{p}[a] + \mathbf{p}[b]\} \pm 25\epsilon$. □

It remains to show that the moralized graph of the graph shown in figure 6 is 3-colorable. The moralized graph and its 3-coloring is shown in figure 7. □

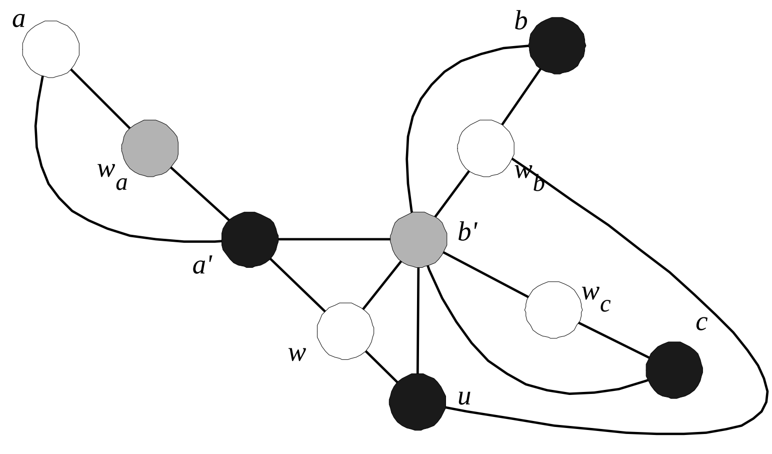

Figure 7: The 3-coloring of the moralized graph of $\mathcal{G}_+$.

3.4 Completing the Proof

Recall the graphical game constructed in the reduction from 3-DIMENSIONAL BROUWER outlined in section 3.1, using the components defined in lemma 1. Note that all games of lemma 1 have moralized graphs that are 3-colorable except for the addition game. Therefore, now that we have constructed a new addition game whose moralized graph is 3-colorable, we are able to construct our graphical game out of components which, by themselves, have 3-colorable moralized graphs. It remains to show that it is possible to "glue" these components together so that the resulting graphical game has moralized graph which is 3-colorable.

Recall that all our gadgets have some distinguished nodes which are the *inputs* and one distinguished node which is the *output* and the construction proceeds by identifying the output nodes of some gadgets as input nodes of other gadgets. It is easy to see that we could get a graphical game with the same functionality if, instead of identifying the output node of some gadget with the input of another gadget, we interposed a sequence of two $\mathcal{G}_=$ games between the two gadgets to be connected, as shown in figure 8. If we "glue" our gadgets in this way then the graphical game

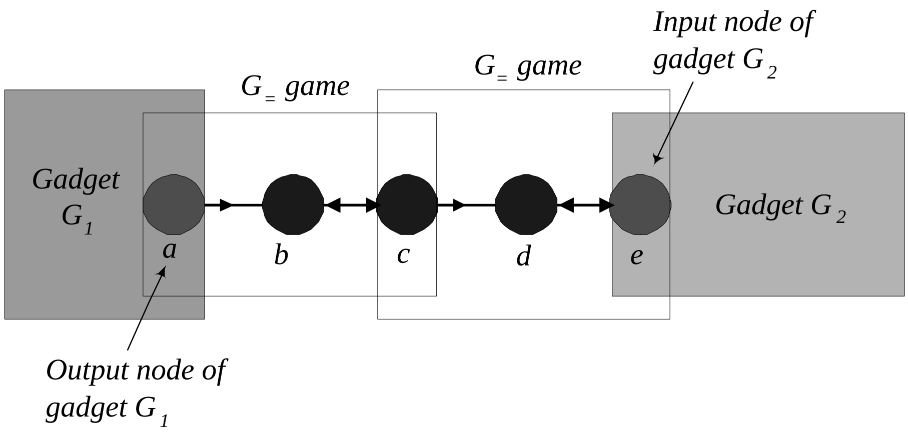

Figure 8: The interposition of two $\mathcal{G}_=$ games between gadgets G_1 and G_2 does not change the game.

$\mathcal{GR}$ that is constructed can be easily shown to have a 3-colorable moralized graph by making the following observations:

- to moralize the graph of $\mathcal{GR}$ it is enough to moralize the graph of every gadget and then moralize the "connections" between gadgets as shown below

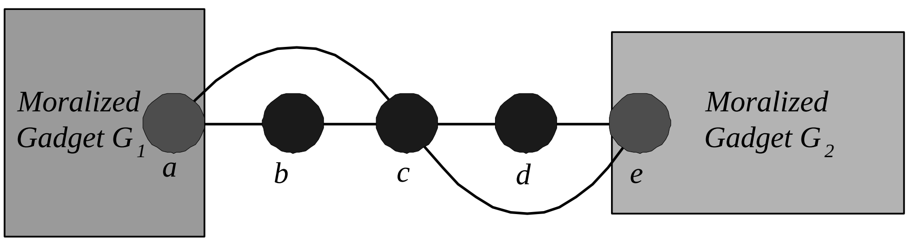

Figure 9: Moralization of the connection.

- a 3 coloring of the moralized graph G^m of $\mathcal{GR}$ can be found as follows:

  i. (stage 1) 3-color the moralized graphs within the "initial gadgets" of the construction

  ii. (stage 2) extend the coloring to the nodes that serve as "connections" between gadgets; any 3-coloring of the gadgets can be extended to a 3-coloring of G^m because, for any pair of gadgets G_1, G_2 which are connected (figure 9) and for any colors assigned to nodes

a and e, nodes b, c and d can be also colored legally. For example, if node a gets color 1 and node e color 2 at stage 1, then, at stage 2, b can be colored 2, c can be colored 3 and d can be colored 1.

This completes the proof of our main result.

4 Discussion

There seems to be absolutely no way that this construction, or any conceivable extension, can be refined to yield a bipartite graph proving that 2-NASH is PPAD-complete. It is conjectured in (2) that 2-NASH can be solved in polynomial time.

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